

# **Superconvergence in Finite Element Methods for the Optimal Control Problem of the Stokes Equations**

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# SUPERCONVERGENCE IN FINITE ELEMENT METHODS FOR THE OPTIMAL CONTROL PROBLEM OF THE STOKES EQUATIONS

A. RÖSCH<sup>†</sup> AND B. VEXLER<sup>‡</sup>

**Abstract.** An optimal control problem for 2-d and 3-d Stokes problem is investigated with pointwise control constraints. This paper is concerned with the discretization of the control by piecewise constant functions. The state and the adjoint state are discretized by stable or stabilized finite element schemes. In the paper a superconvergence based post-processing is suggested, which allows for significant improvement of the accuracy.

**Key words.** PDE-constrained optimization, finite elements, error estimates, Stokes equations, numerical approximation, control constraints.

**AMS subject classifications.** 49K20, 49M25, 65N30

**1. Introduction.** The paper is concerned with the discretization of the optimal control problem

$$\text{Minimize } J(v, q) = \frac{1}{2} \|v - v_d\|_{L^2(\Omega)^d}^2 + \frac{\nu}{2} \|q\|_{L^2(\Omega)^d}^2 \quad (1.1)$$

subject to the Stokes equations (state equation):

$$\begin{aligned} -\Delta v + \nabla p &= f + q && \text{in } \Omega, \\ \nabla \cdot v &= 0 && \text{on } \Omega, \\ v &= 0 && \text{on } \Gamma \end{aligned} \quad (1.2)$$

and subject to the control constraints

$$a \leq q(x) \leq b \quad \text{for a.a. } x \in \Omega, \quad (1.3)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^d$  with  $d = 2, 3$  and  $\Gamma$  is the boundary of  $\Omega$ . The quantities  $a, b \in \mathbb{R}^d$  are constant vectors and the inequality (1.3) is understood componentwise. We denote by  $u = (v, p)$  the solution of (1.2). Moreover, we assume  $v_d, f \in L^\infty(\Omega)^d$  and  $\nu > 0$ .

The set of admissible controls  $Q_{ad}$  is given by:

$$Q_{ad} = \{v \in Q := L^2(\Omega)^d : a \leq q \leq b \text{ a.e. in } \Omega\}.$$

We discuss here the full discretization of the control and the state equations by a finite element method. The asymptotic behavior of the discretized problem is studied.

Approximation properties of discretized optimal control problems are often investigated in the last years. First results were known for piecewise constant functions, we refer to Falk [14], Geveci [15], and Malanowski [23]. A renaissance of this topic was mainly initiated by the papers of Arada, Casas, and Tröltzsch [1] and Casas, Mateos, and Tröltzsch [8]. Error estimates of order  $h$  in the  $L^2$ -norm and in the  $L^\infty$ -norm are established in these articles.

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Piecewise linear control discretizations for elliptic optimal control problems are studied by Casas and Tröltzsch, see [9] and Casas [7]. These papers contain error estimates of order  $h$  and  $o(h)$  in the  $L^2$ -norm for general cases. For more regular cases an approximation order of  $h^{3/2}$  can be proved, see Rösch [26, 27]. An error estimate of order  $h$  in the  $L^\infty$ -norm for an elliptic problem is proved by Meyer and Rösch [25].

However, new discretization concepts have been developed in recent years. The variational approach by Hinze [19] and the superconvergence approach of Meyer and Rösch [24] can achieve approximation order  $h^2$  in the  $L^2$ -norm.

In our opinion the methods of the approaches cited above can be adapted to the Stokes equations. In this paper, we will generalize the superconvergence approach of Meyer and Rösch [24]: The controls are discretized by piecewise constant functions. Clearly, the approximation order of the control cannot be better than  $h$  for the optimal control. However, it will turn out that superconvergence effects appear. The values in the barycenter of the elements are approximated with order  $h^2$ . We will show that state and adjoint state have the approximation error  $h^2$  in the  $L^2$ -norm. A postprocessing step leads to a control with approximation error  $h^2$  in the  $L^2$ -norm, too.

Apart from the fact that the Stokes equations have a more complex structure than the equation investigated in [24], this paper contains an essential generalization in the theory. The theory presented in [24] works only for piecewise linear finite elements. The fact that the second derivative of each ansatz function vanishes identically on each triangle is used in a very explicit manner. Consequently, only piecewise linear finite elements defined on triangles can be handled by that technique. We will prove superconvergence results without such restrictions, i.e., only stability and interpolation properties of the elements are requested. Therefore, our results include many different finite element discretization schemes for the 2d and 3d Stokes equations.

To the best of the authors knowledge this is the first paper discussing the discretization error for the optimal control of the Stokes equations with pointwise control constraints. Of course, several papers are published for the optimal control of the Stokes equations and the Navier-Stokes equations without control constraints, see Gunzburger, Hou, and Svobodny [17, 18] and Bochev and Gunzburger [4].

Let us remark that the investigated optimal control problems governed by the Stokes equations occur as subproblems in several Newton-type methods for control constrained optimal control problems for the Navier-Stokes equations. The convergence theory of such Newton-type methods requires sufficiently accurate numerical solutions of the subproblems.

The paper is organized as follows: In Section 2 a general discretization concept is introduced and the main results are stated. Section 3 contains results from the finite element theory. The proofs of the superconvergence results are placed in Section 4. The assumption of the general discretization concept are verified for a specific discretization in Section 5. The paper ends with numerical experiments shown in Section 6.

**2. Discretization and superconvergence results.** Throughout this paper,  $\Omega$  denotes a bounded convex and polygonal domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ . Moreover, for the case  $d = 3$  we assume that edge openings of the domain  $\Omega$  are smaller than  $2\pi/3$ . This will ensure the  $W^{1,\infty}$ -regularity of the velocity field.

We denote by  $V$  and  $L$  the Hilbert spaces

$$\begin{aligned} V &:= H_0^1(\Omega)^d, \\ L &:= \{p \in L^2(\Omega) : \int_{\Omega} p(x) dx = 0\}. \end{aligned}$$

In all what follows, we will omit the subscript  $L^2$  in the norms and inner products if there is no risk of misunderstanding. We look for solutions of the Stokes equations (1.2) in the sense of a weak formulation: the following equation has to be satisfied for arbitrary  $\phi = (\psi, \xi) \in V \times L$

$$a(u, \phi) := (\nabla v, \nabla \psi) - (p, \nabla \cdot \psi) + (\nabla \cdot v, \xi) = (f + q, \psi) =: (F(q), \phi). \quad (2.1)$$

LEMMA 2.1. *Let  $g$  be a given function in  $L^\infty(\Omega)^d$ . Then there exists a unique solution  $u = (v, p) \in V \times L$  of*

$$\begin{aligned} -\Delta v + \nabla p &= g && \text{in } \Omega, \\ \nabla \cdot v &= 0 && \text{on } \Omega, \\ v &= 0 && \text{on } \Gamma. \end{aligned} \quad (2.2)$$

Moreover, there exist positive constants  $c$  and  $p' > d$  with  $v \in W^{2,p'}(\Omega)^d$ ,  $p \in H^1(\Omega)$  and

$$\|v\|_{W^{2,p'}(\Omega)^d} + \|p\|_{H^1(\Omega)} \leq c \|g\|_{L^\infty(\Omega)} \quad (2.3)$$

for  $d = 2$  or for  $d = 3$  and all edge openings are smaller than  $2\pi/3$ .

*Proof.* For the proof of this result on polygonal domains especially for  $d = 3$ , we refer to [13], Theorem 6.3.  $\square$

We will *assume* that (2.3) is valid for the investigated domain  $\Omega$ . However, it would be enough for the theory presented here to have this regularity for the optimal adjoint velocity  $\bar{w}$  introduced below.

In order to formulate the optimality system, we introduce the adjoint equation

$$\begin{aligned} -\Delta w - \nabla r &= v - v_d && \text{in } \Omega, \\ \nabla \cdot w &= 0 && \text{on } \Omega, \\ w &= 0 && \text{on } \Gamma. \end{aligned} \quad (2.4)$$

We denote by  $z = (w, r) \in V \times L$  the adjoint state. Due to Lemma 2.1 the adjoint velocity  $w$  belongs to  $W^{2,p'}(\Omega)^d$ . This space is embedded in  $W^{1,\infty}(\Omega)^d$ .

We say that  $u = (v, p)$  is the associated state to  $q$  if  $u$  is the solution of (1.2). Analogously, we call the solution  $z = (w, r)$  of (2.4) associated adjoint state to  $q$ .

LEMMA 2.2. *There exists a uniquely determined solution  $\bar{q}$  of the optimal control problem (1.1) – (1.3). Moreover, a necessary and sufficient condition for the optimality of a control  $\bar{q}$  with associated state  $\bar{u}$  and associated adjoint state  $\bar{z}$ , respectively, is that the variational inequality*

$$(\bar{w} + \nu \bar{q}, q - \bar{q})_Q \geq 0 \quad \text{for all } q \in Q_{ad} \quad (2.5)$$

*holds.*

The optimal control problem (1.1)–(1.3) is strictly convex and radially bounded. Hence, there exists a uniquely determined optimal solution and the first-order necessary conditions are also sufficient for optimality. Such basic results and an introduction in optimal control theory governed by partial differential equations can be found

for instance by Lions [22]. We remark that the variational inequality (2.5) can be equivalently formulated as

$$\bar{q} = \Pi_{[a,b]}\left(-\frac{1}{\nu}\bar{w}\right), \quad (2.6)$$

where the projection  $\Pi$  is defined by

$$\Pi_{[a,b]}(f(x)) = \max(a, \min(b, f(x))),$$

see for instance Malanowski [23]. Again, all functions are defined componentwise.

We are now able to introduce the discretized problem: The control  $q$  is discretized piecewise constant on a mesh  $\mathcal{T}_h$  containing open sets  $T$  (finite elements).

$$Q_h := \{q_h \in L^2(\Omega)^d : q_h|_T \in (\mathcal{P}_0)^d \text{ for all } T \in \mathcal{T}_h\}$$

Here  $\mathcal{P}_k$  denotes the polynomials with degree less or equal  $k$ . We require for the mesh  $\mathcal{T}_h$ :

**(A1)** The diameter of the largest element of  $\mathcal{T}_h$  is bounded by  $h$ . Moreover,

$$\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}, \quad T_i \cap T_j = \emptyset \quad \text{for all } T_i, T_j \in \mathcal{T}_h, i \neq j.$$

Next, we introduce a general conform finite element setting for the discretization of the state equation. Let  $V_h \subset V$  and  $L_h \subset L$  be finite dimensional subspaces with the following properties:

**(A2)** The space  $V_h$  and the mesh  $\mathcal{T}_h$  fit in the following sense: Every function  $v_h \in V_h$  is piecewise polynomial on  $\mathcal{T}_h$

$$v_h|_T \in \mathcal{P}^d \quad \text{for all } T \in \mathcal{T}_h,$$

where  $\mathcal{P}$  is a polynomial space.

Please note that there is no corresponding condition for  $L_h$ .

For a given control  $q \in Q$ , the state equation (2.1) is discretized using the spaces  $V_h$  and  $L_h$  as follows: Find  $u_h = (v_h, p_h)$  such that:

$$a(u_h, \phi_h) + s_h(p_h, \xi_h) = (F(q), \phi_h) \quad \text{for all } \phi_h = (\psi_h, \xi_h) \in V_h \times L_h. \quad (2.7)$$

Here, the term  $s_h(\cdot, \cdot)$  denotes a stabilization (continuous, symmetric) bilinear form on  $L_h \times L_h$ . Such stabilization terms are needed if, e.g., finite elements of equal order for the velocities and pressure are used, see [2] or [6]. For this discretization we require the following conditions: We introduce a norm on the space

$$V_h^2(\Omega)^d := \{v \in V : v|_T \in H^2(T)^d \text{ for all } T \in \mathcal{T}_h\},$$

$$\|\psi_h\|'_{H^2(\Omega)^d} := \left( \sum_{T \in \Omega} \|\psi_h\|_{H^2(T)^d}^2 \right)^{1/2}.$$

**(A3)** There exist interpolation operators  $i_h^v : H^2(\Omega)^d \cap V \rightarrow V_h$  and  $i_h^p : H^1(\Omega) \cap L \rightarrow L_h$  with the following approximation properties

$$\begin{aligned} \|v - i_h^v v\|_{L^2(\Omega)^d} + h \|\nabla(v - i_h^v v)\|_{L^2(\Omega)^d} &\leq c_v h^2 \|\nabla^2 v\|_{L^2(\Omega)^{d \times d}}, \\ \|v - i_h^v v\|_{L^\infty(\Omega)^d} + h^{2-d/2} \|v - i_h^v v\|'_{H^2(\Omega)^d} &\leq c_v h^{2-d/2} \|\nabla^2 v\|_{L^2(\Omega)^{d \times d}}, \\ \|p - i_h^p p\|_{L^2(\Omega)} &\leq c_p h \|\nabla p\|_{L^2(\Omega)}. \end{aligned}$$

For the existence of operators  $i_h^p$  we refer to Clément [11].

**(A4)** There exists a finite dimensional space  $\tilde{L}_h \subset L_h$  and a continuous projection operator  $\pi: L_h \rightarrow \tilde{L}_h$  such that:

- For the pair  $(V_h, \tilde{L}_h)$  the inf-sup condition holds, i.e. there exists a positive constant  $\gamma$  independent of  $h$  with

$$\sup_{\phi_h \in \tilde{V}_h} \frac{(p_h, \nabla \cdot \phi_h)}{\|\nabla \phi_h\|_{L^2(\Omega)^d}} \geq \gamma \|p_h\|_{L^2(\Omega)} \quad \text{for all } p_h \in \tilde{L}_h.$$

- There is a positive constant  $c$  independent of  $h$  such that

$$\|\pi p_h\|_{L^2(\Omega)} \leq c \|p_h\|_{L^2(\Omega)} \quad \text{for all } p_h \in L_h.$$

- There is a positive constant  $c$  independent of  $h$  such that

$$\|p_h - \pi p_h\|_{L^2(\Omega)}^2 \leq c s_h(p_h, p_h) \quad \text{for all } p_h \in L_h.$$

- There is a positive constant  $c$  independent of  $h$  such that

$$s_h(i_h^p p, i_h^p p) \leq ch^2 \|\nabla p\|_{L^2(\Omega)} \quad \text{for all } p \in L \cap H^1(\Omega).$$

REMARK 2.3. *If the inf-sup condition is fulfilled for the pair  $(V_h, L_h)$  itself, there is no need for stabilization and we can set  $s(p, \xi) \equiv 0$  and  $\pi = id_{L_h}$ .*

REMARK 2.4. *In the presence of the regularization term  $s_h(p_h, \xi_h)$ , the discretization (2.7) is not a pure Galerkin scheme for (2.1) any more. Therefore, the question arises, if the approaches “discretize-then-optimize” and “optimize-then-discretize” coincide, see the discussion in Collis and Heinkenschloss [12]. However, in our setting these two approaches coincide due to the fact, that  $s_h(\cdot, \cdot)$  is a symmetric bilinear form.*

Moreover, we require the following inverse inequalities:

**(A5)** There is a positive constant  $c$  independent of  $h$  such that for all  $v_h \in V_h$  holds:

$$\|v_h\|'_{H^2(\Omega)^d} \leq ch^{-1} \|v_h\|_V \quad \text{and} \quad \|v_h\|_{L^\infty(\Omega)^d} \leq ch^{-d/2} \|v_h\|_{L^2(\Omega)^d}.$$

Let  $T$  be an arbitrary element of the mesh  $\mathcal{T}_h$ . We define the restriction operator  $R_h: C(\Omega) \rightarrow Q_h$  by

$$(R_h g)(x) = g(S_T)$$

where  $S_T$  denotes the barycenter of the element  $T$ . The restriction operator  $R_h$  is defined componentwise in the case of a vector valued function.

**(A6)** Let  $T \in \mathcal{T}_h$  be an arbitrary element of the discretization and  $g \in H^2(T)$  an arbitrary function. We require the following estimates:

$$\left| \int_T g(x) - (R_h g)(x) dx \right| \leq ch^2 |T|^{1/2} \|\nabla^2 g\|_{L^2(T)},$$

$$\|g - R_h g\|_{L^\infty(\Omega)} \leq ch \|\nabla g\|_{L^\infty(\Omega)}.$$

with a positive constant  $c$  independent of  $h$ .

REMARK 2.5. *Assumptions (A1)–(A6) are standard properties of finite element discretizations. They are fulfilled for many different element pairs with and without stabilization. We will verify these conditions for one specific discretization in Section 5.*

For our superconvergence result an additional assumption is needed. It follows from Lemma 2.1, that the optimal adjoint velocity  $\bar{w}$  belongs to  $W^{2,p'}(\Omega)^d$ . However the regularity of the optimal control is weaker because of the occurrence of kinks caused by the max-function in (2.6). Nevertheless, we can group all elements  $T \in \mathcal{T}_h$  in two classes:

$$K_1 := \bigcup_{T \in \mathcal{T}_h, \bar{q} \notin W^{2,p'}(T)^d} T, \quad K_2 := \bigcup_{T \in \mathcal{T}_h, \bar{q} \in W^{2,p'}(T)^d} T. \quad (2.8)$$

We remark that the properties of the projection operator and Lemma 2.1 imply  $\bar{q} \in W^{1,\infty}(\Omega)^d$ .

(A7)

$$|K_1| \leq ch$$

Assumption (A7) is difficult to verify, but is valid in many practical cases.

The discrete optimization problem is given by the minimization of the cost functional (1.1) subject to the discretized state equation (2.7) and subject to  $q_h \in Q_h^{ad} = Q_h \cap Q_{ad}$ . Similar to the notation for the continuous problem, we denote by  $\bar{q}_h \in Q_h^{ad}$ ,  $\bar{u}_h \in V_h \times L_h$ , and  $\bar{z}_h \in V_h \times L_h$  the optimal control, the associated state, and the adjoint state of the discretized optimal control problem. In the following theorems we formulate our main results:

THEOREM 2.6. *Assume that (A1)–(A7) holds. Then the estimate*

$$\|\bar{q}_h - R_h \bar{q}\|_Q \leq ch^2 (\|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{q}\|_{L^\infty(\Omega)^d}) \quad (2.9)$$

*is valid with a positive constant  $c$  independent of  $h$ .*

THEOREM 2.7. *The estimates*

$$\|\bar{v} - \bar{v}_h\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (2.10)$$

$$\|\bar{w} - \bar{w}_h\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (2.11)$$

*are valid provided that (A1)–(A7) hold.*

THEOREM 2.8. *Assume that (A1)–(A7) holds. Then the estimate*

$$\|\tilde{q}_h - \bar{q}\|_Q \leq ch^2 (\|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{q}\|_{L^\infty(\Omega)^d}) \quad (2.12)$$

*is valid with*

$$\tilde{q}_h = \Pi_{[a,b]} \left( -\frac{1}{\nu} \bar{w}_h \right) \quad (2.13)$$

*and a positive constant  $c$  independent of  $h$ .*

The proofs of the Theorems 2.6 and 2.8 are contained in Section 4.

Let us briefly explain why these are superconvergence results: The best possible rate for approximation of the optimal solution by a piecewise constant function is  $h$ . Therefore we can only expect

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)^d} = \mathcal{O}(h).$$

However, we will show in Theorem 2.6 that the values in the barycenter are approximated with order  $h^2$ . A direct implication of this result will be that the velocity and the adjoint velocity is approximated with order  $h^2$  in the  $L^2$ -norm. The projection in (2.13) increases the accuracy of the calculated control to order  $h^2$ . Hence, the result of Theorem 2.8 provides a possibility to significantly improve the behavior of the error by a simple post-processing step (2.13).

**3. Results from finite element theory.** In this section, we collect results from finite element theory. We define (linear) solution mappings  $(S, S^p)$  of the continuous state equation such that there holds for all  $\phi = (\psi, \xi) \in V \times L$  and  $g \in Q$ :

$$u^g = (S(g), S^p(g)) \in V \times L : a(u^g, \phi) = (g, \psi), \quad (3.1)$$

and the solution mappings  $S_h, S_h^p$  of the discretized state equation such that there holds for all  $\phi_h = (\psi_h, \xi_h) \in V_h \times L_h$  and  $g \in Q$ :

$$u_h^g = (S_h(g), S_h^p(g)) \in V_h \times L_h : a(u_h^g, \phi_h) + s_h(S_h^p(g), \xi_h) = (g, \psi_h). \quad (3.2)$$

Although the solution operators  $S$  and  $S_h$  have better regularity properties, we define them in the space  $Q$ :

$$S : Q \rightarrow Q, \quad S_h : Q \rightarrow Q.$$

In the following we provide some properties of these operators based on the assumptions (A1)–(A7). The following lemma ensures the stability of the discretization of the state equation.

LEMMA 3.1. *Under assumptions (A1)–(A6) the following modified inf-sup condition holds: There exist positive constants  $\tilde{\gamma}$  and  $c$  independent of  $h$  with*

$$\sup_{\phi_h \in V_h} \frac{(p_h, \nabla \cdot \phi_h) + c s(p_h, p_h)}{\|\nabla \phi_h\|_{L^2(\Omega)^d}} \geq \tilde{\gamma} \|p_h\|_{L^2(\Omega)} \quad \text{for all } p_h \in L_h.$$

*Proof.* For the proof we refer to [2].  $\square$

Next, we define the affine linear operators  $P : Q \rightarrow Q$  and  $P_h : Q \rightarrow Q$  by

$$Pq = S^*(S(q + f) - v_d), \quad P_h q = S_h^*(S_h(q + f) - v_d),$$

where  $S^*$  and  $S_h^*$  denote the adjoint operators of  $S$  and  $S_h$  respectively.

LEMMA 3.2. *Assume that the assumption (A1)–(A6) hold. Let  $q \in Q$  be an arbitrary control. Then, the discretization error of the state equation and the adjoint equation can be estimated by*

- (i)  $h \|S_h(q + f) - S(q + f)\|_V + \|S_h(q + f) - S(q + f)\|_Q \leq ch^2 (\|q\|_Q + \|f\|_Q),$
- (ii)  $\|S_h(q + f) - S(q + f)\|_{L^\infty(\Omega)^d} \leq ch^{2-d/2} (\|q\|_Q + \|f\|_Q),$
- (iii)  $\|S_h(q + f) - S(q + f)\|'_{H^2(\Omega)^d} \leq c (\|q\|_Q + \|f\|_Q),$
- (iv)  $\|P_h q - Pq\|_Q \leq ch^2 (\|q\|_Q + \|f\|_Q + \|v_d\|_Q).$

*Proof.* The proof of the error estimate (i) relies on Lemma 3.1 and is given in [2]. The result concerning  $L^2$ -estimate can be obtained by standard techniques, see e.g. [16] for the application of the Aubin-Nitsche trick to the Stokes problem.



For the proof of (ii) we set  $g = f + q$  and use the second inverse inequality from (A5) and an interpolation estimate from (A3):

$$\begin{aligned} & \| (S - S_h)(g) \|_{L^\infty(\Omega)^d} \leq \| S(g) - i_h^v S(g) \|_{L^\infty(\Omega)^d} + \| S_h(g) - i_h^v S(g) \|_{L^\infty(\Omega)^d} \\ & \leq ch^{2-d/2} \| \nabla^2 S(g) \|_{L^2(\Omega)^d} + ch^{-d/2} \| S_h(g) - i_h^v S(g) \|_{L^2(\Omega)^d} \\ \leq & ch^{2-d/2} \| \nabla^2 S(g) \|_{L^2(\Omega)^d} + ch^{-d/2} \{ \| S(g) - i_h^v S(g) \|_{L^2(\Omega)^d} + \| S_h(g) - S(g) \|_{L^2(\Omega)^d} \} \\ & \leq ch^{2-d/2} \| \nabla^2 S(g) \|_{L^2(\Omega)^d} \leq ch^{2-d/2} \| g \|_{L^2(\Omega)^d}. \end{aligned}$$

The estimate (iii) follows in the same manner using the first inverse inequality from (A5) and an interpolation estimate from (A3).

The error estimate (iv) is obtained in a similar way as (i).  $\square$

LEMMA 3.3. *The discretization operators  $S_h$  and  $S_h^*$  are bounded in the following sense:*

$$\begin{array}{llll} (i) & \| S_h \|_{Q \rightarrow V} & \leq c, & \| S_h^* \|_{Q \rightarrow V} & \leq c, \\ (ii) & \| S_h \|_{Q \rightarrow L^\infty(\Omega)^d} & \leq c, & \| S_h^* \|_{Q \rightarrow L^\infty(\Omega)^d} & \leq c, \\ (iii) & \| S_h \|_{Q \rightarrow V_h^2(\Omega)^d} & \leq c, & \| S_h^* \|_{Q \rightarrow V_h^2(\Omega)^d} & \leq c. \end{array}$$

*Proof.* We sketch only the proof for the operator  $S_h$ . The results for the adjoint operator  $S_h^*$  can be derived by the same techniques. In order to prove the first estimate we set  $\phi = (S_h(g), 0)$  in (3.2) and obtain:

$$\| \nabla S_h(g) \|_{L^2(\Omega)^d}^2 + s_h(S_h^p(g), S_h^p(g)) = (g, S_h(g)).$$

Due to (A4) we have:

$$\| \nabla S_h(g) \|_{L^2(\Omega)^d}^2 \leq (g, S_h(g)).$$

The assertion follows then by Poincaré inequality.

The second estimate is obtained using (ii) from Lemma 3.2:

$$\| S_h(g) \|_{L^\infty(\Omega)^d} \leq \| S(g) - S_h(g) \|_{L^\infty(\Omega)^d} + \| S(g) \|_{L^\infty(\Omega)^d} \leq c(1+h) \| g \|_{L^2(\Omega)}.$$

The third estimate is obtained similarly using (iii) from Lemma 3.2.  $\square$

LEMMA 3.4. *Let  $p' > d$  the regularity parameter of Lemma 2.1, i.e. in particular,  $w \in W^{2,p'}(\Omega)^d$ . Then the inequality*

$$(\psi_h, \bar{q} - R_h \bar{q}) \leq ch^2 (\| \psi_h \|_{L^\infty(\Omega)^d} + \| \psi_h \|'_{H^2(\Omega)^d}) (\| \bar{q} \|_{L^\infty(\Omega)^d} + \| \bar{w} \|_{W^{2,p'}(\Omega)^d})$$

*is satisfied for all  $\psi_h \in V_h$  provided that the assumptions (A1)-(A7) are fulfilled.*

*Proof.* With the sets  $K_1$  and  $K_2$  introduced by (2.8), we obtain

$$(\psi_h, \bar{q} - R_h \bar{q}) = \int_{K_1} \psi_h \cdot (\bar{q} - R_h \bar{q}) dx + \int_{K_2} \psi_h \cdot (\bar{q} - R_h \bar{q}) dx. \quad (3.3)$$

Using the  $W^{1,\infty}$ -regularity of  $\bar{q}$ , the  $K_1$ -part can be estimated as follows:

$$\begin{aligned} \int_{K_1} \psi_h \cdot (\bar{q} - R_h \bar{q}) dx &= \sum_{T \subset K_1} \int_T \psi_h \cdot (\bar{q} - R_h \bar{q}) dx \\ &\leq \| \psi_h \|_{L^\infty(\Omega)^d} \sum_{T \subset K_1} \| \bar{q} - R_h \bar{q} \|_{L^\infty(T)^d} \int_T dx \\ &\leq ch \| \psi_h \|_{L^\infty(\Omega)^d} | \bar{q} |_{W^{1,\infty}(\Omega)^d} | K_1 |. \end{aligned} \quad (3.4)$$

Assumption (A7) and the properties of the projection (2.6) yield

$$\int_{K_1} \psi_h \cdot (\bar{q} - R_h \bar{q}) dx \leq ch^2 \|\psi_h\|_{L^\infty(\Omega)^d} \|\bar{w}\|_{W^{1,\infty}(\Omega)^d} \leq ch^2 \|\psi_h\|_{L^\infty(\Omega)^d} \|\bar{w}\|_{W^{2,p'}(\Omega)^d} \quad (3.5)$$

On the  $K_2$ -part, we proceed as follows:

$$\left| \int_{K_2} \psi_h \cdot (\bar{q} - R_h \bar{q}) dx \right| \leq \left| \int_{K_2} (\psi_h \cdot R_h \bar{q} - R_h(\psi_h \cdot \bar{q})) dx \right| + \left| \int_{K_2} \psi_h \cdot \bar{q} - R_h(\psi_h \cdot \bar{q}) dx \right| \quad (3.6)$$

Using  $R_h(\psi_h \cdot \bar{q}) = R_h \psi_h \cdot R_h \bar{q}$ , we find for the first integral

$$\left| \int_{K_2} (\psi_h \cdot R_h \bar{q} - R_h(\psi_h \cdot \bar{q})) dx \right| \leq \sum_{T \subset K_2} \left| \int_T (\psi_h - R_h \psi_h) \cdot R_h \bar{q} dx \right|$$

Note, that  $R_h \bar{q}$  is constant on every element  $T$ . Hence, we can continue with

$$\left| \int_{K_2} (\psi_h \cdot R_h \bar{q} - R_h(\psi_h \cdot \bar{q})) dx \right| \leq \sum_{T \subset K_2} \left| R_h \bar{q} \cdot \int_T (\psi_h - R_h \psi_h) dx \right|.$$

Consequently, we find by means of (A6)

$$\begin{aligned} \left| \int_{K_2} (\psi_h \cdot R_h \bar{q} - R_h(\psi_h \cdot \bar{q})) dx \right| &\leq ch^2 \sum_{T \subset K_2} |T|^{1/2} \|\bar{q}\|_{L^\infty(T)^d} |\psi_h|_{H^2(T)^d} \\ &\leq c|\Omega|^{1/2} h^2 \|\bar{q}\|_{L^\infty(\Omega)^d} \|\psi_h\|'_{H^2(\Omega)^d} \\ &\leq ch^2 \|\bar{w}\|_{L^\infty(\Omega)^d} \|\psi_h\|'_{H^2(\Omega)^d}. \end{aligned} \quad (3.7)$$

It remains the second integral in (3.6). Again, we can use (A6):

$$\left| \int_{K_2} \psi_h \cdot \bar{q} - R_h(\psi_h \cdot \bar{q}) dx \right| \leq \sum_{T \subset K_2} \left| \int_T \psi_h \cdot \bar{q} - R_h(\psi_h \cdot \bar{q}) dx \right| \leq ch^2 \sum_{T \subset K_2} |T|^{1/2} |\psi_h \cdot \bar{q}|_{H^2(T)}. \quad (3.8)$$

We will estimate this seminorm by

$$|\psi_h \cdot \bar{q}|_{H^2(T)} \leq \|\psi_h\|_{L^\infty(T)^d} |\bar{q}|_{H^2(T)^d} + |\psi_h|_{H^2(T)^d} \|\bar{q}\|_{L^\infty(T)^d} + 2\|\psi_h\|_{H^1(T)^d} |\bar{q}|_{W^{1,\infty}(T)^d}. \quad (3.9)$$

The projection formula (2.6) and the fact that  $\bar{q}$  smooth is on every  $T \subset K_2$  imply

$$\begin{aligned} |\psi_h \cdot \bar{q}|_{H^2(T)} &\leq c(\|\psi_h\|_{L^\infty(T)^d} |\bar{w}|_{H^2(T)^d} + |\psi_h|_{H^2(T)^d} \|\bar{q}\|_{L^\infty(T)^d} + |\psi_h|_{H^1(T)^d} |\bar{w}|_{W^{1,\infty}(T)^d}) \\ &\leq c(\|\psi_h\|_{L^\infty(\Omega)^d} |\bar{w}|_{H^2(T)^d} + |\psi_h|_{H^2(T)^d} \|\bar{q}\|_{L^\infty(\Omega)^d} + |\psi_h|_{H^1(T)^d} |\bar{w}|_{W^{1,\infty}(\mathfrak{R}^d)}) \end{aligned}$$

Combining (3.8) with (3.10) we find

$$\begin{aligned} \left| \int_{K_2} \psi_h \cdot \bar{q} - R_h(\psi_h \cdot \bar{q}) dx \right| &\leq ch^2 \sum_{T \subset K_2} |T|^{1/2} (\|\psi_h\|_{L^\infty(\Omega)^d} |\bar{w}|_{H^2(T)^d} \\ &\quad + |\psi_h|_{H^2(T)^d} \|\bar{q}\|_{L^\infty(\Omega)^d} + |\psi_h|_{H^1(T)^d} |\bar{w}|_{W^{1,\infty}(\Omega)^d}) \\ &\leq ch^2 (\|\psi_h\|_{L^\infty(\Omega)^d} \|\bar{w}\|_{H^2(\Omega)^d} + \\ &\quad + \|\psi_h\|'_{H^2(\Omega)^d} \|\bar{q}\|_{L^\infty(\Omega)^d} + \|\psi_h\|_{H^1(\Omega)^d} \|\bar{w}\|_{W^{1,\infty}(\Omega)^d}). \end{aligned}$$

By imbedding arguments we end up with

$$\left| \int_{K_2} \psi_h \cdot \bar{q} - R_h(\psi_h \cdot \bar{q}) \, dx \right| \leq ch^2 (\|\psi_h\|_{L^\infty(\Omega)^d} + \|\psi_h\|'_{H^2(\Omega)^d}) (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (3.11)$$

Inserting (3.7) and (3.11) in (3.6), we obtain

$$\left| \int_{K_2} (\psi_h \cdot R_h \bar{q} - R_h(\psi_h \cdot \bar{q})) \, dx \right| \leq ch^2 (\|\psi_h\|_{L^\infty(\Omega)^d} + \|\psi_h\|'_{H^2(\Omega)^d}) (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}). \quad (3.12)$$

From (3.3), (3.5), and (3.12), the assertion follows immediately.  $\square$

LEMMA 3.5. *Let  $p' > d$  the regularity parameter of Lemma 2.1 and (A1)–(A7) be fulfilled. Then the estimates*

$$\|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (3.13)$$

$$\|P_h \bar{q} - P_h R_h \bar{q}\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (3.14)$$

are valid.

*Proof.* We start with

$$\begin{aligned} \|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q^2 &= (S_h(\bar{q} + f) - S_h(R_h \bar{q} + f), S_h(\bar{q} + f) - S_h(R_h \bar{q} + f))_Q \\ &= (S_h(\bar{q} - R_h \bar{q}), (S_h(\bar{q} + f) - v_d) - (S_h(R_h \bar{q} + f) - v_d))_Q \\ &= (\bar{q} - R_h \bar{q}, P_h \bar{q} - P_h R_h \bar{q})_Q \\ &\leq ch^2 (\|P_h \bar{q} - P_h R_h \bar{q}\|_{L^\infty(\Omega)^d} + \|P_h \bar{q} - P_h R_h \bar{q}\|'_{H^2(\Omega)^d}) \\ &\quad \cdot (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \end{aligned} \quad (3.15)$$

where we have used Lemma 3.4 with  $\psi_h = P_h \bar{q} - P_h R_h \bar{q}$ . We benefit now from the fact that  $P_h \bar{q}$  and  $P_h R_h \bar{q}$  are solutions of the discretized adjoint equation, that means  $\psi_h = P_h \bar{q} - P_h R_h \bar{q} = S_h^*(S_h(\bar{q} + f) - v_d) - S_h^*(S_h(R_h \bar{q} + f) - v_d) = S_h^*(S_h(\bar{q} + f) - S_h(R_h \bar{q} + f))$ . Therefore we obtain by Lemma 3.3

$$\|P_h \bar{q} - P_h R_h \bar{q}\|_{L^\infty(\Omega)^d} \leq c \|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q. \quad (3.16)$$

and

$$\|P_h \bar{q} - P_h R_h \bar{q}\|'_{H^2(\Omega)^d} \leq c \|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q. \quad (3.17)$$

Inserting (3.16) and (3.17) in (3.15) and dividing by  $\|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q$ , the assertion (3.13) is obtained. Inequality (3.13) and the continuity of  $S_h$  in  $Q$  yield (3.14).  $\square$

COROLLARY 3.6. *Let  $p' > d$  the regularity parameter of Lemma 2.1 and (A1)–(A7) be fulfilled. Then the inequality*

$$\|\bar{w} - P_h R_h \bar{q}\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \quad (3.18)$$

is valid.

*Proof.* We apply Lemma 3.2 for  $q = \bar{q}$ . Using  $\bar{w} = P\bar{q}$ , we obtain:

$$\|\bar{w} - P_h \bar{q}\|_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d})$$

The assertion follows now from (3.14) and the triangle inequality.  $\square$

**4. Superconvergence properties.** In this section, we prove the main results stated in Section 2. We start with an auxiliary result.

LEMMA 4.1. *The inequality*

$$\nu \|R_h \bar{q} - \bar{q}_h\|_Q^2 \leq (R_h \bar{w} - \bar{w}_h, \bar{q}_h - R_h \bar{q})_Q \quad (4.1)$$

is valid provided that the assumptions (A1)–(A7) hold.

*Proof.* First, we recall the optimality condition (2.5):

$$(\bar{w} + \nu \bar{q}, q - \bar{q})_Q \geq 0 \quad \text{for all } q \in Q_{ad}.$$

This formula is true for all  $q \in Q_{ad}$ . Therefore, this formula holds also pointwise a.e. in  $\Omega$ :

$$(\bar{w}(x) + \nu \bar{q}(x)) \cdot (q(x) - \bar{q}(x)) \geq 0 \quad \text{for all } q \in Q_{ad}.$$

Consider any element  $T$  with center of gravity  $S_T$  and apply this formula for  $x = S_T$  and  $q = \bar{q}_h$ . This can be done because of the continuity of the functions  $\bar{w}$ ,  $\bar{q}$ , and  $\bar{q}_h$  in this point:

$$(\bar{w}(S_T) + \nu \bar{q}(S_T)) \cdot (\bar{q}_h(S_T) - \bar{q}(S_T)) \geq 0 \quad \text{for all } T \in \mathcal{T}_h.$$

Due to the definition of  $R_h$ , this is equivalent to

$$(R_h \bar{w}(S_T) + \nu R_h \bar{q}(S_T)) \cdot (\bar{q}_h(S_T) - R_h \bar{q}(S_T)) \geq 0 \quad \text{for all } T \in \mathcal{T}_h$$

We integrate this formula over  $T$ , add over all  $T$  and get

$$(R_h \bar{w} + \nu R_h \bar{q}, \bar{q}_h - R_h \bar{q})_Q \geq 0. \quad (4.2)$$

Otherwise, the optimal control  $\bar{q}_h$  of the discretized problem fulfills the optimality condition:

$$(\bar{w}_h + \nu \bar{q}_h, q - \bar{q}_h)_Q \geq 0 \quad \text{for all } q \in Q_h^{ad}. \quad (4.3)$$

We apply this formula for  $q = R_h \bar{q}$ :

$$(\bar{w}_h + \nu \bar{q}_h, R_h \bar{q} - \bar{q}_h)_Q \geq 0. \quad (4.4)$$

Adding (4.2) and (4.4), we obtain

$$(R_h \bar{w} - \bar{w}_h + \nu(R_h \bar{q} - \bar{q}_h), \bar{q}_h - R_h \bar{q})_Q \geq 0 \quad (4.5)$$

This completes the proof.  $\square$

REMARK 4.2. *Lemma 4.1 is the key to prove our main results. The presented technique benefits from the fact that the controls are discretized by piecewise constant functions. The derivation of the estimate (4.1) motivates our choice for the control discretization.*

Now we are able to prove Theorem 2.6.

*Proof.* (Theorem 2.6) We begin with rewriting formula (4.1):

$$\begin{aligned} \nu \|R_h \bar{q} - \bar{q}_h\|_Q^2 &\leq (R_h \bar{w} - \bar{w}_h, \bar{q}_h - R_h \bar{q})_Q \\ &= (R_h \bar{w} - \bar{w}, \bar{q}_h - R_h \bar{q})_Q + (\bar{w} - P_h R_h \bar{q}, \bar{q}_h - R_h \bar{q})_Q \\ &\quad + (P_h R_h \bar{q} - \bar{w}_h, \bar{q}_h - R_h \bar{q})_Q. \end{aligned} \quad (4.6)$$

Let us now estimate these three terms. We start with the first term using (A6) and the fact that  $\bar{q}_h - R_h\bar{q}$  is piecewise constant on each element

$$\begin{aligned}
(R_h\bar{w} - \bar{w}, \bar{q}_h - R_h\bar{q})_Q &= \sum_{T \in \mathcal{T}_h} \int_T (R_h\bar{w}(x) - \bar{w}(x)) \cdot (\bar{q}_h(x) - R_h\bar{q}(x)) \, dx \\
&= \sum_{T \in \mathcal{T}_h} (\bar{q}_h(S_T) - \bar{q}(S_T)) \cdot \int_T (\bar{w}(S_T) - \bar{w}(x)) \, dx \\
&\leq \sum_{T \in \mathcal{T}_h} ch^2 |\bar{q}_h(S_T) - \bar{q}(S_T)| |T|^{1/2} |\bar{w}|_{H^2(\Omega)^d} \\
&\leq ch^2 \|\bar{q}_h - R_h\bar{q}\|_Q \|\bar{w}\|_{W^{2,p'}(\Omega)^d}. \tag{4.7}
\end{aligned}$$

The second term in (4.6) is estimated by Corollary 3.6 and the Cauchy-Schwartz inequality:

$$(\bar{w} - P_h R_h \bar{q}, \bar{q}_h - R_h \bar{q})_Q \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \|\bar{q}_h - R_h \bar{q}\|_Q \tag{4.8}$$

The third term can be omitted because of

$$\begin{aligned}
(P_h R_h \bar{q} - \bar{w}_h, \bar{q}_h - R_h \bar{q})_Q &= (P_h R_h \bar{q} - P_h \bar{q}_h, \bar{q}_h - R_h \bar{q})_Q \\
&= (S_h^*(S_h(R_h \bar{q} + f) - v_d) - S_h^*(S_h(\bar{q}_h + f) - v_d), \bar{q}_h - R_h \bar{q})_Q \\
&= (S_h^* S_h(R_h \bar{q} - \bar{q}_h), \bar{q}_h - R_h \bar{q})_Q \\
&= (S_h(R_h \bar{q} - \bar{q}_h), S_h(\bar{q}_h - R_h \bar{q}))_Q \\
&\leq 0. \tag{4.9}
\end{aligned}$$

Inserting (4.7)–(4.9) in (4.6), we end up with

$$\nu \|R_h \bar{q} - \bar{q}_h\|_Q^2 \leq ch^2 (\|\bar{q}\|_{L^\infty(\Omega)^d} + \|v_d\|_{L^\infty(\Omega)^d} + \|f\|_{L^\infty(\Omega)^d} + \|\bar{w}\|_{W^{2,p'}(\Omega)^d}) \|\bar{q}_h - R_h \bar{q}\|_Q. \tag{4.10}$$

This inequality is equivalent to the assertion (2.9).  $\square$

*Proof.* (Theorem 2.7) Using the triangle inequality, we find

$$\begin{aligned}
\|\bar{v} - \bar{v}_h\|_Q &= \|S(\bar{q} + f) - S_h(\bar{q}_h + f)\|_Q \\
&\leq \|S(\bar{q} + f) - S_h(\bar{q} + f)\|_Q + \|S_h(\bar{q} + f) - S_h(R_h \bar{q} + f)\|_Q + \|S_h(R_h \bar{q} - \bar{q}_h)\|_Q.
\end{aligned}$$

The first term is estimated using Lemma 3.2, for the second term we use the assertion from Lemma 3.5 and for the third term we apply Theorem 2.6 and the boundedness of  $S_h$ . This yields estimate (2.10).

Inequality (2.11) can be similarly obtained by Corollary 3.6, Theorem 2.6, and the boundedness of  $S_h$  and  $S_h^*$  in  $\mathcal{L}(Q)$ .  $\square$

Next, we prove Theorem 2.8.

*Proof.* (Theorem 2.8) We use the Lipschitz continuity of the projection operator and find

$$\begin{aligned}
\|\tilde{q}_h - \bar{q}\|_Q &= \left\| \Pi_{[a,b]} \left( -\frac{1}{\nu} \bar{w}_h \right) - \Pi_{[a,b]} \left( -\frac{1}{\nu} \bar{w} \right) \right\|_Q \\
&\leq \frac{1}{\nu} \|\bar{w}_h - \bar{w}\|_Q.
\end{aligned}$$

Inequality (2.11) implies now the assertion.  $\square$

**5. Verification of the assumptions for concrete numerical schemes.** In this section we check the assumptions (A1) – (A6) for some discretization schemes. Let  $\mathcal{T}_h$  be a shape regular quasi-uniform mesh (see e.g. Braess [5]) consisting of triangles or quadrilaterals for  $d = 2$  or tetrahedrons or hexahedrons for  $d = 3$ . Then, the assumption (A1) is automatically fulfilled. If the control is defined on the same mesh then assumption (A2) is fulfilled, too.

Let  $P_h^k$  denote the space of finite elements of order  $k$  on a triangle/tetrahedron mesh  $\mathcal{T}_h$  and  $Q_h^k$  denote the space of finite elements of order  $k$  (bi/tri-linear, bi/tri-quadratic etc.) on a quadrilateral/hexahedron mesh  $\mathcal{T}_h$ .

If  $(P_h^k)^d \subset V_h$  and  $P_h^l \subset L_h$  (or  $(Q_h^k)^d \subset V_h$  and  $Q_h^l \subset L_h$ ) for  $k \geq 1$ ,  $l \geq 0$ , then the assumptions (A3) and (A5) follow by standard arguments, see e.g. [5] or [10]. Assumption (A6) is also fulfilled on shape regular quasi-uniform meshes, which can be seen by virtue of Bramble-Hilbert-Lemma and a transformation argument.

It remains to discuss the assumption (A4). As mentioned in Remark 2.3, this assumption is obviously fulfilled, if the pair  $(V_h, L_h)$  is stable, i.e. if the inf-sup condition for is directly fulfilled for  $(V_h, L_h)$ . Therefore, our results are justified for all such pairs, as e.g. “Taylor-Hood element”, see [20], different bubble elements  $(P_1^b/P_0, Q_1^b/Q_0)$  etc., see, e.g., [16]) etc.

In the sequel we want to recall another discretization scheme, introduced in Becker & Braack [2], which also fulfills the assumption (A4). This scheme will be used in the next section for the numerical example.

For this discretization we assume that the (quadrilateral or hexahedron) mesh  $\mathcal{T}_h$  is organized in a patch-wise manner. This means, that it results from a coarser regular mesh  $\mathcal{T}_{2h}$  by one uniform refinement. By a “patch” of elements we denote a group of four cells (in 2D) or eight cells (in 3D) in  $\mathcal{T}_h$  which results from a common coarser cell in  $\mathcal{T}_{2h}$ . The finite element spaces are chosen as:

$$V_h = (Q_h^1)^d \quad L_h = Q_h^1.$$

The space  $\tilde{L}_h$  is defined as the space of bilinear/trilinear elements on the patch-mesh  $\mathcal{T}_{2h}$ , i.e.  $\tilde{L}_h = Q_{2h}^1$ . The stabilization form  $s_h(\cdot, \cdot)$  is defined as:

$$s_h(p_h, \xi_h) = \delta_0 h^2 \sum_{P \in \mathcal{T}_{2h}} (\nabla p_h - \widetilde{\nabla} p_h, \nabla \xi_h - \widetilde{\nabla} \xi_h)_{L^2(P)},$$

where

$$\widetilde{\nabla} p_h = \frac{1}{|P|} \int_P \nabla p_h \, dx.$$

We refer to [2] for the proof that this scheme fulfills the assumption (A4). We note, that also other equal order stabilized schemes are included in our setting, see e.g. [6].

**6. Numerical examples.** In this section we present two numerical examples (2D and 3D) confirming our results. In both examples the Stokes equation are discretized by equal order elements (bilinear in 2D and trilinear in 3D) with a stabilization term as described in the previous section. The resulting finite dimensional optimal control problem is solved by primal-dual active set method, see e.g. [3] or [21].

**6.1. Example in 2D.** We consider an optimal control problem as stated in Section 1 with

$$\Omega = (0, 1)^2, \quad \nu = 1, \quad a = (-0.1, -0.1)^t, \quad b = (0.25, 0.25)^t,$$

and a given solution:

$$\begin{aligned}\bar{v}_1 &= \bar{w}_1 = \sin^2(\pi x) \sin(\pi y) \cos(\pi y), \\ \bar{v}_2 &= \bar{w}_2 = -\sin^2(\pi y) \sin(\pi x) \cos(\pi x), \\ \bar{p} &= \bar{r} = \sin(2\pi x) \sin(2\pi y),\end{aligned}$$

and

$$\bar{q} = \Pi_{[a,b]}(-\frac{1}{\nu}\bar{w}).$$

The data of the problem is then given by:

$$\begin{aligned}f &= -\Delta\bar{v} + \nabla\bar{p} - \bar{q}, \\ v_d &= \bar{v} + \Delta\bar{w} + \nabla\bar{r}.\end{aligned}$$

In Figure 6.1 we show the first component of the optimal solution  $\bar{q}$ . The second component of  $\bar{q}$  has a similar structure.

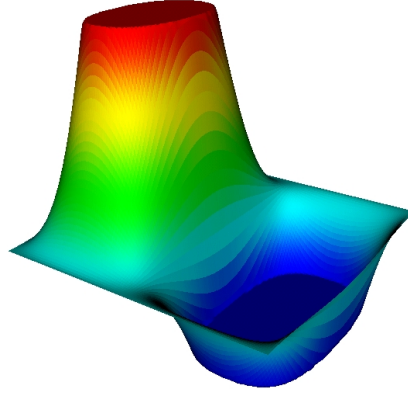


FIG. 6.1. *The first component of the optimal solution  $\bar{q}$*

Let us remark that the assumption (A7) is fulfilled for this example: Let

$$\gamma_{ex} := \{x \in \Omega : (\bar{w}_1(x) - a_1)(\bar{w}_1(x) - b_1)(\bar{w}_2(x) - a_2)(\bar{w}_2(x) - b_2) = 0\},$$

i.e., the curve (consisting of four connected parts) that separates active and inactive parts of the optimal control. Then we find the estimate

$$|K_1| \leq 2h|\gamma_{ex}|$$

and consequently (A7) holds.

In Table 6.1 we show the behavior of the error  $\|\bar{q} - \bar{q}_h\|_Q$  and the error after the post-processing step, i.e.  $\|\bar{q} - \tilde{q}_h\|_Q$  on a sequence of uniformly refined meshes. As expected, we observe first order convergence for  $\|\bar{q} - \bar{q}_h\|_Q$  and second order convergence for  $\|\bar{q} - \tilde{q}_h\|_Q$ .

In Table 6.2 we show the corresponding results concerning the error behavior with respect to  $\|\cdot\|_{L^\infty(\Omega)}$ . Although we only proved the results concerning the convergence with respect to  $\|\cdot\|_{L^2(\Omega)}$ , we observe similar behavior also for  $\|\cdot\|_{L^\infty(\Omega)}$ .

TABLE 6.1  
Convergence of  $\|\bar{q} - \bar{q}_h\|_Q$  and  $\|\bar{q} - \tilde{q}_h\|_Q$  for  $h$  tending to zero.

$h/\sqrt{2}$	$\ \bar{q} - \bar{q}_h\ _Q$	reduction rate	$\ \bar{q} - \tilde{q}_h\ _Q$	reduction rate
$2^{-2}$	7.65e-2	–	3.84e-2	–
$2^{-3}$	5.06e-2	1.50	8.59e-3	4.47
$2^{-4}$	2.73e-2	1.85	1.82e-3	4.72
$2^{-5}$	1.39e-2	1.97	4.23e-4	4.29
$2^{-6}$	6.99e-3	1.99	1.02e-4	4.15
$2^{-7}$	3.51e-3	1.99	2.51e-5	4.06

TABLE 6.2  
Convergence of  $\|\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)^d}$  and  $\|\bar{q} - \tilde{q}_h\|_{L^\infty(\Omega)^d}$  for  $h$  tending to zero.

$h/\sqrt{2}$	$\ \bar{q} - \bar{q}_h\ _{L^\infty(\Omega)^d}$	reduction rate	$\ \bar{q} - \tilde{q}_h\ _{L^\infty(\Omega)^d}$	reduction rate
$2^{-2}$	2.60e-1	–	1.20e-1	–
$2^{-3}$	2.19e-1	1.18	2.43e-2	4.96
$2^{-4}$	1.27e-1	1.73	7.62e-3	3.19
$2^{-5}$	6.47e-2	1.96	1.77e-3	4.30
$2^{-6}$	3.25e-2	1.99	4.74e-4	3.74
$2^{-7}$	1.63e-2	2.00	1.22e-4	3.88

**6.2. Example in 3D.** For the 3D case we construct a similar example as in 2D by setting:

$$\Omega = (0, 1)^3, \quad \nu = 1, \quad a = (-0.1, -0.1, -0.1)^t, \quad b = (0.25, 0.25, 0.01)^t.$$

The exact solution is given by:

$$\begin{aligned} \bar{v}_1 = \bar{w}_1 &= 2 \sin^2(\pi x) \sin(2\pi y) \sin(2\pi z), \\ \bar{v}_2 = \bar{w}_2 &= -\sin^2(\pi y) \sin(2\pi x) \sin(2\pi z), \\ \bar{v}_3 = \bar{w}_3 &= -\sin^2(\pi z) \sin(2\pi x) \sin(2\pi y), \\ \bar{p} = \bar{r} &= \sin(2\pi x) \sin(2\pi y) \sin(2\pi z), \end{aligned}$$

and

$$\bar{q} = \Pi_{[a,b]} \left( -\frac{1}{\nu} \bar{w} \right).$$

The data of the problem is then determined by:

$$\begin{aligned} f &= -\Delta \bar{v} + \nabla \bar{p} - \bar{q}, \\ v_d &= \bar{v} + \Delta \bar{w} + \nabla \bar{r}. \end{aligned}$$

For this problem, Assumption (A7) is valid for similar reasons as in the previous example. As for the 2D example, we present the behavior of error  $\|\bar{q} - \bar{q}_h\|_Q$  and the error after the post-processing step, i.e.  $\|\bar{q} - \tilde{q}_h\|_Q$  in Table 6.3, and for the corresponding  $L^\infty$ -norm in Table 6.4.



TABLE 6.3  
Convergence of  $\|\bar{q} - \bar{q}_h\|_Q$  and  $\|\tilde{q} - \tilde{q}_h\|_Q$  for  $h$  tending to zero.

$h/\sqrt{3}$	$\ \bar{q} - \bar{q}_h\ _Q$	reduction rate	$\ \tilde{q} - \tilde{q}_h\ _Q$	reduction rate
$1/3 \cdot 2^{-1}$	8.68e-2	–	2.13e-2	–
$1/3 \cdot 2^{-2}$	5.81e-2	1.49	5.27e-3	4.04
$1/3 \cdot 2^{-3}$	3.49e-2	1.66	1.08e-3	4.87
$1/3 \cdot 2^{-4}$	1.83e-2	1.91	2.43e-4	4.44

TABLE 6.4  
Convergence of  $\|\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)^d}$  and  $\|\tilde{q} - \tilde{q}_h\|_{L^\infty(\Omega)^d}$  for  $h$  tending to zero.

$h/\sqrt{3}$	$\ \bar{q} - \bar{q}_h\ _{L^\infty(\Omega)^d}$	reduction rate	$\ \tilde{q} - \tilde{q}_h\ _{L^\infty(\Omega)^d}$	reduction rate
$1/3 \cdot 2^{-1}$	3.29e-1	–	7.96e-2	–
$1/3 \cdot 2^{-2}$	2.99e-1	1.10	2.76e-2	2.88
$1/3 \cdot 2^{-3}$	2.59e-1	1.15	5.69e-3	4.85
$1/3 \cdot 2^{-4}$	1.39e-1	1.86	1.30e-3	4.37

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