

Strict Contractivity of the 2-Wasserstein distance for the porous medium equation by mass-centering

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RICAM-Report 2005-12

Johann Radon Institute

for Computational and Applied Mathematics
Austrian Academy of Sciences

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RICAM-Report No. 00-00

March 2004

A-4040 LINZ, ALTENBERGERSTRASSE 69, AUSTRIA

<http://www.ricam.oeaw.ac.at>

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Abstract

We show that the Euclidean Wasserstein distance between two compactly supported solutions of the one-dimensional porous medium equation having the same center of mass decays to zero for large times. As a consequence, we detect an improved L^1 -rate of convergence of solutions of the one-dimensional porous medium equation towards *well centered* self-similar Barenblatt profiles, as time goes to infinity.

1 Introduction and result.

We consider the Cauchy problem for the one dimensional porous medium equation posed on the whole real line \mathbb{R}

$$\begin{cases} u_t = (u^m)_{xx} & m > 1 \\ u(x, 0) = u_0(x), \end{cases} \quad (1)$$

where the initial datum u_0 is integrable, nonnegative and compactly supported. Under these assumptions, it is well known that there exists a unique weak solution $u(\cdot, t)$ of (1) globally defined in time. Moreover, it is also known that the profile $u(\cdot, t)$ is bounded and compactly supported at any positive time t . We refer to the surveys [2, 22, 18] for a good account of the literature concerning existence theory, free boundary regularity, qualitative and asymptotic properties of the solutions to the porous medium equations. Let us denote the *total mass* of u_0 by

$$\int_{-\infty}^{+\infty} u_0(x) dx = M > 0$$

and its *center of mass* by

$$x_0 = \frac{1}{M} \int_{-\infty}^{+\infty} x u_0(x) dx.$$

We recall that both quantities M and x_0 are invariant with respect to time [18]. The long time asymptotics for equation (1) is described by the family of self similar *source type* Barenblatt–Pattle solutions

$$U_C(x, t) = t^{-\frac{1}{m+1}} \left(C - \frac{m-1}{2m(m+1)} x^2 t^{-\frac{2}{m+1}} \right)_+^{\frac{1}{m-1}},$$

where the constant C must be chosen in order to match the initial mass M . More precisely, it is well known, for instance [18], that

$$\|u(\cdot, t) - U_C(\cdot, t)\|_{L^1(\mathbb{R})} = o(1) \quad \text{as } t \rightarrow \infty, \quad (2)$$

for any solution $u(t)$ having nonnegative initial datum in L^1 . Without further assumptions on the initial datum, it has been proven in [18] that no algebraic rate of convergence can be determined in (2). If we restrict the class of initial data to those having finite L^m -norm and finite second moment, it is possible to obtain an improved rate of convergence [5]

$$\|u(\cdot, t) - U_C(\cdot, t)\|_{L^1(\mathbb{R})} \leq C t^{-\alpha} \quad \text{as } t \rightarrow \infty, \quad (3)$$

where

$$\alpha = \frac{2}{(m+1) \max(2, m)}, \quad (4)$$

which is sharp for $1 < m \leq 2$.

Solutions to the porous medium equation can also be interpreted as a one parameter curve in the space of probability densities having finite second moment [17]. In this framework, it is also known that the flow map $u(\cdot, t)$ induces a 1-Lipschitz semigroup with respect to the Euclidean Wasserstein distance [17, 7, 8]

$$d_2^2(U, V) = \inf_{T_\# U = V} \int |x - T(x)|^2 U(x) dx, \quad (5)$$

where the push-forward condition $T_\# U = V$ is expressed by the relation

$$\int_{-\infty}^{+\infty} \psi(x) V(x) dx = \int_{-\infty}^{+\infty} \psi(T(x)) U(x) dx \quad \text{for all } \psi \in C_0(\mathbb{R}) \quad (6)$$

More precisely, we have

$$d_2(u_1(\cdot, t), u_2(\cdot, t)) \leq d_2(u_1(\cdot, 0), u_2(\cdot, 0)), \quad (7)$$

where u_1 and u_2 are two solutions to the porous medium equation with initial data satisfying the above hypothesis. In the one dimensional case, the above result (7) can be easily proven to

be sharp by considering (for instance) two any solutions of the form U_{C,x_0}, U_{C,x_1} with $x_0 \neq x_1$, where U_{C,x_0} is the shifted Barenblatt profile

$$U_{C,x_0}(x,t) = t^{-\frac{1}{m+1}} \left(C - \frac{m-1}{2m(m+1)}(x-x_0)^2 t^{-\frac{2}{m+1}} \right)_+^{\frac{1}{m-1}}. \quad (8)$$

Moreover, by taking as $u(t)$ in (3) any shifted Barenblatt profile U_{C,x_0}^∞ with $x_0 \neq 0$ in case of $1 < m \leq 2$, we achieve the optimal rate $t^{-\alpha}$ in the class of initial data where (3) is true. We observe that $U_{C,x_0}^\infty(\cdot, t)$ has center of mass x_0 .

Fixing the center of mass by using well-centered Barenblatt profiles avoids spatial translations and thus, one would expect to improve the convergence rate towards them. This intuition coming from the spatial translation invariance of the equation has already been quantified in certain cases. As it is well-known, one can improve the rate of convergence towards a well centered gaussian kernel for the heat equation by fixing the center of mass of the initial data. Indeed, one can fix a certain number of initial moments in order to recover a higher order asymptotic approximation for the solutions with a faster rate of convergence [13, 14]. Mass-centering for the heat equation also speeds up the entropy decay [11]. In general, this suggests the possibility of detecting a more accurate asymptotic description even in the nonlinear case by fixing a suitable number of initial moments [24].

The occurrence of such phenomenon was first established by J.L. Vázquez in [19]. He proved both that the support of a general solution takes the shape of the support of its corresponding shifted Barenblatt profile for large times, and that a faster convergence rate in L^∞ towards such profile can be obtained. In particular he recovered an optimal rate t^{-1} of convergence in L^1 in case of solutions with radial symmetry. We shall describe those results in detail later on (see Theorem 2.1 and Remark 2.4).

In the fast diffusion case, the spatial translation invariance is related to an explicit eigenvalue of the linearization of the rescaled equation in self-similar variables [9, 12]. Therefore, we expect an improvement of the rate of convergence for the fast diffusion equation by mass-centering to the rate t^{-1} in L^1 in the translation dominated range, which is given by the next eigenvalue of the linearization. A nearly optimal convergence rate in the fast diffusion range has been obtained in [15, 16] based on mass-centering.

In this paper we improve estimate (7) by mass-centering in the 1D porous medium equation. More precisely, we shall prove that it is possible to detect an algebraic rate of decay to zero of the 2-Wasserstein distance between any two solutions having the same center of mass. As an easy consequence of that, we get a faster algebraic L^1 rate of convergence towards well centered Barenblatt profiles for compactly supported solution with prescribed center of mass, than that in (3). However, we cannot achieve the optimal rate t^{-1} of symmetric solutions provided in [19]. Our result is summarized in the following theorem.

Theorem 1.1 *Let $u(x,t)$ be the solution to the Cauchy problem (1) where the initial datum u_0 is nonnegative, compactly supported and with unit mass. Let x_0 be the center of mass of u_0 . Let U_{C,x_0} be the shifted Barenblatt profile (8) with unit mass. Then, there exists two constants $\varepsilon, \delta > 0$ such that the following inequalities holds for large t ,*

$$d_2(u(\cdot, t), U_{C,x_0}(\cdot, t)) = O((t+1)^{-\varepsilon}) \quad (9)$$

$$\|u(\cdot, t) - U_{C,x_0}(\cdot, t)\|_{L^1(\mathbb{R})} \leq C(t+1)^{-\alpha-\delta}. \quad (10)$$

2 Preliminaries

2.1 Time dependent scaling and asymptotic behavior

Our starting point is the standard time dependent scaling

$$\begin{aligned} u(x,t) &= \alpha(t)^{-\lambda} v(y,s) & y &= \alpha(t)^{-\lambda} (x-x_0) & s &= \beta(t) \\ \alpha(t) &= \left(1 + \frac{t}{\lambda}\right) & \beta(t) &= \lambda \log \alpha(t) & \lambda &= \frac{1}{m+1}, \end{aligned} \quad (11)$$

which transforms problem (1) into the Cauchy problem for the nonlinear Fokker Planck equation

$$\begin{cases} v_s = (yv + (v^m)_y)_y \\ v(y,0) = u_0(y). \end{cases} \quad (12)$$

The unique stationary solution to (12) with prescribed mass (the dependence on the mass is included in the positive constant C_M below) is

$$v_M^\infty(y) = \left(C_M - \frac{m-1}{2m} |y|^2 \right)_+^{\frac{1}{m-1}}, \quad (13)$$

which in the original variables $u(x, t)$ corresponds to a slight modification of the Barenblatt–Pattle profile (8), namely $U_{M, x_0}^\infty(x, t) = \lambda^{-\frac{1}{m-1}} U_{\bar{C}, x_0}^\infty(x, \alpha(t))$, where $\bar{C} = C_M \lambda$. Such a solution is explicitly given by the formula

$$U_{M, x_0}^\infty(x, t) = \frac{1}{\alpha(t)^\lambda} \left(C_M - \frac{m-1}{2m} \left(\frac{x-x_0}{\alpha(t)^\lambda} \right)^2 \right)_+^{\frac{1}{m-1}}. \quad (14)$$

We recall that the difference $U_{M, x_0}^\infty(x, t) - U_{\bar{C}, x_0}^\infty(\cdot, \alpha(t))$ decays to zero in L^1 with a faster rate than that in (10). This can be proven with similar arguments as in [6]. We now introduce the pressure variable $\omega = \frac{m}{m-1} u^{m-1}$. In terms of the pressure, the shifted self similar solution (8) reads

$$\begin{aligned} \omega_{M, x_0}^\infty(x, t) &= \frac{m}{m-1} \alpha(t)^{-\frac{m-1}{m+1}} \left(C_M - \frac{m-1}{2m} \left(\frac{x-x_0}{\alpha(t)^\lambda} \right)^2 \right)_+ \\ &= \frac{1}{2\alpha(t)} (r(t) - (x-x_0)^2)_+, \quad r(t) = \frac{2m}{m-1} C_M \alpha(t)^{2\lambda}. \end{aligned} \quad (15)$$

Since we shall mostly work in the similarity variables $v(y, s)$, it is convenient to introduce the following rescaled pressure variable $z(y, s) = \frac{m}{m-1} v(y, s)^{m-1}$. In terms of this new variable z , the stationary solution (13) reads

$$z^\infty(y) = \left(\frac{m}{m-1} C_M - \frac{y^2}{2} \right)_+$$

As mentioned in the introduction, we recall that if the initial datum u_0 is compactly supported so is the solution of (1) at any time $t > 0$. Moreover, after a waiting time t^* , the support of the solution takes the shape of a connected interval (see [18]). For $t > t^*$, we denote the support of $u(t)$ by $[\underline{\zeta}(t), \bar{\zeta}(t)]$. Then, it is known that the curve $\{\underline{\zeta}(t)\}_{t>t^*}$ is strictly decreasing and the curve $\{\bar{\zeta}(t)\}_{t>t^*}$ strictly increasing [18]. Let us denote the support of the rescaled solution $v(s)$ by $[\underline{\eta}(s), \bar{\eta}(s)]$. The following theorem is due to Vázquez [19, 21, 20]

Theorem 2.1 *The following properties hold as $t \rightarrow \infty$ with $r(t)$ defined by (15),*

$$\underline{\zeta}(t) = -r(t) + x_0 + o(1) \quad \bar{\zeta}(t) = r(t) + x_0 + o(1), \quad (16)$$

$$\omega(x, t) - \omega_{x_0}^\infty(x, t) = o\left(t^{-\frac{m}{m+1}}\right) \quad \text{uniformly on } \mathbb{R}, \quad (17)$$

$$\omega_x(x, t) = -\frac{x-x_0}{(m+1)t} + o\left(\frac{1}{t}\right) \quad \text{uniformly on } \text{supp } \omega(\cdot, t), \quad (18)$$

$$\omega_{xx}(x, t) = -\frac{1}{(m+1)t} + O\left(\frac{1}{t^2}\right) \quad \text{uniformly on } \text{supp } \omega(\cdot, t). \quad (19)$$

Next we reformulate the above result in similarity variables.

Corollary 2.2 *Given $c(m, M) = \frac{2mC_M}{m-1}$, the following properties hold as $s \rightarrow \infty$,*

$$\underline{\eta}(s) = -c(m, M) + o(1) \quad \bar{\eta}(s) = c(m, M) + o(1), \quad (20)$$

$$z(y, s) - z_{x_0}^\infty(y) = o(e^{-s}) \quad \text{uniformly on } \mathbb{R}, \quad (21)$$

$$z_y(y, s) = -y + o(e^{-s}) \quad \text{uniformly on } \text{supp } z(\cdot, s), \quad (22)$$

$$z_{yy}(y, s) = -1 + O(e^{-(m+1)s}) \quad \text{uniformly on } \text{supp } z(\cdot, s). \quad (23)$$

As a consequence of the above corollary, we state the following useful result.

Corollary 2.3 Let $v(y, s)$ be a solution to (12). Let $F(y, s) = \int_{-\infty}^y v(\xi, s) d\xi$. Then, there exists a small constant $\delta > 0$ and a fixed $s_0 > 0$ such that, for any $s \geq s_0$, $i = 1, 2$ we have

$$y \in [\underline{\eta}_i(s), \underline{\eta}_i(s) + \delta] \Rightarrow K_0(y - \underline{\eta}_i(s))^{\frac{1}{m-1}} \leq v(y, s) \leq K_1(y - \underline{\eta}_i(s))^{\frac{1}{m-1}}, \quad (24)$$

$$y \in [\bar{\eta}_i(s) - \delta, \bar{\eta}_i(s)] \Rightarrow K_0(\bar{\eta}_i(s) - y)^{\frac{1}{m-1}} \leq v(y, s) \leq K_1(\bar{\eta}_i(s) - y)^{\frac{1}{m-1}}, \quad (25)$$

$$y \in [\underline{\eta}_i(s), \underline{\eta}_i(s) + \delta] \Rightarrow K_0(y - \underline{\eta}_i(s))^{\frac{m}{m-1}} \leq F(y, s) \leq K_1(y - \underline{\eta}_i(s))^{\frac{m}{m-1}}, \quad (26)$$

$$y \in [\bar{\eta}_i(s) - \delta, \bar{\eta}_i(s)] \Rightarrow K_0(\bar{\eta}_i(s) - y)^{\frac{m}{m-1}} \leq F(y, s) \leq K_1(\bar{\eta}_i(s) - y)^{\frac{m}{m-1}}, \quad (27)$$

where K_0 and K_1 are fixed positive constants independent on s .

Proof. Relation (23) implies

$$z_{yy}(y, s) \geq -1 + \varepsilon, \quad \text{as } s \geq s_0, \quad (28)$$

for $\varepsilon \ll 1$ and for some $s_0 > 0$. A double integration upon the interval $[\underline{\eta}_i(s), y]$ of the above relation (28) yields

$$z(y, s) - z(\underline{\eta}(s), s) \geq (y - \underline{\eta}(s)) z_y(\underline{\eta}(s), s) - \frac{(1 - \varepsilon)^2}{2} (y - \underline{\eta}(s))^2$$

We then use (20) and (22) to obtain, for $s \geq s_0$,

$$z(y, s) \geq (y - \underline{\eta}(s)) \left[c(m, M) + o(1) - \frac{(1 - \varepsilon)^2}{2} (y - \underline{\eta}(s)) \right].$$

By choosing $\delta > 0$ small enough, we then obtain $z(y, s) \geq O(1)(y - \underline{\eta}(s))$ as $|\underline{\eta}(s) - y| \leq \delta$, and the first inequality in (24) is proven. The second inequality in (24) and the two inequalities in (25) can be proven in a similar fashion. We skip the details about these computations. The inequalities in (26) and (27) easily follow from (24) and (24) after integration in $(-\infty, y]$. \square

Remark 2.4 (L^1 -rates of convergence in the literature) As observed in [19], (17) together with (16) imply

$$\|u(t) - U_{M, x_0}^\infty(t)\|_{L^1(\mathbb{R})} = o(t^{-\gamma}), \quad \gamma = \frac{2}{(m+1) \max(2, 2(m-1))}. \quad (29)$$

We observe that, in case $1 < m \leq 2$, the above estimate improves (3) not providing a better rate. An explicit faster rate of convergence is obtained in [19, 18] in case of solutions with symmetric initial datum, i.e. $u_0(x_0 - x) = u_0(x_0 + x)$, namely

$$\|u(t) - U_{M, x_0}^\infty(t)\|_{L^1(\mathbb{R})} = O(t^{-1}).$$

We observe that such a rate of convergence can be heuristically recovered by means of spectral analysis of the linearized non linear Fokker-Planck equation (12) around the stationary state (see [1, 24]).

2.2 One dimensional Wasserstein distance

Let U_1 and U_2 be two probability densities on \mathbb{R} having compact support. The Euclidean Wasserstein distance between U_1 and U_2 can be analytically expressed as follows (the following procedure can be performed only in one space dimension). Consider the primitive variables

$$F_i(x) = \int_{-\infty}^x U_i(\xi) d\xi, \quad i = 1, 2,$$

and their pseudo-inverses $F_i^{-1} : [0, 1] \rightarrow \mathbb{R}$, $F_i^{-1}(\rho) = \inf\{\omega \in \mathbb{R} : F_i(\omega) > \rho\}$, $i = 1, 2$. Then, the following identity is satisfied (see [23, 8, 3])

$$d_2(U_1, U_2) = \left[\int_0^1 (F_1^{-1}(\rho) - F_2^{-1}(\rho))^2 d\rho \right]^{1/2}. \quad (30)$$

2.3 Two useful inequalities

We now recall a weighted Poincaré inequality we will need in the sequel (see the paper [10] for the proof).

Theorem 2.5 (Weighted Poincaré inequalities) *Let μ and w be two nonnegative measurable functions on $[a, b]$ which are finite almost everywhere, and ν be a nonnegative finite Borel measure on $[a, b]$, with $\nu[a, b] > 0$. Let $1 < p \leq q < \infty$, with conjugate exponents p' and q' respectively, and define*

$$K_{p,q}(\mu, \nu, w) = \frac{1}{\nu[a, b]} \left(\sup_{a < x < b} \left\{ \mu[x, b]^{1/q} \left(\int_a^x \nu[a, t]^{p'} w(t)^{1-p'} dt \right)^{1/p'} \right\} + \sup_{a < x < b} \left\{ \mu[a, x]^{1/q} \left(\int_x^b \nu[t, b]^{p'} w(t)^{1-p'} dt \right)^{1/p'} \right\} \right). \quad (31)$$

Then,

$$\left(\int_a^b \left| f(x) - \frac{1}{\nu[a, b]} \int_a^b f d\nu \right|^q \mu(x) dx \right)^{1/q} \leq C \left(\int_a^b |f'(x)|^p w(x) dx \right)^{1/p} \quad (32)$$

for all absolutely continuous functions f on $[a, b]$ if and only if $K_{p,q} < \infty$. Moreover, the smallest possible constant C in (32) satisfies the estimate

$$\frac{K_{p,q}}{2} \leq C \leq (1 + q/p')^{1/q} (1 + p'/q)^{1/p'} K_{p,q}. \quad (33)$$

We shall also make use of the following interpolation inequality involving L^1 -norm and d_2 -distance, which is an easy consequence of the generalized HWI inequality (Theorem 2.1) in [4], of the results in [5, 17] about the decay of the relative Fisher information and of the generalized Csiszár–Kullback inequality proven in [5].

Theorem 2.6 (Interpolation inequality) *Let $u_0 \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with second moment bounded and let v^∞ be as in (13). Then, there exists a positive constant C depending only on u_0 such that the solution $v(s)$ to (12) satisfies*

$$\|v(s) - v^\infty\|_{L^1(\mathbb{R})}^{\max(2,m)} \leq C d_2(v(s), v^\infty) e^{-2s}. \quad (34)$$

3 Proof of Theorem 1.1

We use the rescaled formulation (12) of problem (1). We suppose the initial datum u_0 to be compactly supported with unit mass and zero center of mass. As in the case of (1), the solution $v(s)$ to (12) at time s maintains the same mass as the initial datum. The conservation of the zero first moment comes from the estimate

$$\frac{d}{ds} \int yv(y, s) dy = \int y(yv + (v^m)_y)_y dy = - \int yv(y, s) dy.$$

Let us define the primitive variable

$$F(y, s) = \int_{-\infty}^y v(\xi, s) d\xi \quad (35)$$

and its pseudo inverse $F^{-1} : [0, 1] \rightarrow \mathbb{R}$ as in the previous section. Then, it can be easily computed that $F^{-1}(\rho, s)$ satisfies the following equation in $(0, 1) \times \mathbb{R}_+$

$$\frac{\partial F^{-1}}{\partial s} = -F^{-1} - \frac{\partial}{\partial \rho} \left[\left(\frac{\partial F^{-1}}{\partial \rho} \right)^{-m} \right]. \quad (36)$$

We remark that F^{-1} satisfies (36) in classical sense at least after a waiting time (see [3]). Let us consider then two solutions v_1 and v_2 to (12) and let F_1 and F_2 be their corresponding primitive

variables as in the above definition (35). Due to (30), we compute the L^2 norm of the difference $F_1^{-1} - F_2^{-1}$ as follows (see also [8, 3]).

$$\begin{aligned} \frac{d}{ds} \int_0^1 [F_1^{-1}(\rho, s) - F_2^{-1}(\rho, s)]^2 d\rho &= 2 \int_0^1 [F_1^{-1} - F_2^{-1}] \partial_s [F_1^{-1} - F_2^{-1}] d\rho \\ &= -2 \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho - 2 \int_0^1 (F_1^{-1} - F_2^{-1}) \partial_\rho \left[(F_{1,\rho}^{-1})^{-m} - (F_{2,\rho}^{-1})^{-m} \right] d\rho. \end{aligned} \quad (37)$$

Then, integration by parts in the last term of (37) yields

$$\begin{aligned} \frac{d}{ds} \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho &= -2 \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho - 2 (F_1^{-1} - F_2^{-1}) (\tilde{v}_1 - \tilde{v}_2) \Big|_{\rho=0}^{\rho=1} \\ &\quad + 2 \int_0^1 (F_{1,\rho}^{-1} - F_{2,\rho}^{-1}) (\tilde{v}_1^m - \tilde{v}_2^m) d\rho =: I + B + J. \end{aligned} \quad (38)$$

The boundary term B above disappears because of the relation

$$(\tilde{v}_1 - \tilde{v}_2) = 0 \quad \text{at } \rho = 0, 1$$

and because of the finite speed of propagation property of the support of v , which implies that F_1^{-1} and F_2^{-1} are bounded at any time (see also [8]). We now consider the term

$$J = 2 \int_0^1 (F_{1,\rho}^{-1} - F_{2,\rho}^{-1}) (\tilde{v}_1^m - \tilde{v}_2^m) d\rho = 2 \int_0^1 (F_{1,\rho}^{-1} - F_{2,\rho}^{-1}) \left((F_{1,\rho}^{-1})^{-m} - (F_{2,\rho}^{-1})^{-m} \right) d\rho.$$

Since the function $\phi(u) = u^m$ is strictly increasing, then the integrand in J is not positive at any (ρ, s) . Therefore, as a first step (see [3]) we obtain exponential decay for $d_2(v_1(s), v_2(s))$ with rate e^{-s} , and the contraction property (7) when turning back to the original variables u, x, t . Our intention now is to handle the term J in order to recover a better exponential rate for $d_2(v_1(s), v_2(s))$. To this aim, let us introduce the following notations

$$\begin{aligned} J &= 2 \int_0^1 (F_{1,\rho}^{-1} - F_{2,\rho}^{-1}) (\tilde{v}_1^m - \tilde{v}_2^m) d\rho = -2 \int_0^1 [F_{1,\rho}^{-1} - F_{2,\rho}^{-1}]^2 \Sigma(\rho, s) \\ \Sigma(\rho, s) &= \left[\frac{\tilde{v}_1^m - \tilde{v}_2^m}{\tilde{v}_1 - \tilde{v}_2} \tilde{v}_1 \tilde{v}_2 \right] (\rho, s) \\ \tilde{v}_i(\rho, s) &= v_i(F_i^{-1}(\rho, s), s) = [\partial_\rho F_i^{-1}(\rho, s)]^{-1}, \quad i = 1, 2. \\ \text{supp}(v_i(s)) &= (\underline{\eta}_i(s), \bar{\eta}_i(s)), \quad i = 1, 2. \end{aligned}$$

Proposition 3.1 *The weight function $\Sigma(\rho, s)$ satisfies the following properties.*

- (i) $\Sigma(0, s) = \Sigma(1, s) = 0$ for all $s \geq 0$
- (ii) $\Sigma(\rho, s) > 0$ for all $0 < \rho < 1, s \geq 0$.
- (iii) *There exist a small positive constant $\rho_0 \ll 1$ and a $s_0 > 0$ such that, for any $s \geq s_0, i = 1, 2$ we have*

$$\rho \in [0, \rho_0] \Rightarrow C_0 \rho^{\frac{m-1}{m}} \leq F_i^{-1}(\rho, s) - \underline{\eta}_i(s) \leq C_1 \rho^{\frac{m-1}{m}}, \quad (39)$$

$$\rho \in [1 - \rho_0, 1] \Rightarrow C_0 (1 - \rho)^{\frac{m-1}{m}} \leq F_i^{-1}(\rho, s) - \bar{\eta}_i(s) \leq C_1 (1 - \rho)^{\frac{m-1}{m}}, \quad (40)$$

$$\Sigma(\rho, s) \geq C [\rho(1 - \rho)]^{\frac{m+1}{m}} =: \sigma(\rho). \quad (41)$$

where C, C_0 and C_1 are fixed positive constants independent on s .

Proof. Let us start with (i). By definition of \tilde{v}_i we easily obtain

$$\tilde{v}_i(0, s) = v_i(\underline{\eta}_i(s), s) = 0, \quad \tilde{v}_i(1, s) = v_i(\bar{\eta}_i(s), s) = 0, \quad i = 1, 2. \quad (42)$$

We then compute

$$\lim_{\rho \searrow 0} \frac{\tilde{v}_1^m - \tilde{v}_2^m}{\tilde{v}_1 - \tilde{v}_2} = m \lim_{\rho \searrow 0} \xi^{m-1}, \quad (43)$$

for some ξ belonging in the interval with extremals $\tilde{v}_1(\rho, s)$ and $\tilde{v}_2(\rho, s)$. Hence, the limit in (43) equals zero. This fact, together with (42), proves (i). In order to prove (ii) we only need to observe that the function of two nonnegative real variables

$$(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \frac{x^m - y^m}{x - y}, \quad m > 1$$

is always non positive and it equals zero if and only if $x = y = 0$. Therefore, the weight function $\Sigma(\rho, s)$ is zero if and only if both \tilde{v}_1 and \tilde{v}_2 are zero, and this occurs if and only if $\rho \in \{0, 1\}$. The proof of (iii) is a straight forward consequence of (26) and (27) in Corollary 2.3. Moreover, as a simple consequence of (ii) and (iii), we have the following inequalities for $i = 1, 2$ and for $s \geq s_0$,

$$\tilde{v}_i(\rho, s) \geq O(1)\rho^{\frac{1}{m}} \text{ as } \rho \in [0, \rho_0]; \quad \tilde{v}_i(\rho, s) \geq O(1)(1 - \rho)^{\frac{1}{m}} \text{ as } \rho \in [1 - \rho_0, 1]. \quad (44)$$

Moreover, by means of the limiting relation (21) together with (39) and (40) and because F_i^{-1} are nondecreasing w.r.t. ρ , we can easily deduce that

$$\tilde{v}_i(\rho, s) \geq C, \quad \text{as } \rho \in [\rho_0, 1 - \rho_0], \quad i = 1, 2, \quad (45)$$

where C is a fixed constant independent on s . Hence, from (45), (44) one can found a suitable fixed constant C such that

$$\tilde{v}_i \geq C [\rho(1 - \rho)]^{1/m}, \quad i = 1, 2. \quad (46)$$

Finally, by means of the mean value formula

$$\frac{\tilde{v}_1(\rho, s)^m - \tilde{v}_2(\rho, s)^m}{\tilde{v}_1(\rho, s) - \tilde{v}_2(\rho, s)} = m\xi^{m-1}$$

for some ξ belonging in the interval with extremal points $\tilde{v}_1(\rho, s)$ and $\tilde{v}_2(\rho, s)$, we can use the lower estimate (46) in the definition of Σ and this proves the desired estimate (41). By combining (41) and (38), we obtain

$$\frac{d}{ds} \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho \leq -2 \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho - 2 \int_0^1 [F_{1,\rho}^{-1} - F_{2,\rho}^{-1}]^2 \sigma(\rho) d\rho. \quad (47)$$

Next we exploit the weighted Poincaré inequality (32) in (47) in order to increase the exponential rate of convergence to zero of the term $d_2(v_1, v_2)$. Following the same notation as in theorem 2.5, we perform the following substitutions

$$\begin{array}{llll} p & \longrightarrow & 2 & q & \longrightarrow & 2 \\ \mu, \nu & \longrightarrow & \text{Lebesgue measure} & w & \longrightarrow & \sigma \\ x \in [a, b] & \longrightarrow & \rho \in [0, 1] & f(\cdot) & \longrightarrow & F_1^{-1}(\cdot, s) - F_2^{-1}(\cdot, s). \end{array}$$

Since $v(s)$ has zero first moment, the change of variable $\rho = F(y, s)$ yields

$$\int_0^1 F^{-1}(\rho, s) d\rho = \int_{-\infty}^{+\infty} y F_y(y, s) dy = \int_{-\infty}^{+\infty} y v(y, s) dy = 0.$$

Hence, with the notation of Theorem 2.5, the term $\int_a^b f d\nu$ in (32) equals zero. We then compute the constant $K_{2,2}$ in (31). From simple calculations, it is easy to see that the condition

$$\sup_{0 \leq \rho \leq 1} \left\{ \rho^{1/2} \left(\int_\rho^1 (1-t)^{1-\frac{1}{m}} t^{-1-\frac{1}{m}} dt \right)^{1/2} \right\} < \infty \quad (48)$$

is sufficient to have $K_{2,2} < \infty$. But (48) is clearly satisfied for $m > 1$. We can then employ (32) and we obtain

$$\frac{d}{ds} \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho = -(2 + 2\varepsilon_0) \int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho,$$

for some positive constant ε_0 , and this implies

$$\int_0^1 [F_1^{-1} - F_2^{-1}]^2 d\rho \leq C e^{-(2+2\varepsilon_0)s}. \quad (49)$$

By expressing (49) in the original variables u, x, t of the porous medium equation, according to the time dependent scaling (11), we obtain (9). Inequality (10) then follows from (9) and the interpolation inequality (34).

Acknowledgements

This work has been partially supported by the EU-funded IHP-project ‘HYKE’. The work of MDF has been partially supported by the Wittgenstein 2000 Award of Peter A. Markowich. The authors would like to thank Prof. Juan Luis Vázquez for useful discussions and suggestions.

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