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Abstract

Shift radix systems provide a unified notion to study two important types of number systems. In this paper, we briefly review the origin of this notion.

1. Introduction

Let $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$. Consider the mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$, which maps each element $(z_1, \ldots, z_d)$ to $(z_2, \ldots, z_{d+1})$, provided that

$$0 \le r_1 z_1 + r_2 z_2 + \cdots + r_d z_d + z_{d+1} < 1.$$

Obviously, $\tau_{\mathbf{r}}$ is defined by

$$\tau_{\mathbf{r}}\big((z_1, \ldots, z_d)\big) = (z_2, \ldots, z_d, -\lfloor r_1 z_1 + \cdots + r_d z_d \rfloor). \tag{1.1}$$

We say that $\tau_{\mathbf{r}}$ has the *finiteness property* if for every $\mathbf{z} \in \mathbb{Z}^d$ there exists a k, such that $\tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$.

This concept unifies the notations for two important number systems, namely *canonical number systems* and *β-expansions.* For these number systems, the

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finiteness property means that all numbers of a certain set admit finite expansions. This property also plays an important role for constructing the tilings giving the Markoff partitions of the dynamical systems associated to these number systems.

If $\tau_{\mathbf{r}}$ has the finiteness property, then $(\mathbb{Z}, \tau_{\mathbf{r}})$ is called a *shift radix system* (for short SRS). It turned out to be hard problem to characterize all $\mathbf{r} \in \mathbb{R}^d$ which give raise to a SRS and currently, a complete solution seems to be out of range. However, in the last years, several partial results have been established.

For an introduction to SRS, we do not aim to give an exact account. For detailed information, we refer to the original papers. Emphasis will be put on the results of Gilbert and Hollander who developed the essential structure of the SRS algorithm.

As an example, we consider Knuth's number system. He proposed a way to express Gaussian integers by digits from $\mathcal{A} = \{0, 1\}$ with base $\alpha = -1 + \sqrt{-1}$:

$$\begin{aligned}
1 &= \alpha^0 = (1)_\alpha \\
2 &= \alpha^3 + \alpha^2 = (1100)_\alpha \\
3 &= \alpha^3 + \alpha^2 + 1 = (1101)_\alpha \\
4 &= \alpha^8 + \alpha^7 + \alpha^6 + \alpha^4 = (111010000)_\alpha \\
5 &= \alpha^8 + \alpha^7 + \alpha^6 + \alpha^4 + 1 = (111010001)_\alpha
\end{aligned}$$

How do we find such expressions ? This is done by the so called *residual algorithm.* Take for example the expansion of 3 in base α: As $\mathcal{A}$ forms a complete system of representatives modulo α, and $3 \equiv 1 \pmod{\alpha}$, the lowest digit is 1. We subtract 1 and then divide by α. Then we restart this process from $(3-1)/\alpha = -1-\sqrt{-1}$. Iterating this, we generate the expansion in base α. After four steps, we terminate this process since we run into the trivial cycle $0 \to 0$ which generates infinitely many leading zeros.

Generally, we may start with an algebraic integer α and a complete residue system $\mathcal{A}$ modulo α in $\mathbb{Z}[\alpha]$. If all algebraic conjugates of α have modulus greater than 1, then the residual algorithm can be viewed as a contractive map on $\mathbb{Z}[\alpha]$. Since $\mathbb{Z}[\alpha]$ is isomorphic to $\mathbb{Z}^d$ as an additive group, this means that each orbit must be eventually periodic. However, it is not trivial that this process must terminate in finitely many steps. In fact, it is possible that the residual algorithm runs into a non trivial cycle. An easy example is given by the well known b-ary number system. In this case, we have $b \in \mathbb{Z}$ with $b \geq 2$ and $\mathcal{A} = \{0, \ldots, b-1\}$. It is well known that every non negative integer admits a finite representation. However, for negative integers the residual algorithm does not terminate. In this case, the residual algorithm produces an infinitely, but eventually periodic expansion.

For Knuth's number system, it is proved that the process terminates for every starting value. This is exactly the above mentioned *finiteness property.* In this case, the pair $(\alpha, \mathcal{A})$ is called a *canonical number system* (CNS for short).

Before explaining Gilbert's idea, we summarize the known results on the characterization of CNS. The first systematic treatment was given in Kátai-Szabó [18], where all CNS bases for the Gaussian integers have been characterized. This result was generalized to quadratic integers in Kátai-Kovács [16, 17] and independently in Gilbert [13]. Körmendi [19] dealt with a special class of cubic integers and recently Brunotte [9, 10] characterized all CNS whose bases are roots of trinomials. In the general case Kovács [20] proved that an algebraic integer α gives rise to a CNS if its minimal polynomial

$$P(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$$

satisfies

$$1 \leq b_{n-1} \leq b_{n-2} \leq \cdots \leq b_1 \leq b_0, \qquad b_0 \geq 2.$$

However, this characterization is far from complete. Examples not contained in this class are given in Kovács-Pethő [21], where also an algorithm is established to decides whether a given α is a CNS base or not. Akiyama-Pethő [4] developed a much faster algorithm than the one in [21]. Akiyama-Rao [5] and Scheicher-Thuswaldner [25] studied CNS under the so called *Dominant Condition*

$$|b_1| + \ldots + |b_n| < b_0.$$

In particular, all CNS up to degree five with this additional property have been characterized. Finally, we mention that Brunotte [9, 10] provided the fastest known algorithm to determine if an arbitrary polynomial gives a CNS or not.

2. Gilbert's Clearing Algorithm

In [13], W. J. Gilbert introduced his *Clearing Algorithm* which is one of the origins of the SRS setting. We explain his idea by using Knuth's number system.

Let $\alpha = -1 + \sqrt{-1}$. Then $\mathbb{Z}[\alpha] = \mathbb{Z}[\sqrt{-1}]$. As α satisfies $x^2 + 2x + 2 = 0$, we have the isomorphism $\mathbb{Z}[\alpha] \simeq \mathbb{Z}[x]/(x^2 + 2x + 2)$. The residual algorithm is interpreted by the division algorithm on $\mathbb{Z}[x]$. A polynomial $Q(x) = \sum_i q_i x^i \in \mathbb{Z}[x]$ will be called *cleared* if $q_i \in \{0, 1\}$. For a given $A(x) \in \mathbb{Z}[x]$, we wish to find $B(x)$ and $Q(x)$ with

$$A(x) = (x^2 + 2x + 2)B(x) + Q(x),$$

such that $Q(x) = \sum_i q_i x^i$ is cleared. If this holds, then $Q(\alpha) = \sum_i q_i \alpha^i$ is the desired expression.

This process is visualized for $A(x) = 5$ in the Table 1. The first line contains

								5
						-2	-4	-4
					2	4	4	
				-1	-2	-2		
			0	0	0			
		1	2	2				
	-1	-2	-2					
1	2	2						
1	1	1	0	1	0	0	0	1

Table 1: Gilbert's Clearing Algorithm

the coefficients of $A(x) = 5$. The last line gives the coefficients of the cleared polynomial $Q(x)$ and each intermediate horizontal line contains a multiple of $x^2 + 2x + 2$.

Now we come to the main point. We only keep track of the sequence $-2, 2, -1, 0, 1, -1, 1$ and this is nothing but the sequence of coefficients of $B(x)$. Denote this sequence by $z_1, z_2, z_3, \cdots$. The key step is to switch to this sequence instead of observing the output reminders $1, 0, 0, 0, 1, 0, 1, 1, 1$. The z_i are determined by the inequality

$$0 \le z_i + 2z_{i+1} + 2z_{i+2} < 2.$$

Dividing by 2, we get

$$0 \le \tfrac{1}{2} z_i + z_{i+1} + z_{i+2} < 1.$$

Thus $\mathbf{r} = (\frac{1}{2}, 1)$. If $\mathbf{r}$ gives a SRS, then Knuth's number system has the expected finiteness property. A straightforward generalization yields

Theorem 2.1 *The polynomial* $P(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0$ *gives a canonical number system if and only if* $\left(\frac{1}{b_0}, \frac{b_{d-1}}{b_0}, \ldots, \frac{b_1}{b_0}\right)$ *gives a* d*-dimensional SRS.*

The clearing algorithm was reformulated by a suitable base change of $\mathbb{Z}[\alpha]$ into a SRS. An important idea, *the set of witnesses*, was invented independently by Brunotte [9, 10] and Scheicher-Thuswaldner [24, 25] under this base change. This method gives the easiest and fastest algorithm to determine whether α gives a CNS or not. Consult [2] to see how this idea works in SRS framework.

3. β-expansions

A second type of number system, the so called β-expansions, are related to SRS as well. Given $\beta > 1$, we wish to expand a positive real number x in the form

$$x = x_{-m}\beta^m + x_{-m+1}\beta^{m-1} + \dots \tag{3.1}$$

with $x_i \in \mathcal{A} = [0, \beta) \cap \mathbb{Z}$. This is done as follows. Find the largest integer m such that $\beta^m \leq x < \beta^{m+1}$. Compute $x - x_{-m}\beta^m$ with

$$x_{-m} = \max\{a \in \mathcal{A} : x - a\beta^m \geq 0\}.$$

Iterating this process will lead to an expansion of form (3.1). This setting is also a natural generalization of the decimal expansions. Therefore, the β-expansion corresponds to a *greedy algorithm.*

For $x \in [0, 1)$, the *greedy algorithm* is equivalent to the following setting: Define the *β-transformation* $T : x \to \beta x - \lfloor \beta x \rfloor$. By iterating this map and considering its trajectory

$$x \xrightarrow{x_1} T(x) \xrightarrow{x_2} T^2(x) \xrightarrow{x_3} \dots$$

with $x_j = \lfloor \beta T^{j-1} x \rfloor$, we obtain

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots.$$

We will call the sequence

$$d_\beta(x) = .x_1 x_2 \dots$$

the β-expansion of x. We say that $d_\beta(x)$ is finite when $x_i = 0$ for all sufficiently large i. This is the case when there in an integer $i \geq 0$ such that $T^i x = 0$. For an arbitrary $x \geq 1$, there is a maximal integer m such that $\beta^{-m} x \in [0, 1)$ with $d_\beta(\beta^{-m} x) = .x_{-m} x_{-m+1} \cdots$. By shifting, we obtain

$$d_\beta(x) = x_{-m} x_{-m+1} \cdots x_{-1} x_0 . x_1 x_2 \cdots .$$

Formally, by applying T, one can expand 1 and obtains $d_\beta(1) = .c_1 c_2 \dots$. In contrast to CNS, in β-expansions some subwords of $\mathcal{A}^*$ never appear. For example, the subword $w = c_1(c_2+1)$ can not appear even if $c_1(c_2+1) \in \mathcal{A}^2$. This is because w is too large and we would have already removed it earlier by the greedy algorithm. Thus, a word of $\mathcal{A}^*$ appears as a subword of a β-expansion if and only if all its suffices are less than $d_\beta(1)$ in natural lexicographic order (cf. [22, 15]).

β-expansions were introduced by Rényi [23], who proved that the β-transformation is ergodic. The invariant measure has been computed independently by Gelfond and Parry [12, 22]. As a non-trivial example for a dynamical system where the invariant measure is explicitly known, its arithmetic, diophantine and ergodic properties have been extensively studied. Let

$$\begin{aligned} \mathbf{Per}(\beta) &= \{x \in \mathbb{R}_+ \ : \ d_\beta(x) \text{ is eventually periodic}\} \quad \text{and} \\ \mathbf{Fin}(\beta) &= \{x \in \mathbb{R}_+ \ : \ d_\beta(x) \text{ is finite}\}. \end{aligned}$$

Recall that a Pisot number is an algebraic integer $\beta > 1$ for which all algebraic conjugates $\gamma \neq \beta$ satisfy $|\gamma| < 1$.

If β is a Pisot number, then $\mathbf{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+$ (cf. Bertrand and Schmidt [8, 26]), which is a generalization of the fact that the decimal expansions of rational numbers are eventually periodic.

We say that a number $\beta > 1$ has the *finiteness property* or property (F), if

$$\mathbf{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+. \tag{F}$$

The inclusion $\mathbf{Fin}(\beta) \subset \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$ is trivial. Obviously a rational number with a denominator 10^n has a finite decimal expansion. Therefore we are just expecting to have finite expansions for all reasonable candidates. The notion of property (F) was introduced in [11]. If β has property (F) then β is a Pisot number (cf. [11]). Even the weaker condition $\mathbb{Z}_+ \subset \mathbf{Fin}(\beta)$ implies that β is a Pisot number (cf. [1]). On the other hand, there exist Pisot numbers which do not have property (F).

The characterization of bases β with finiteness property (F) is a hard question. In [2] it is proved, that this question is equivalent to the problem of the characterization of CNS. Before this, the following partial results have been established: Let $x^d - a_{d-1}x^{d-1} - \ldots - a_0$ be the minimal polynomial of β. Frougny and Solomyak [11] proved that

$$a_{d-1} \geq a_{d-2} \geq \cdots \geq a_0 > 0$$

is a sufficient condition for (F). Hollander proved that

$$a_{d-1} \geq a_{d-2} + \cdots + a_0, \quad a_i \geq 0,$$

is also sufficient. Akiyama [1] classified all cubic Pisot units with (F). Akiyama-Rao-Steiner [6] includes further progress. However, the problem of characterizing all cubic Pisot numbers with property (F) is still open.

4. Hollander's Carry Sequence

The second originator of SRS is M. Hollander who was a student of B. Solomyak. As his thesis [14] did not yet appear in publication, we emphasize that the idea of SRS is basically due to him. Start from the relation:

$$\beta^d = a_{d-1}\beta^{d-1} + a_{d-2}\beta^{d-2} + \cdots + a_0$$

which is not necessary irreducible. Let

$$\begin{aligned} r_1 &= \frac{a_0}{\beta} \\ r_2 &= \frac{a_1}{\beta} + \frac{a_0}{\beta^2} \\ &\vdots \\ r_d &= \frac{a_{d-1}}{\beta} + \frac{a_{d-2}}{\beta^2} + \ldots + \frac{a_0}{\beta^d}. \end{aligned}$$

Then $r_d = 1$ and $\{r_1, \ldots, r_d\}$ spans $\mathbb{Z}[\beta]$ as a $\mathbb{Z}$-module and gives a basis if the relation is irreducible. Therefore, each $\gamma \in \mathbb{Z}[\beta]$ has a representation $\gamma = \sum_{i=1}^{d} z_i r_i$ with $z_i \in \mathbb{Z}$. Let $\gamma \in \mathbb{Z}[\beta] \cap [0, 1)$, i.e.,

$$0 \le z_1 r_1 + z_2 r_2 + \ldots + z_d r_d < 1.$$

Then the β-transform of γ can be written as

$$T(\gamma) = \sum_{i=1}^{d} z_{i+1}\, r_i$$

with z_{d+1} satisfying

$$0 \le z_2 r_1 + z_3 r_2 + \ldots + z_{d+1} r_d < 1.$$

Note that z_{d+1} is uniquely determined by this inequality, i.e.

$$z_{d+1} = -\lfloor z_2 r_1 + z_3 r_2 + \ldots + z_d r_{d-1} \rfloor.$$

Hollander called this sequence $z_1, z_2 \ldots$ the *Carry Sequence*. We clearly have

Theorem 4.1 *β has property (F) if and only if $(r_1, r_2, \ldots, r_{d-1})$ gives a $d-1$-dimensional SRS.*

As an example, let $\beta^3 = 3\beta^2 + 0\beta + 1$. Thus $\beta \approx 3.1038$. In Table 2, we expand $5\beta^{-2} + 5\beta^{-3} \approx 0.2010$. The reader will see that Hollander's idea is similar to

0	5	5				
2	-6	0	-2			
	1	-3	0	-1		
		-1	3	0	1	
			-1	3	0	1
2	0	1	0	2	1	1

Table 2: Hollander's Carry Sequence

Gilbert's. In this case, we keep track of the sequence $2, 1, -1, -1$ instead of $2, 0, 1, 0, \ldots$. For example, by considering the second line to 4-th, the sequence $2, 1, -1$ (sign-changed) appears because

$$(-2)r_3 - r_2 + r_1 = -2\left(\frac{1}{\beta}\right) - \left(\frac{0}{\beta} + \frac{1}{\beta^2}\right) + \left(\frac{3}{\beta} + \frac{0}{\beta^2} + \frac{1}{\beta^3}\right) \in [0, 1)$$

gives the fractional part, the image by T. From the third to 5-th, we have

$$(-1)r_3 + r_2 + r_1 = -\left(\frac{1}{\beta}\right) + \left(\frac{0}{\beta} + \frac{1}{\beta^2}\right) + \left(\frac{3}{\beta} + \frac{0}{\beta^2} + \frac{1}{\beta^3}\right) \in [0, 1).$$

Now the relation to SRS is obvious. The key idea of Gilbert and Hollander is simply summarized:

Observe quotients instead of reminders.

As the algorithm of CNS and β-expansions are poles apart, it is surprising that both systems could be unified to the SRS setting.

5. Recent developments

For $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d$, let $\varrho(\mathbf{r})$ be the maximum absolute value of all roots of the polynomial $x^d + r_d x^{d-1} + \ldots + r_1$. Let

$$\begin{aligned}
\mathcal{D}_d^0 &= \left\{\mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0\right\} \\
\mathcal{D}_d &= \left\{\mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \text{ the sequence } \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k \geq 0} \text{ is ultimately periodic}\right\} \\
\mathcal{E}_d &= \{\mathbf{r} \in \mathbb{R}^d \,|\, \varrho(\mathbf{r}) < 1\}.
\end{aligned}$$

Thus, $\mathcal{D}_d^0$ is the set of all vectors $\mathbf{r} \in \mathbb{R}^d$, such that $(\mathbb{Z}, \tau_{\mathbf{r}})$ is a SRS. Note that $\mathcal{E}_d$ is a bounded subset of $\mathbb{R}^d$ which can be explicitly described by certain polynomial inequalities. Furthermore, $\mathcal{E}_d \subset \mathcal{D}_d \subset \overline{\mathcal{E}}_d$ holds (see [2]).

Let H be a compact subset of $\mathcal{E}_d$. In [2], an algorithm to determine $\mathcal{D}_d^0 \cap H$ was derived. The set $\mathcal{D}_d^0 \cap H$ can be constructed efficiently from H by cutting out finitely many convex polyhedra. However this algorithm does not work if H intersects with $\partial(E_d)$. The closer H is to $\partial(E_d)$, the harder becomes the computation. We have $\mathcal{E}_2 = \{(x,y) \in \mathbb{R}^2 \ : \ x < 1, |y| < x+1\}$. Figure 1. shows an approximation of $\mathcal{D}_2^0$. In [2, 3], several parts of the shaded (resp. white) areas are proved to be in (resp. not in) $\mathcal{D}_2^0$. For example,

$$\left\{(r_1, r_2) \in \mathbb{R}^2 \ \middle|\ -r_1 \le r_2 < r_1 + 1, \quad 0 \le r_1 \le \tfrac{2}{3}\right\} \subset \mathcal{D}_2^0,$$

which contains the point $(1/2, 1)$ for Knuth's number system. The characterization of $\mathcal{D}_2^0$ is a hard problem. The main difficulties occur near the line $L = \{(1,y) : -1 \le y \le 2\}$.

A *symmetric* version of SRS can be defined by shifting by 1/2:

$$-\frac{1}{2} \le r_1 z_1 + r_2 z_2 + \cdots + r_d z_d + z_{d+1} < \frac{1}{2}.$$

This change gives rise to other number systems. They are the symmetric canonical number systems and symmetric β-expansions. Imagine the ternary expression using digits $\{-1, 0, 1\}$ instead of $\{0, 1, 2\}$. Symmetric β-expansions of real numbers are generated by the transformation $x \to \beta x - \lfloor \beta x + \frac{1}{2} \rfloor$ on $[-\frac{1}{2}, \frac{1}{2})$. Let

$$\mathcal{D'}_d^0 = \left\{ \mathbf{r} \in \mathbb{R}^d \ \middle|\ \forall \mathbf{a} \in \mathbb{Z}^d \, \exists k > 0 : {\tau'}_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\}.$$

In [7], $\mathcal{D'}_2^0$ was completely characterized in this case since the regions near $\partial(E_2)$ were already cut out! A picture of $\mathcal{D'}_2^0$ is given in Figure 2. The outcome of this slight shift should be compared with Figure 1.

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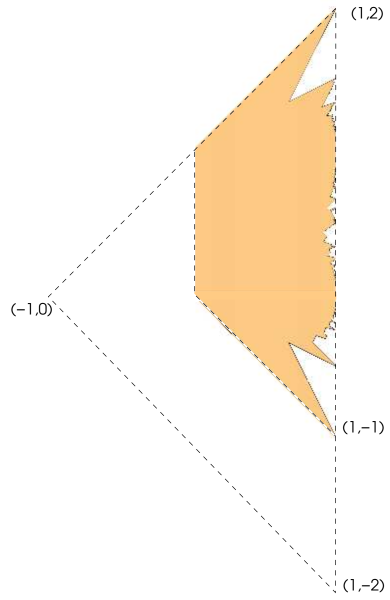

Figure 1: The sets $\mathcal{D}_2^0$ and $\mathcal{E}_2$.

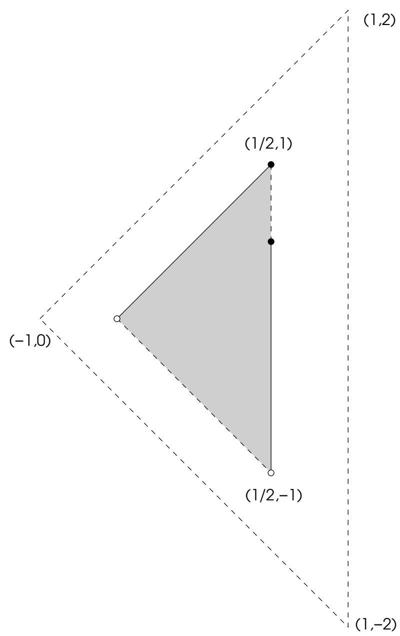

Figure 2: The sets $\mathcal{D}'^0_2$ and $\mathcal{E}_2$.

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