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Parallel *h-p* Spectral Element Methods for Elliptic Problems on Non-smooth Domains

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Abstract

We propose a new parallel *h-p* Spectral element method to solve elliptic boundary value problems with mixed Neumann and Dirichlet boundary conditions on non-smooth domains. The method is shown to be exponentially accurate and asymptotically faster than the standard *h-p* finite element method. We use the *auxiliary mapping* of the form of $z = \ln \xi$. The spectral element functions we use are fully non-conforming for pure Dirichlet problems. However, for mixed problems we need to make it conforming *only* at the vertices of the elements. The dimension of the resulting Schur complement matrix is quite small and this enables us to construct an accurate approximation of the Schur system. The method is a *least-squares collocation* method and the resulting *normal equations* are solved using *preconditioned conjugate gradient method*. We state the differentiability, stability and error estimates and discuss the numerical scheme and parallelization techniques. Using the stability estimates a parallel preconditioner with *optimal condition number (polylogarithmic)* is obtained. The algorithm is suitable for a distributed memory parallel computer. Load balancing issues are discussed and inter-processor communication involved is shown to be small. Finally, we provide the numerical results of a number of model problems which confirm the theoretical estimates and demonstrate the robustness of the method.

Key words: Corner singularities, geometric mesh, least-squares method, vertex based methods, Schur complement, preconditioners, polylogarithmic condition number, exponential accuracy.
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1 Introduction

Current formulations of spectral methods to solve elliptic problems in non-smooth domains allow us to recover only algebraic convergence [10]. The conformal mapping (also referred to as *auxiliary mapping*) $z = \xi^\alpha$ yields relatively fast convergence by smoothing out the singularity that occurs at the corner but exponential convergence can not be fully restored[10]. A method for obtaining a numerical solution to exponential accuracy for elliptic problems with analytic coefficients posed on curvilinear polygons whose boundary is piecewise analytic with mixed Neumann and Dirichlet boundary conditions, was first proposed by Babuska and Guo [1,2] within the framework of the finite element method. They were able to resolve the singularities which arise at the corners by using a *geometric mesh*.

We also use a geometric mesh to solve the same class of problems to exponential accuracy using *h-p* spectral element methods but with an important difference. To remove the singularity at the origin we use the auxiliary mapping of the form of $z = \ln \xi$ and this enables us to obtain the solution with exponential accuracy. The geometric mesh becomes geometrically fine in a neighborhood of each of the corners. In a neighborhood of the corner A we switch to new variables (τ, θ) where $\tau = \ln r$ and (r, θ) are the usual polar coordinates with origin at A. In doing so the geometric mesh is reduced to a *quasi-uniform* mesh in a sectoral neighborhood of the corners and so *Sobolev's embedding theorems* and the *trace theorems* for Sobolev spaces apply for functions defined on mesh elements in these new variables with a uniform constant. These new variables, which we shall refer to as *modified polar coordinates*, were first introduced by Kondratiev in his seminal paper [11]. Away from these sectoral neighborhoods of the corners we retain the Cartesian coordinate system (x, y).

We now seek a solution to elliptic boundary value problems as in [4] which minimizes the sum of the squares of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the sum of the squares of the jump in the function and its derivatives at inter-element boundaries in appropriate Sobolev norms to the functional being minimized.

To solve the minimization problem we need to solve the *normal equations* for the *least-squares* problem. To compute the residuals in the normal equations we do not need to compute *mass* and *stiffness* matrices. Moreover we do not need to filter the coefficients of the differential and boundary operators or the data. Of course, the evaluation of these residuals on each element requires the interchange of boundary values between neighboring elements.

For Dirichlet problems we choose *nonconforming* spectral element functions (SEF). We solve the normal equations by the *preconditioned conjugate gradient method*

(PCGM) using a *block diagonal preconditioner* which is nearly optimal as the *condition number* of the preconditioned system is *polylogarithmic* in N, the number of elements in the radial direction. This follows from the *stability theorem* proved in [7,14] and we can solve the normal equations to exponential accuracy in N using $O\left(N \ln N\right)$ iterations of the PCGM.

For problems with mixed Neumann and Dirichlet boundary conditions we choose the SEF to be continuous *only* at the vertices of the elements and nonconforming elsewhere. In [8,14] we proved a stability theorem for the quadratic form restricted to those SEF which vanish at the set of the *common boundary values* (CBV). The constant in this stability theorem is as good as that of the stability theorem for Dirichlet problems. We divide the vector composed of the values of the SEF at the quadrature points into two subvectors - one consisting of the common values of the SEF at the vertices of all the elements and the other consisting of the remaining values. The dimension of the first subvector, which can be thought of as a vector of CBV, is quite small as compared to the standard finite element methods. To solve the normal equations we first need to solve the system of equations corresponding to the Schur complement S of the first subvector of CBV. However, to compute the residual for S we need to be able to compute the *action* of the inverse of the matrix corresponding to the above restricted quadratic form in the right hand side of the stability theorem. We can precondition this matrix by a block diagonal matrix with the same matrix occurring repeatedly as its block diagonal elements. Thus we can compute the action of the inverse of the block diagonal matrix on a given vector to exponential accuracy in N using $O\left(N \ln N\right)$ iterations of the PCGM. Now since the dimension of S is small $(N_B \times N_B,$ where $N_B = O\left(N\right))$ we can compute an accurate approximation to S from it's definition and solve the normal equations corresponding to S using only $O\left(N^{3/2} \ln N\right)$ iterations of the PCGM.

We then make a correction to the approximate solution so that the solution is conforming and the error between the corrected and actual solution is exponentially small in N in the $H^1\left(\Omega\right)$ norm. The proposed method is shown to be *asymptotically* faster than the *h-p* finite element method. The method applies to elliptic problems with analytic coefficients with mixed Neumann and Dirichlet boundary conditions on curvilinear polygons whose sides and the data are piecewise analytic, provided that the *inf-sup* conditions are satisfied [8,14]. For numerical results in this paper, however, we restrict ourselves to the Poisson equation on a polygon with Dirichlet or mixed boundary conditions to keep our presentation simple.

The outline of this paper is as follows. In Section 2 we briefly discuss the differentiability estimates. Our main results, stability theorems for Dirichlet and mixed problems, are discussed in the Section 3. In Section 4 we present our numerical scheme and least-squares formulation. Section 5 briefly states the error estimate and discusses bilinear corrections so that the computed solution is conforming. In Section 6 we discuss the parallelization techniques for Dirichlet and mixed problems and give the computational complexities of the proposed scheme. This is followed

by the discussion on load balancing issues. In Section 7 we provide computational results and make concluding remarks in Section 8.

2 Differentiability estimates

In this paper, to keep the presentation simple, we shall examine the solution to the problem

$$\Delta u = f \qquad \text{for } (x,y) \in \Omega, \tag{1a}$$

with the boundary conditions

$$u = g_j \qquad \text{for } (x,y) \in \Gamma_j, \Gamma_j \subseteq \Gamma^{[0]}, \tag{1b}$$

$$\frac{\partial u}{\partial n} = g_j \qquad \text{for } (x,y) \in \Gamma_j, \Gamma_j \subseteq \Gamma^{[1]}. \tag{1c}$$

Let Ω be a polygonal domain with vertices $A_1, A_2 \cdots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$, where Γ_i joins the points A_{i-1} and A_i. In addition let the angle subtended at A_j be ω_j. Here $\Gamma^{[0]} \bigcup \Gamma^{[1]} = \partial\Omega$ and we shall assume that $\Gamma^{[0]} \neq \emptyset$. For the case when $\Gamma^{[0]} = \emptyset$ the solution is indeterminate upto a constant and so some additional condition has to be specified for a unique solution to exist. $\Gamma^{[0]}$ will be called the Dirichlet part of the boundary and $\Gamma^{[1]}$ the Neumann part of the boundary.

We assume that f is analytic on $\overline{\Omega}$ and g_j is analytic on every closed arc $\overline{\Gamma}_j$. Moreover if Γ_j and Γ_{j+1} are both contained in $\Gamma^{[0]}$ then $g_j(A_j) = g_{j+1}(A_j)$. Here $g_{p+1}(A_p) = g_1(A_p)$. Let $\mathcal{D} = \left\{ j : \Gamma_j \subseteq \Gamma^{[0]} \right\}$ and $\mathcal{N} = \left\{ j : \Gamma_j \subseteq \Gamma^{[1]} \right\}$.

Let (r_k, θ_k) denote polar coordinates with origin at the vertex A_k. Let $\tau_k = \ln r_k$. Then (τ_k, θ_k) are the modified polar coordinates we define in a neighborhood of A_k. These systems of local coordinates were first introduced by Kondratiev in [11]. Let $a_k = u(A_k)$. Now the image of the sector $S_k^\mu = \{(x,y) \in \Omega : r_k < \mu\}$, for μ small enough, is a semi infinite strip. Define

$$\widetilde{S}_k^\mu = \left\{ (\tau_k, \theta_k) : -\infty < \tau_k < \ln \mu_k,\ \psi_k^l < \theta_k < \psi_k^u \right\}.$$

Let

$$\|u(x,y)\|_{l,\Delta}^2 = \int_\Delta \sum_{\alpha_1+\alpha_2 \leq l} \left| \partial_x^{\alpha_1} \partial_y^{\alpha_2} u(x,y) \right|^2 dxdy.$$

By $H^l(\Delta)$ we denote the usual Sobolev space of order l with the norm $\|.\|_{l,\Delta}$ as defined above. Here Δ is a domain contained in $\mathbb{R}^2$.

Now it has been shown in [7,14] that

$$\|u(\tau_k, \theta_k) - a_k\|_{m,\widetilde{S}_k^\mu}^2 \leq \mu^{2(1-\beta_k)} \left(C d^{m-2} (m-2)! \right)^2.$$

Here $0 < \beta_k < 1$ and $\beta_k > 1 - \frac{\pi}{\omega_k}$ respectively, ($\beta_k > 1 - \frac{\pi}{2\omega_k}$ if Dirichlet and Neumann boundary conditions are imposed on the edges Γ_k, Γ_{k+1}, $\overline{\Gamma}_k \bigcap \overline{\Gamma}_{k+1} = A_k$) and C is a constant.

3 Stability estimates

We divide Ω into p subdomains $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_p$ where $\mathcal{S}_k$ denotes a domain which contains the vertex A_k and no other, and on each $\mathcal{S}_k$ we define the geometric mesh as has been done in [13]. Let $\mathcal{G}_k = \left\{\Omega_k^{i,j}, j = 1, \ldots, J_k, i = 1, \ldots I_k^j\right\}$ be a partition of $\mathcal{S}_k$ and let $\mathcal{G} = \bigcup_{k=1}^p \mathcal{G}_k$. We shall assume $I_k^j = I_k$ for $j \leq N+1$ and $I_k^j \leq I$, a constant, for all j and k. Let Ω_k be the sector of radius ρ with center at A_k, i.e. $\Omega_k =$

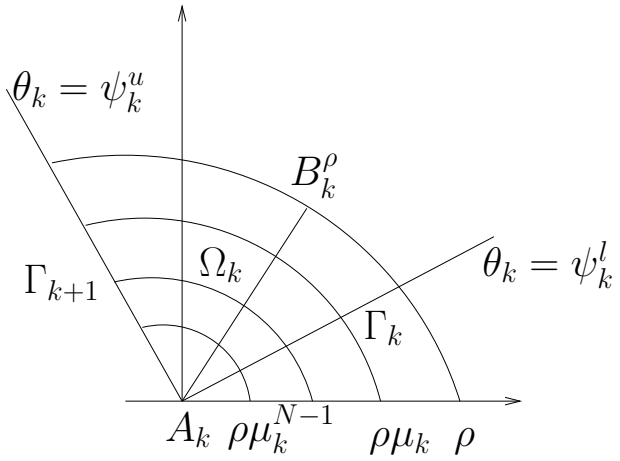

Fig. 1. Geometric mesh at a sector with vertex A_k

$\{(x,y) \in \Omega : r_k < \rho\}$. Let $\Omega_k^{i,j} = \left\{(x,y) : \sigma_k^j < r_k < \sigma_k^{j+1}, \psi_k^i < \theta_k < \psi_k^{i+1}\right\}$ for $1 \leq i \leq I_k$, $1 \leq j \leq N$. Here $\sigma_k^j = \rho\,(\mu_k)^{N+1-j}$ for $2 \leq j \leq N+1$ and $\sigma_k^1 = 0$. The geometric ratio μ_k of the mesh is given by $0 < \mu_k < 1$. Now $\widetilde{\Omega}_k^{i,j}$, the image of $\Omega_k^{i,j}$ in (τ_k, θ_k) coordinates, is given by

$$\widetilde{\Omega}_k^{i,j} = \left\{(\tau_k, \theta_k) : \eta_k^j < \tau_k < \eta_k^{j+1}, \psi_k^i < \theta_k < \psi_k^{i+1}\right\}.$$

Here $\eta_k^j = \ln \sigma_k^j = \ln \rho + (N+1-j) \ln \mu_k$ for $2 \leq j \leq N+1$. Moreover, $\eta_k^1 = -\infty$. Hence the geometric mesh $\left\{\Omega_k^{i,j}\right\}_{i,2 \leq j \leq N}$ becomes a quasi-uniform mesh in modified polar coordinates. $\widetilde{\Omega}_k^{i,1}$ is a semi-infinite strip, however. Now $J_k - N \leq M$, a constant for all k. Some of the elements $\Omega_k^{i,j}$ for $j > N+1$ may be triangles.

We shall represent $u_k^{i,1}$ in the form

$$u_k^{i,1} \equiv b_k, \qquad 1 \leq i \leq I_k,$$

where b_k is a constant. We shall represent the remaining $u_k^{i,j}$ as

$$u_k^{i,j}(\tau_k, \theta_k) = \sum_{m=1}^{N^j} \sum_{n=1}^{N^j} a_{m,n} \tau_k^m \theta_k^n, \qquad \text{for } 1 \le i \le I_k, 2 \le j \le N.$$

Here $1 \le N^j \le N$ and will be specified later.

Remark 1 *It is necessary to choose the circular arc $r_k = \rho$ as one of the boundaries of the geometric mesh. However, in the sector Ω_k we may impose any geometric mesh as in [1–3]. The mesh we have chosen though is the most simple and natural one.*

Let $\Omega_{p+1} = \Omega \setminus \left\{\bigcup_{k=1}^{p} \overline{\Omega}_k\right\}$. Then $\Omega_k^{i,j} \subseteq \Omega_{p+1}$ for $j > N$. We shall denote $\Omega_k^{i,j}$ for $j > N$ by Ω_{p+1}^l where $1 \le l \le \mathcal{M}$. Here $\mathcal{M}$ denotes the cardinality of the set $\left\{\Omega_k^{i,j}\right\}_{i,j>N,k}$. To proceed further we need to specify the form of u_{p+1}^l defined on Ω_{p+1}^l. Now u_{p+1}^l corresponds to $u_k^{i,j}$, for some $j > N$. Hence there is a mapping M_{p+1}^l, also denoted as $M_k^{i,j}$, which maps the master square $\mathcal{Q} = (-1,1) \times (-1,1)$ onto Ω_{p+1}^l. We let

$$u\left(X_{p+1}^l(\xi,\eta), Y_{p+1}^l(\xi,\eta)\right) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_{m,n} \xi^m \eta^n.$$

In case Ω_{p+1}^l is triangular M_{p+1}^l maps the master triangle $\mathcal{T}$ to Ω_{p+1}^l [1–3].

Now let

$$\|w\|_{s,I}^2 = \int_I w^2(x)\,dx + \int_I \int_I \frac{|w(x) - w(x')|^2}{|x - x'|^{1+2s}} dx dx'$$

denote the fractional Sobolev norm of order s, where $0 < s < 1$. Here I denotes an interval contained in $\mathbb{R}$. Before we state the stability theorem for the polygon we shall state a version of it for the sector Ω_k which has straight lines Γ_k and Γ_{k+1} for sides and is bounded by the circular arc B_k^ρ with center at A_k and radius ρ. Let $\widetilde{\gamma}$ denote a side of one of the elements $\widetilde{\Omega}_k^{i,j}$.

Theorem 2 *For the sectoral domain Ω_k the following stability estimate holds:*

$$\begin{aligned}
&\sum_{i=1}^{I_k}\left|u_k^{i,1}\right|^2+\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|u_k^{i,j}\left(\tau_k,\theta_k\right)\right\|_{2,\widetilde{\Omega}_k^{i,j}}^2 \qquad (2)\\
\leq & C\left(\ln N\right)^2\Bigg(\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta u_k^{i,j}\left(\tau_k,\theta_k\right)\right\|_{0,\widetilde{\Omega}_k^{i,j}}^2\\
&+\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty\\ \widetilde{\gamma}\subseteq\widetilde{\Omega}_k}}\left(\left\|\left[u_k\right]\right\|_{0,\widetilde{\gamma}}^2+\left\|\left[\left(u_k\right)_{\tau_k}\right]\right\|_{1/2,\widetilde{\gamma}}^2+\left\|\left[\left(u_k\right)_{\theta_k}\right]\right\|_{1/2,\widetilde{\gamma}}^2\right)\\
&+\sum_{\widetilde{\gamma}\subseteq\widetilde{B}_k^{\rho}}\left(\left\|\left(u_k\right)\right\|_{0,\widetilde{\gamma}}^2+\left\|\left(u_k\right)_{\theta_k}\right\|_{1/2,\widetilde{\gamma}}^2\right)\\
&+\sum_{m=k}^{k+1}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty\\ \widetilde{\gamma}\subseteq\widetilde{\Gamma}_m}}\left(\left\|\left(u_k\right)\right\|_{0,\widetilde{\gamma}}^2+\left\|\left(u_k\right)_{\tau_k}\right\|_{1/2,\widetilde{\gamma}}^2\right)\Bigg).
\end{aligned}$$

Here $[.]\,|_{\widetilde{\gamma}}$ denotes the jump in w across $\widetilde{\gamma}$, $\kappa\left(\widetilde{\gamma}\right)$ denotes the length of the curve $\widetilde{\gamma}$ and C is a constant. ■

The above stability theorem is similar to the *shift theorems* for elliptic problems with Dirichlet boundary conditions. However, the factor multiplying the right hand side is not a constant but grows slowly with N like $\left(\ln N\right)^2$.

Before we state the stability theorem for the polygon we need to introduce some notation. Consider the domain Ω_{p+1}^l. Now

$$\int_{\Omega_{p+1}^l}\int\left|\Delta u_{p+1}^l\left(x,y\right)\right|^2dxdy=\int_{\mathcal{Q}}\int\left|L_{p+1}^l u_{p+1}^l\left(\xi,\eta\right)\right|^2d\xi d\eta,$$

where

$$L_{p+1}^l w=A_{p+1}^l w_{\xi\xi}+2B_{p+1}^l w_{\xi\eta}+C_{p+1}^l w_{\eta\eta}+D_{p+1}^l w_{\xi}+E_{p+1}^l w_{\eta}+F_{p+1}^l w. \quad (3)$$

Here $\mathcal{Q}$ denotes the master square. We then define

$$\widehat{L_{p+1}^l}\,w=\widehat{A}_{p+1}^l w_{\xi\xi}+2\widehat{B}_{p+1}^l w_{\xi\eta}+\widehat{C}_{p+1}^l w_{\eta\eta}+\widehat{D}_{p+1}^l w_{\xi}+\widehat{E}_{p+1}^l w_{\eta}+\widehat{F}_{p+1}^l w, \quad (4)$$

where the differential operator $\widehat{L_{p+1}^l}$ is obtained from the differential operator L_{p+1}^l (with analytic coefficients) by choosing the coefficients $\widehat{A}_{p+1}^l,\ldots,\widehat{F}_{p+1}^l$ to be the orthogonal projections of $A_{p+1}^l,\ldots,F_{p+1}^l$ into the space of polynomials of degree $(N-1)$ in ξ and η with respect to the usual inner product in $H^2\left(\mathcal{Q}\right)$. Now let γ be a side of the element Ω_{p+1}^l and let it be the image of $\xi=-1$ under the mapping

M_{p+1}^l. Clearly,

$$\frac{\partial u_{p+1}^l}{\partial x} = \left(u_{p+1}^l\right)_\xi \xi_x + \left(u_{p+1}^l\right)_\eta \eta_x.$$

We now define

$$\left.\widehat{\frac{\partial u_{p+1}^l}{\partial x}}\right|_\gamma = \left(\left(u_{p+1}^l\right)_\xi \widehat{\xi}_x + \left(u_{p+1}^l\right)_\eta \widehat{\eta}_x\right)(-1,\eta).$$

Here $\widehat{\xi}_x(\xi,\eta)$ and $\widehat{\eta}_x(\xi,\eta)$ are the unique polynomials which are the orthogonal projections of $\xi_x(\xi,\eta)$ and $\eta_x(\xi,\eta)$ into the space of polynomials of degree $(N-1)$ in ξ and η with respect to the usual inner product in $H^2(\mathcal{Q})$. In the same way we can define $\widehat{\dfrac{\partial u_{p+1}^l}{\partial y}}$. Now let γ be a side common to Ω_{p+1}^l and Ω_{p+1}^m and let it be the image of $\xi=-1$ under the mapping M_{p+1}^l and the image of $\xi=1$ under the mapping M_{p+1}^m. Then we define

$$\left\|\left[\widehat{\frac{\partial u}{\partial x}}\right]\right\|_{1/2,\gamma}^2 = \left\|\widehat{\frac{\partial u_{p+1}^l}{\partial x}}(-1,\eta) - \widehat{\frac{\partial u_{p+1}^m}{\partial x}}(1,\eta)\right\|_{1/2,(-1,1)}^2,$$

$$\left\|\left[\widehat{\frac{\partial u}{\partial y}}\right]\right\|_{1/2,\gamma}^2 = \left\|\widehat{\frac{\partial u_{p+1}^l}{\partial y}}(-1,\eta) - \widehat{\frac{\partial u_{p+1}^m}{\partial y}}(1,\eta)\right\|_{1/2,(-1,1)}^2.$$

Finally, consider a side Γ_k of the polygon Ω as shown in Figure 2. Then we denote

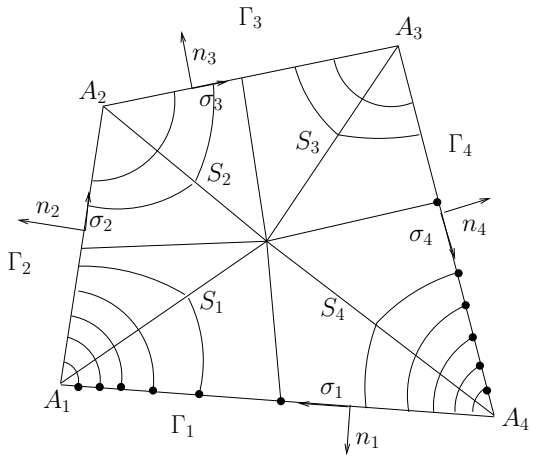

Fig. 2. Polygon with mesh at sectors

the unit tangent vector to Γ_k by σ_k and the unit normal vector by n_k. Let γ be a side of Ω_{p+1}^l such that $\gamma \subseteq \Gamma_k$ and γ is the image of $\xi=-1$ under the mapping M_{p+1}^l.

Then we can define $\widetilde{\frac{\partial u_{p+1}^l}{\partial \sigma_k}}$ and $\widetilde{\frac{\partial u_{p+1}^l}{\partial n_k}}$ in the same way. Finally, we define

$$\left\| \widetilde{\frac{\partial u_{p+1}^l}{\partial \sigma_k}} \right\|_{1/2,\gamma}^2 = \left\| \widetilde{\frac{\partial u_{p+1}^l}{\partial \sigma_k}} \right\|_{1/2,(-1,1)}^2 .$$

We remark that though we filter the coefficients of the boundary and differential operators, and the stability theorems are stated in terms of these filtered operators we use the unfiltered coefficients in our computations as will be shown in the next section. We shall let $\mathcal{E}$ denote either $\mathcal{Q}$, the master square, or $\mathcal{T}$, the master triangle.

Let

$$\mathcal{F}_u = \left(\left\{ u_k^{i,j}\left(\tau_k, \theta_k\right) \right\}_{i,j,k}, \left\{ u_k^{i,j}\left(\xi, \eta\right) \right\}_{i,j,k} \right) \tag{5}$$

denote the representation of the SEF in the polygon. We now define the quadratic forms

$$\begin{aligned} \mathcal{V}_{interior}^N\left(\mathcal{F}_u\right) = & \sum_{l=1}^{\mathcal{M}} \left\| \widetilde{L_{p+1}^l}\, u_{p+1}^l\left(\xi,\eta\right) \right\|_{0,\mathcal{E}}^2 \\ & + \sum_{\gamma \subseteq \Omega_{p+1}} \left(\left\| \left[u_{p+1} \right] \right\|_{0,\gamma}^2 + \left\| \widetilde{\left[\frac{\partial u_{p+1}}{\partial x} \right]} \right\|_{1/2,\gamma}^2 + \left\| \widetilde{\left[\frac{\partial u_{p+1}}{\partial y} \right]} \right\|_{1/2,\gamma}^2 \right) \\ & + \sum_{k \in \mathcal{D}} \sum_{\gamma \subseteq \partial\Omega_{p+1} \bigcap \Gamma_k} \left(\left\| u_{p+1} \right\|_{0,\gamma}^2 + \left\| \widetilde{\frac{\partial u_{p+1}}{\partial \sigma_k}} \right\|_{1/2,\gamma}^2 \right) \\ & + \sum_{k \in \mathcal{N}} \sum_{\gamma \subseteq \partial\Omega_{p+1} \bigcap \Gamma_k} \left(\left\| \widetilde{\frac{\partial u_{p+1}}{\partial n_k}} \right\|_{1/2,\gamma}^2 \right), \text{ and} \end{aligned} \tag{6}$$

$$\mathcal{V}^N_{vertices}\left(\mathcal{F}_u\right) = \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta u_k^{i,j}\left(\tau_k,\theta_k\right)\right\|^2_{0,\widetilde{\Omega}_k^{i,j}}$$
$$+\sum_{k=1}^{p}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty\\ \widetilde{\gamma}\subseteq\widetilde{\Omega}_k}}\left(\left\|[u_k]\right\|^2_{0,\widetilde{\gamma}}+\left\|\left[\left(u_k\right)_{\tau_k}\right]\right\|^2_{1/2,\widetilde{\gamma}}+\left\|\left[\left(u_k\right)_{\theta_k}\right]\right\|^2_{1/2,\widetilde{\gamma}}\right)$$
$$+\sum_{m\in\mathcal{D}}\sum_{k=m-1}^{m}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty\\ \widetilde{\gamma}\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}_k}}\left(\left\|u_k\right\|^2_{0,\widetilde{\gamma}}+\left\|\left(u_k\right)_{\tau_k}\right\|^2_{1/2,\widetilde{\gamma}}\right)$$
$$+\sum_{m\in\mathcal{N}}\sum_{k=m-1}^{m}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty\\ \widetilde{\gamma}\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}_k}}\left(\left\|\left(u_k\right)_{\theta_k}\right\|^2_{1/2,\widetilde{\gamma}}\right)$$
$$+\sum_{k=1}^{p}\sum_{\widetilde{\gamma}\subseteq\widetilde{B}_k^{\rho}}\left(\left\|[u_k]\right\|^2_{0,\widetilde{\gamma}}+\left\|\left[\widehat{\frac{\partial u_k}{\partial\tau_k}}\right]\right\|^2_{1/2,\widetilde{\gamma}}+\left\|\left[\widehat{\frac{\partial u_k}{\partial\theta_k}}\right]\right\|^2_{1/2,\widetilde{\gamma}}\right).$$

Here $\kappa\left(\widetilde{\gamma}\right)$ denotes the length of the curve $\widetilde{\gamma}$. We next define the quadratic form $\mathcal{V}^N\left(\mathcal{F}_u\right)=\mathcal{V}^N_{vertices}\left(\mathcal{F}_u\right)+\mathcal{V}^N_{interior}\left(\mathcal{F}_u\right)$.

Now let $\mathcal{P}^N$ denote the space of SEF $\mathcal{F}_v$. Let $\mathcal{P}^N_V$ denote the subspace of $\mathcal{P}^N$ consisting of SEF which are continuous at the vertices of their elements. Finally, let $\mathcal{P}^N_0$ denote the subspace of $\mathcal{P}^N_V$ consisting of SEF which are zero at the vertices of their elements. We can now state the stability theorems which we shall use in this paper [7,8].

Theorem 3 *Consider the case when the boundary conditions are of Dirichlet type. Then for N large enough there exists a constant C such that*

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|v_k^{i,1}\right|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|v_k^{i,j}\left(\tau_k,\theta_k\right)\right\|^2_{2,\widetilde{\Omega}_k^{i,j}}+\sum_{l=1}^{\mathcal{M}}\left\|v^l_{p+1}\left(\xi,\eta\right)\right\|^2_{2,\mathcal{E}} \qquad (7)$$
$$\leq C\left(\ln N\right)^2\mathcal{V}^N\left(\mathcal{F}_v\right)$$

for all $\mathcal{F}_v\in\mathcal{P}^N$.■

For problems with mixed Neumann and Dirichlet boundary conditions the factor multiplying the right hand side may not grow slowly like $\left(\ln N\right)^2$, but can grow like N^4.

Theorem 4 *Consider the case when the boundary conditions are of mixed Neumann and Dirichlet type. Then for N large enough there exists a constant C such*

that

$$\begin{aligned}&\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|v_k^{i,1}\right|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|v_k^{i,j}\left(\tau_k,\theta_k\right)\right\|_{2,\widetilde{\Omega}_k^{i,j}}^2+\sum_{l=1}^{\mathcal{M}}\left\|v_{p+1}^{l}\left(\xi,\eta\right)\right\|_{2,\mathcal{E}}^2 \qquad (8)\\ \leq& CN^4\mathcal{V}^N\left(\mathcal{F}_v\right)\end{aligned}$$

for all $\mathcal{F}_v \in \mathcal{P}^N$.■

The rapid growth of the factor multiplying the right hand side can create difficulties in devising an efficient parallel numerical scheme. This can be overcome by restricting $\mathcal{F}_v$ to belong to the subspace $\mathcal{P}_V^N$. The values of the SEF at the vertices of their elements become a set of CBV. It should be noted that the set of CBV we have defined has a much smaller cardinality than the set of CBV for the *h-p* finite element method where the finite element functions have to be continuous along the sides of their elements.

We now state a stability theorem when the boundary conditions are of mixed type and the SEF vanish at the vertices of their elements.

Theorem 5 *Consider the case when the boundary conditions are of mixed Neumann and Dirichlet type. Then for* N *large enough there exists a constant* C *such that*

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|v_k^{i,j}\left(\tau_k,\theta_k\right)\right\|_{2,\widetilde{\Omega}_k^{i,j}}^2+\sum_{l=1}^{\mathcal{M}}\left\|v_{p+1}^{l}\left(\xi,\eta\right)\right\|_{2,\mathcal{E}}^2 \leq C\left(\ln N\right)^2\mathcal{V}^N\left(\mathcal{F}_v\right) \qquad (9)$$

for all $\mathcal{F}_v \in \mathcal{P}_0^N$.■

Remark 6 *For Theorem 5 to hold we need to make the values of the spectral element functions continuous only at those vertices which lie on the boundary* $\partial\Omega$ *of* Ω*. Further, if Dirichlet boundary conditions are imposed on one of the sides* Γ_k *or* Γ_{k+1} *of the domain* $\mathcal{S}_k$ *then we do not need to make the SEF continuous at the vertices lying on* $\Gamma_k \bigcap \overline{\mathcal{S}}_k$ *or* $\Gamma_{k+1} \bigcap \overline{\mathcal{S}}_k$ *either. It is only if Neumann boundary conditions are imposed on both* Γ_k *and* Γ_{k+1} *that we need to make the SEF continuous at the vertices lying on one of the sides, say* $\Gamma_k \bigcap \mathcal{S}_k$*. Thus for Dirichlet boundary conditions we do not need to make the SEF continuous at any of the vertices of the elements for Theorem 5 to hold and this is the content of Theorem 3. As an example, consider the domain* Ω *shown in Figure 2 and suppose we impose Neumann boundary conditions on* Γ_1*,* Γ_2 *and* Γ_4 *and Dirichlet boundary conditions on* Γ_3*. Then we need to make the SEF continuous only on the vertices on the boundary which are marked by dots for Theorem 5 to hold. Hence for mixed problems we need to make the SEF continuous only on a subset of the vertices lying on the boundary of* Ω*.*

4 The Numerical scheme and symmetric formulation

We can now formulate the numerical scheme. Let

$$f_{p+1}^{l}(\xi,\eta) = f\left(X_{p+1}^{l}(\xi,\eta), Y_{p+1}^{l}(\xi,\eta)\right)$$

for $1 \leq l \leq \mathcal{M}$ and let $J_{p+1}^{l}(\xi,\eta)$ denote the Jacobian of the mapping M_{p+1}^{l}. Define $F_{p+1}^{l}(\xi,\eta) = f_{p+1}^{l}(\xi,\eta)\sqrt{J_{p+1}^{l}(\xi,\eta)}$ and let $\widehat{F}_{p+1}^{l}(\xi,\eta)$ denote the unique polynomial which is the orthogonal projection of $F_{p+1}^{l}(\xi,\eta)$ into the space of polynomials of degree $(2N-1)$ in ξ and η with respect to the usual inner product in $H^2(\mathcal{E})$.

Next, let the vertex $A_k = (x_k, y_k)$. Let

$$F_k^{i,j}(\tau_k,\theta_k) = e^{2\tau_k} f\left(x_k + e^{\tau_k}\cos\theta_k, y_k + e^{\tau_k}\sin\theta_k\right),$$

for $1 \leq k \leq p$, $2 \leq j \leq N$, $1 \leq i \leq I_k$. We let $\widehat{F}_k^{i,j}(\tau_k,\theta_k)$ be the unique polynomial which is the orthogonal projection of $F_k^{i,j}(\tau_k,\theta_k)$ into the space of polynomials of degree $(2N^j-1)$ in τ_k and θ_k with respect to the usual inner product in $H^2\left(\widetilde{\Omega}_k^{i,j}\right)$. Now consider the boundary conditions $u = g_k$ on Γ_k for $k \in \mathcal{D}$, and $\frac{\partial u}{\partial n} = g_k$ on Γ_k for $k \in \mathcal{N}$. Let

$$l_k^1(\tau_k) = \begin{cases} u & = g_k\left(x_k + e^{\tau_k}\cos\left(\psi_k^l\right), y_k + e^{\tau_k}\sin\left(\psi_k^l\right)\right) \text{ for } k \in \mathcal{D} \\ \frac{\partial u}{\partial \theta_k} & = e^{\tau_k} g_k\left(x_k + e^{\tau_k}\cos\left(\psi_k^l\right), y_k + e^{\tau_k}\sin\left(\psi_k^l\right)\right) \text{ for } k \in \mathcal{N} \end{cases}.$$

We define $\widehat{l}_k^{1,j}(\tau_k)$ to be the unique polynomial which is the orthogonal projection of $l_k^1(\tau_k)$ into the space of polynomials of degree $2N^j - 1$ in τ_k with respect to the usual norm $H^2\left(\eta_k^j, \eta_k^{j+1}\right)$ for $2 \leq j \leq N$. Let

$$l_k^2(\tau_{k-1}) = \begin{cases} u = & g_k\left(x_{k-1} + e^{\tau_{k-1}}\cos\left(\psi_{k-1}^u\right), y_{k-1} + e^{\tau_{k-1}}\sin\left(\psi_{k-1}^u\right)\right) \\ & \text{for } k \in \mathcal{D} \\ \frac{\partial u}{\partial \theta_k} = & e^{\tau_{k-1}} g_k\left(x_{k-1} + e^{\tau_{k-1}}\cos\left(\psi_{k-1}^u\right), y_{k-1} + e^{\tau_{k-1}}\sin\left(\psi_{k-1}^u\right)\right) \\ & \text{for } k \in \mathcal{N} \end{cases}.$$

We define $\widehat{l}_k^{2,j}(\tau_{k-1})$ to be the unique polynomial which is the orthogonal projection of $l_k^2(\tau_{k-1})$ into the space of polynomials of degree $2N^j - 1$ in τ_{k-1} with respect to the usual norm $H^2\left(\eta_{k-1}^j, \eta_{k-1}^{j+1}\right)$ for $2 \leq j \leq N$. Finally, let $\Gamma_k \bigcap \partial\Omega_{p+1}^j = C_k^j$ be the image of the mapping M_{p+1}^j corresponding to $\xi = -1$. Let $l_k^j(\eta) = g_k\left(X_{p+1}^j(0,\eta), Y_{p+1}^j(0,\eta)\right)$ where $-1 \leq \eta \leq 1$. We shall let $\widehat{l}_k^j(\eta)$ denote the unique polynomial which is the orthogonal projection of l_k^j into the space of polynomials of degree $2N-1$ in η with respect to the usual norm in $H^2(-1,1)$.

We now define the functional $\mathfrak{r}^N(\mathcal{F}_v)$ as follows:

Let

$$\begin{aligned}
\mathfrak{r}^N_{vertices}\left(\mathcal{F}_v\right) = & \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta v_k^{i,j}\left(\tau_k,\theta_k\right)-\widehat{F}_k^{i,j}\left(\tau_k,\theta_k\right)\right\|^2_{0,\widetilde{\Omega}_k^{i,j}} \\
& +\sum_{k=1}^{p}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty \\ \widetilde{\gamma}\subseteq\widetilde{\Omega}_k}}\left(\left\|[v_k]\right\|^2_{0,\widetilde{\gamma}}+\left\|\left[\left(v_k\right)_{\tau_k}\right]\right\|^2_{1/2,\widetilde{\gamma}}+\left\|\left[\left(v_k\right)_{\theta_k}\right]\right\|^2_{1/2,\widetilde{\gamma}}\right) \\
& +\sum_{m\in\mathcal{D}}\sum_{k=m-1}^{m}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty \\ \widetilde{\gamma}\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}_k}}\left(\left\|v_k-\widehat{l}_m^{m-k+1}\right\|^2_{0,\widetilde{\gamma}}+\left\|v_k-\widehat{l}_m^{m-k+1}\right\|^2_{1/2,\widetilde{\gamma}}\right) \\
& +\sum_{m\in\mathcal{N}}\sum_{k=m-1}^{m}\sum_{\substack{\kappa\left(\widetilde{\gamma}\right)<\infty \\ \widetilde{\gamma}\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}_k}}\left(\left\|\left(v_k\right)_{\theta_k}-\widehat{l}_m^{m-k+1}\right\|^2_{1/2,\widetilde{\gamma}}\right) \\
& +\sum_{k=1}^{p}\sum_{\widetilde{\gamma}\subseteq\widetilde{B}_k^{\rho}}\left(\left\|[v_k]\right\|^2_{0,\widetilde{\gamma}}+\left\|\left[\widetilde{\frac{\partial v_k}{\partial\tau_k}}\right]\right\|^2_{1/2,\widetilde{\gamma}}+\left\|\left[\widetilde{\frac{\partial v_k}{\partial\theta_k}}\right]\right\|^2_{1/2,\widetilde{\gamma}}\right), \text{ and}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{r}^N_{interior}\left(\mathcal{F}_v\right) = & \sum_{l=1}^{\mathcal{M}}\left\|\widetilde{L^l_{p+1}}\,v^l_{p+1}\left(\xi,\eta\right)-\widehat{F}^l_{p+1}\left(\xi,\eta\right)\right\|^2_{0,\mathcal{E}} \\
& +\sum_{\gamma\subseteq\Omega_{p+1}}\left(\left\|[v_{p+1}]\right\|^2_{0,\gamma}+\left\|\left[\widetilde{\frac{\partial v_{p+1}}{\partial x}}\right]\right\|^2_{1/2,\gamma}+\left\|\left[\widetilde{\frac{\partial v_{p+1}}{\partial y}}\right]\right\|^2_{1/2,\gamma}\right) \\
& +\sum_{k\in\mathcal{D}}\sum_{\gamma\subseteq\partial\Omega_{p+1}\bigcap\Gamma_k}\left(\left\|v_{p+1}-\widehat{l}_k\right\|^2_{0,\gamma}+\left\|\widetilde{\frac{\partial v_{p+1}}{\partial\sigma_k}}-\widetilde{\frac{\partial\widehat{l}_k}{\partial\sigma_k}}\right\|^2_{1/2,\gamma}\right) \\
& +\sum_{k\in\mathcal{N}}\sum_{\gamma\subseteq\partial\Omega_{p+1}\bigcap\Gamma_k}\left(\left\|\widetilde{\frac{\partial v_{p+1}}{\partial n_k}}-\widehat{l}_k\right\|^2_{1/2,\gamma}\right).
\end{aligned}$$

We now define $\mathfrak{r}^N\left(\mathcal{F}_v\right)=\mathfrak{r}^N_{vertices}\left(\mathcal{F}_v\right)+\mathfrak{r}^N_{interior}\left(\mathcal{F}_v\right)$. The functional $\mathfrak{r}^N\left(\mathcal{F}_v\right)$ is closely related to the quadratic form $\mathcal{V}^N\left(\mathcal{F}_u\right)$ as defined after (6). For the Dirichlet problem we choose our approximate solution as the unique $\mathcal{F}_w\in\mathcal{P}^N$ which minimizes $\mathfrak{r}^N$ over all $\mathcal{F}_v\in\mathcal{P}^N$. For problems with mixed boundary conditions we choose our approximate solution as the unique $\mathcal{F}_w\in\mathcal{P}^N_V$ which minimizes $\mathfrak{r}^N$ over all $\mathcal{F}_v\in\mathcal{P}^N_V$.

In minimizing the functional $\mathfrak{r}^N\left(\mathcal{F}_v\right)$ we seek a solution which minimizes the sum of the squares of weighted L^2 norms of the residuals in the partial differential equa-

tion and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the sum of the squares of the jumps in the function and its derivatives at inter-element boundaries, in appropriate Sobolev norms.

The above method is a *least-squares* method and we can obtain a solution by using the PCGM for solving the normal equations. To be able to do so we must be able to compute the residuals in the normal equations inexpensively and efficiently. Now

$$\mathfrak{r}^N\left(U+\epsilon V\right)=\mathfrak{r}^N\left(U\right)+2\epsilon V^t\left(SU-TG\right)+O\left(\epsilon^2\right)$$

for all V, where U is a vector assembled from the values of

$$\left\{\left\{u_k^{i,j}\left(\tau_k^{j,l},\theta_k^{i,m}\right)\right\}_{2\leq j\leq N,1\leq i\leq I_k}^{0\leq l,m\leq N^j},\left\{u_k^{i,j}\left(\xi^l,\eta^l\right)\right\}_{N<j\leq J_k,1\leq i\leq I_k^j}^{1\leq l\leq(N+1)^2}\right\}_{1\leq k\leq p}$$

and G is a similar vector assembled from the data. We now show how to compute the integrals arising in the above formulation.

4.1 Integrals on the element domain

We first show how to compute the integrals on the element domain

$$\int_{\mathcal{Q}}\int\widehat{L_k^{i,j}}\,v_k^{i,j}\left(\widehat{L_k^{i,j}}\,u_k^{i,j}-\widehat{g}_k^{i,j}dsdt\right) \tag{10}$$

for some i,j,k. Here $\widehat{L_k^{i,j}}$ is defined in a similar way as $\widehat{L_{p+1}^l}$ in (4) and $\widehat{g}$ is a filtered representation of g. Moreover,

$$v_k^{i,j}\left(s,t\right)=\sum_{l=0}^{N^j}\sum_{m=0}^{N^j}a_{m,l}s^mt^l,$$
$$u_k^{i,j}\left(s,t\right)=\sum_{l=0}^{N^j}\sum_{m=0}^{N^j}b_{m,l}s^mt^l.$$

Let $\left(L_k^{i,j}\right)^t$ denote the formal adjoint of the differential operator $\widehat{L_k^{i,j}}$. Then

$$\left(L_k^{i,j}\right)^t w=\left(\widehat{A}_k^{i,j}w\right)_{ss}+\left(\widehat{B}_k^{i,j}w\right)_{st}+\left(\widehat{C}_k^{i,j}w\right)_{tt}-\left(\widehat{D}_k^{i,j}w\right)_s-\left(\widehat{E}_k^{i,j}w\right)_t+\widehat{F}_k^{i,j}w.$$

Now all the integrands are polynomials of degree $4N-2$ and so the integral may be exactly evaluated by the Legendre-Gauss-Lobatto (LGL) quadrature formula with $2N+1$ points. Let $t_0^{2N},\ldots t_{2N}^{2N}$ and $s_0^{2N},\ldots s_{2N}^{2N}$ represent the $(2N+1)$ quadrature points in each direction and $w_0^{2N},\ldots w_{2N}^{2N}$ the corresponding weights.

Proposition 7 *Let the matrix $D^{2N} = d_{i,j}^{2N}$ denotes the differentiation matrix. If l is a polynomial of degree less than or equal to $2N$, then*

$$\frac{dl}{dt}\left(t_i^{2N}\right) = \sum_{j=0}^{2N} d_{i,j}^{2N} l\left(t_j^{2N}\right). \tag{11}$$

Let $z_k^{i,j} = L_k^{i,j} u_k^{i,j} - g_k^{i,j}$. Integrating (10) by parts and using the LGL quadrature rule we obtain

$$\begin{aligned}
&\int_{\mathcal{Q}} \int \widehat{L_k^{i,j}}\, v_k^{i,j} \left(\widehat{L_k^{i,j}}\, u_k^{i,j} - \widehat{g}_k^{i,j} ds dt \right) \\
=&\sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N}, w_j^{2N} L^t z\left(s_i^{2N}, t_j^{2N}\right)\right) \\
&+\sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_j^{2N} D_{2N,i}^{2N} A\left(1, t_j^{2N}\right) z\left(1, t_j^{2N}\right)\right) \\
&+\sum_{j=0}^{2N} v\left(1, t_j^{2N}\right) w_j^{2N}\left((Dz - (Bz)_t - (Az)_s)\left(1, t_j^{2N}\right)\right) \\
&-\sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_j^{2N} D_{0,i}^{2N} A\left(-1, t_j^{2N}\right) z\left(-1, t_j^{2N}\right)\right) \\
&-\sum_{j=0}^{2N} v\left(-1, t_j^{2N}\right) w_j^{2N}\left((Dz - (Bz)_t - (Az)_s)\left(-1, t_j^{2N}\right)\right) \\
&+\sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N} D_{2N,j}^{2N} C\left(s_i^{2N}, 1\right) z\left(s_i^{2N}, 1\right)\right) \\
&+\sum_{i=0}^{2N} v\left(s_i^{2N}, 1\right) w_i^{2N}\left((Ez - (Bz)_s - (Cz)_t)\left(s_i^{2N}, 1\right)\right) \\
&-\sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N} D_{0,j}^{2N} C\left(s_i^{2N}, -1\right) z\left(s_i^{2N}, -1\right)\right) \\
&-\sum_{i=0}^{2N} v\left(s_i^{2N}, -1\right) w_i^{2N}\left((Ez - (Bz)_s - (Cz)_t)\left(s_i^{2N}, -1\right)\right) \\
&+v(-1,-1)\, Bz(-1,-1) - v(1,-1)\, Bz(1,-1) \\
&-v(-1,1)\, Bz(-1,1) + v(1,1)\, Bz(1,1).
\end{aligned}$$

Note here that we have used unfiltered coefficients in the right hand side and represented $v_{i,j}^k$, $z_{i,j}^k$ by v and z etc.

Remark 8 *Of course, in writing the above we commit an error. It can be argued as in [5] that this error is spectrally small. In fact if we assume that the boundary of the domain Ω is composed of analytic curves, the coefficients of the differential*

operator are analytic and the data is analytic then the error committed is exponentially small in N. Hence, there is never any need to filter the coefficients of the differential and boundary operators or the data in any of our computations.

We now rewrite the $u\left(s_i^N, t_j^N\right)$ by arranging them in lexicographic order and denote

$$U^N_{(N+1)i+j+1} = u\left(s_i^N, t_j^N\right) \text{ for } 0 \leq i \leq N, 0 \leq j \leq N.$$

Similarly $U^{2N}_{(2N+1)i+j+1}$ and $Z^{2N}_{(2N+1)i+j+1} = (Lu - g)\left(s_i^{2N}, t_j^{2N}\right)$ for $0 \leq i \leq 2N$, $0 \leq j \leq 2N$ are arranged in lexicographic order. Then we may write

$$\int_{\mathcal{Q}} \int \widehat{L_k^{i,j}}\, v_k^{i,j} \left(\widehat{L_k^{i,j}}\, u_k^{i,j} - \widehat{g}_k^{i,j} ds dt\right) = \left(V^{2N}\right)^t R Z^{2N},$$

where R is matrix such that RZ^{2N} is easily computed.

4.2 Integrals on the boundary of the elements

We now show how to evaluate the boundary terms. For this we have to examine the norm $H^{1/2}(-1,1)$. We have

$$\|l\|^2_{1/2,(-1,1)} \simeq \int_{-1}^{1} l^2(t)\, dt + \int_{\mathcal{Q}} \int \frac{(l(x) - l(x'))^2}{(x - x')^2} dx dx'.$$

Let $l(t)$ be a polynomial of degree less than or equal to $2N-1$. Then

$$(l(x) - l(x')) / (x - x')$$

is a polynomial of degree less than or equal to $2N-1$ in x and x'. Now using the LGL quadrature rule we can evaluate $\|l\|^2_{1/2,(-1,1)}$ as

$$\begin{aligned}\|l\|^2_{1/2,(-1,1)} =& \sum_{i=0}^{2N} w_i^{2N} l^2\left(t_i^{2N}\right) + \sum_{i=0}^{2N} \left(w_i^{2N}\right)^2 \left(\frac{dl}{dt}\left(t_i^{2N}\right)\right)^2 \\ &+ \sum_{j=0}^{2N} \sum_{\substack{i=0 \\ i \neq j}}^{2N} w_i^{2N} w_j^{2N} \left(\frac{l\left(t_i^{2N}\right) - l\left(t_j^{2N}\right)}{t_i^{2N} - t_j^{2N}}\right)^2.\end{aligned}$$

Thus there is a symmetric positive definite matrix H^{2N} such that

$$\|l\|^2_{1/2,(-1,1)} = \sum_{i=0}^{2N} \sum_{j=0}^{2N} l\left(t_i^{2N}\right) H^{2N}_{i,j} l\left(t_j^{2N}\right). \tag{12}$$

Now a typical boundary term will be of the form

$$\left\| \left(\widehat{P} u_s - \widehat{Q} u_t\right)(s,1) - \widehat{h}(s) \right\|^2_{1/2,(-1,1)}$$

and its variation is given by

$$\sum_{i=0}^{2N}\sum_{j=0}^{2N}\left(\left(\widehat{P}v_s-\widehat{Q}v_t\right)\left(s_i^{2N},1\right)\right)H_{i,j}^{2N}\left(\left(\widehat{P}u_s-\widehat{Q}u_t\right)\left(s_j^{2N},1\right)-\widehat{h}\left(s_j^{2N}\right)\right).$$

Let

$$\sigma_i^{2N}=\sum_{j=0}^{2N}H_{i,j}^{2N}\left(\left(\widehat{P}u_s-\widehat{Q}u_t\right)\left(s_j^{2N},1\right)-\widehat{h}\left(s_j^{2N}\right)\right).$$

Then

$$\begin{aligned}&\sum_{i=0}^{2N}\left(\widehat{P}v_s-\widehat{Q}v_t\right)\left(s_i^{2N},1\right)\sigma_i^{2N}\\=&\sum_{j=0}^{2N}v\left(s_j^{2N},1\right)\left(\sum_{i=0}^{2N}d_{i,j}^{2N}\widehat{P}\left(s_i^{2N},1\right)\sigma_i^{2N}\right)\\&-\sum_{i=0}^{2N}\sum_{j=0}^{2N}v\left(s_i^{2N},t_j^{2N}\right)\left(\widehat{Q}\left(s_i^{2N},1\right)d_{2N,j}^{2N}\sigma_i^{2N}\right).\end{aligned}$$

Proposition 9 *Let*

$$Y_i^{2N}=\left(Pu_s-Qu_t-h\right)\left(s_i^{2N},1\right),\qquad X^{2N}=H^{2N}Y^{2N},$$

where the matrix H is defined in (12). Then we may write

$$\begin{aligned}&\sum_{i=0}^{2N}\sum_{j=0}^{2N}\left(\left(\widehat{P}v_s-\widehat{Q}v_t\right)\left(s_i^{2N},1\right)\right)H_{i,j}^{2N}\left(\left(\widehat{P}u_s-\widehat{Q}u_t\right)\left(s_j^{2N},1\right)-\widehat{h}\left(s_j^{2N}\right)\right)\\=&\sum_{j=0}^{2N}v\left(s_j^{2N},1\right)\left(\sum_{i=0}^{2N}d_{i,j}^{2N}P\left(s_i^{2N},1\right)X_i^{2N}\right)\\&-\sum_{i=0}^{2N}\sum_{j=0}^{2N}v\left(s_i^{2N},t_j^{2N}\right)\left(Q\left(s_i^{2N},1\right)d_{2N,j}^{2N}X_i^{2N}\right)\\=&\left(V^{2N}\right)^tTX^{2N}.\end{aligned}$$

Here T is a $(2N+1)^2\times(2N+1)$ matrix and TX^{2N} can be easily computed. In writing the above we are again committing an error and this error can be shown to be exponentially small in N once more.

Adding all the terms we obtain

$$\begin{aligned}
&\int_{\mathcal{Q}} \int \widehat{L_k^{i,j}}\, v_k^{i,j} \left(\widehat{L_k^{i,j}}\, u_k^{i,j} - \widehat{g}_k^{i,j} dsdt \right) \\
&+ \sum_{i=0}^{2N} \sum_{j=0}^{2N} \left(\left(\widehat{P} u_s - \widehat{Q} u_t \right) \left(s_i^{2N}, 1 \right) \right) H_{i,j}^{2N} \left(\left(\widehat{P} u_s - \widehat{Q} u_t \right) \left(s_j^{2N}, 1 \right) - \widehat{h} \left(s_j^{2N} \right) \right) + \ldots \\
&= \left(V^{2N} \right)^t \left(RZ^{2N} + TX^{2N} + \ldots \right) \\
&= \left(V^{2N} \right)^t O^{2N}
\end{aligned}$$

where $O^{2N} = RZ^{2N} + TX^{2N} + \ldots$ is a $(2N+1)^2$ vector which can be easily computed. Now there exists a matrix G^N such that $V^{2N} = G^N V^N$. Hence

$$\left(V^{2N} \right)^t O^{2N} = \left(V^N \right)^t \left(\left(G^N \right)^t O^{2N} \right).$$

In [6] it has been shown how $\left(G^N \right)^t O^{2N}$ can be computed. Here we just describe the steps involved in brief and refer the reader to [6] for further details.

Let λ_k^N be the normalizing factors used in computing the discrete Legendre transform as

$$\lambda_k^N = \begin{cases} (k+1/2)^{-1} & \text{if } k < N \\ 2/N & \text{if } k = N \end{cases}.$$

Let $\{O_{i,j}\}_{0 \le i \le 2N, 0 \le j \le 2N}$ be defined as $O_{i,j} = O^{2N}_{i(2N+1)+j+1}$. Now we perform the following set of operations:

Procedure

Algorithm 1 *Define* $O_{i,j} \leftarrow O_{i,j} / w_i^{2N} w_j^{2N}$.

Calculate $\{\mathcal{O}\}_{0 \le i \le 2N, 0 \le j \le 2N}$ *the Legendre transform of* $O_{i,j}$. *Define*

$$\mathcal{O}_{i,j} \leftarrow \lambda_i^{2N} \lambda_j^{2N} \mathcal{O}_{i,j}.$$

Calculate $\mathcal{M}_{i,j} \leftarrow \mathcal{O}_{i,j} / \lambda_i^N \lambda_j^N$, $0 \le i \le N$, $0 \le j \le N$.

Compute $\mathcal{I}$, *the inverse Legendre transform of* $\mathcal{M}$. *Define*

$$\mathcal{I}_{i,j} \leftarrow w_i^N w_j^N \mathcal{I}_{i,j}, \qquad 0 \le i \le N, 0 \le j \le N.$$

Define a vector J *of dimension* $(N+1)^2$ *as*

$$J^N_{i(N+1)+j+1} = \mathcal{I}_{i,j} \qquad \textit{for } 0 \le i \le N, 0 \le j \le N.$$

Then $J^N = \left(G^N\right)^t O^{2N}$. Thus we see to compute J^N we do not need to compute and store any mass and stiffness matrices.

In order to compute the residual vector we need to pass the values of $u_k^{i,j}$ and its derivatives defined on $\partial\Omega_k^{i,j}$, or $\partial\widetilde{\Omega}_k^{i,j}$, to its neighboring elements as well as to communicate the corresponding values defined on neighboring elements to the element on which $u_k^{i,j}$ is defined. Moreover, when computing the two scalars, required to update the approximate solution and the search direction during each step of the conjugate gradient process, some scalars have to be exchanged among all the processors. Thus we see that communication among the processors is small.

Finally, to compute the residual vector requires $O\left(N^3\right)$ operations on a processor which is mapped onto a rectangular element and $O\left(N^4\right)$ operations on a processor which is mapped onto a triangle. Hence we require $O\left(N^4\right)$ time to compute the residual vector on a parallel computer with N processors if some of the elements are triangles.

5 Error Estimates

Let $U_k^{i,j}\left(\tau_k,\theta_k\right) = u\left(X_k^{i,j}\left(\tau_k,\theta_k\right), Y_k^{i,j}\left(\tau_k,\theta_k\right)\right)$ for $1 \leq i \leq I_k$, $2 \leq j \leq N$, $1 \leq k \leq p$ and let $U_k^{i,1} = u\left(x_k, y_k\right)$ for $1 \leq i \leq I_k$, $1 \leq k \leq p$. Here the vertex $A_k = \left(x_k, y_k\right)$. Moreover, let $U_{p+1}^l\left(\xi,\eta\right) = u\left(X_{p+1}^l\left(\xi,\eta\right), Y_{p+1}^l\left(\xi,\eta\right)\right)$ for $1 \leq l \leq \mathcal{M}$. Then the following error estimate has been proven in [14].

Theorem 10 *Let $\mathcal{F}_w$, defined in (5), minimize $\mathfrak{r}^N\left(\mathcal{F}_v\right)$ over all $\mathcal{F}_v \in \mathcal{P}^N$ for Dirichlet problems and over all $\mathcal{F}_v \in \mathcal{P}_V^N$ for mixed problems. Then there exist positive constants C and b such that for N large enough*

$$\begin{aligned} &\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|U_k^{i,1} - w_k^{i,1}\right|^2 + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\left(U_k^{i,j} - w_k^{i,j}\right)\left(\tau_k,\theta_k\right)\right\|_{2,\widetilde{\Omega}_k^{i,j}}^2 \\ &+ \sum_{l=1}^{\mathcal{M}}\left\|\left(U_{p+1}^l - w_{p+1}^l\right)\left(\xi,\eta\right)\right\|_{2,\mathcal{E}}^2 \\ \leq & Ce^{-bN}. \end{aligned} \tag{13}$$

provided $N \geq N^j \geq \alpha j$ for $2 \leq j \leq N$. Here α is a constant which depends on the data.■

After having obtained a non-conforming approximation on all the elements we can make a correction to the SEF $\mathcal{F}_w$ so that the corrected SEF $\mathcal{F}_{\mathring{w}}$ are conforming and the error between the exact solution and the corrected approximation in the $H^1\left(\Omega\right)$ norm is exponentially small in N. We do this in two steps:

Step 1: First, we make a bilinear correction $\left\{\alpha_k^{i,j}(\xi,\eta)\right\}_{i,j,k}$ so that

$$\left\{\left(w_k^{i,j}+\alpha_k^{i,j}\right)(\xi,\eta)\right\}_{i,j,k}$$

are continuous at the vertices of the rectangles on which they are defined. For this we define $\alpha_k^{i,1}\equiv 0$ for all i and k. If P is a vertex of $\Omega_k^{i,j}$ for $j\geq 2$ and $P\notin\partial\Omega_k^{i,1}$ then we choose $\alpha_k^{i,j}(P)$ so that $\left(w_k^{i,j}+\alpha_k^{i,j}\right)(P)=\overline{u}(P)$, where $\overline{u}(P)$ denotes the average of the values of u at P. If however, $P\in\partial\Omega_k^{i,1}$ we choose $\alpha_k^{i,j}(P)$ so that $\left(w_k^{i,j}+\alpha_k^{i,j}\right)(P)=w_k^{i,1}$.

Step 2: Next, we make a correction $\left\{\beta_k^{i,j}(\xi,\eta)\right\}_{i,j,k}$ so that

$$\left\{\left(w_k^{i,j}+\alpha_k^{i,j}+\beta_k^{i,j}\right)(\xi,\eta)\right\}_{i,j,k}$$

are conforming. Once again, we define $\beta_k^{i,1}\equiv 0$ for all i and k. If γ is a side of $\Omega_k^{i,j}$ for $j\geq 2$ and $\gamma\subseteq\partial\Omega_k^{i,1}$ then we choose $\beta_k^{i,j}$ so that $\left(w_k^{i,j}+\alpha_k^{i,j}+\beta_k^{i,j}\right)(P)=w_k^{i,1}$ for $P\in\gamma$. Otherwise, we choose $\beta_k^{i,j}$ so that $\left(w_k^{i,j}+\alpha_k^{i,j}+\beta_k^{i,j}\right)(P)=\left(\overline{u}+\overline{\alpha}\right)(P)$ for $P\in\gamma$. Now $\beta_k^{i,j}$ has its traces defined on the sides of the square $\mathcal{Q}$. Moreover, the traces of $\beta_k^{i,j}$ are polynomials on the sides of $\mathcal{Q}$ which vanish at the vertices. Let

$$\begin{aligned}\beta_k^{i,j}(\xi,-1)&=\phi_1(\xi), &\qquad \beta_k^{i,j}(\xi,1)&=\phi_3(\xi), \text{ and}\\ \beta_k^{i,j}(-1,\eta)&=\phi_4(\eta), &\qquad \beta_k^{i,j}(1,\eta)&=\phi_2(\eta).\end{aligned}$$

We then define a lifting of $\beta_k^{i,j}$ as follows

$$\beta_k^{i,j}(\xi,\eta)=\frac{1}{2}\left(\phi_1(\xi)(1-\eta)+\phi_3(\xi)(1+\eta)+\phi_2(\eta)(1+\xi)+\phi_4(\eta)(1-\xi)\right).$$

We can now define the corrected set of SEF as

$$\mathring{w}_k^{i,j}(\xi,\eta)=\left(w_k^{i,j}+\alpha_k^{i,j}+\beta_k^{i,j}\right)(\xi,\eta), \qquad \text{for } (\xi,\eta)\in\mathcal{Q}.$$

Now, once again, it can be shown that there exist positive constants C and b such that

$$\left\|u(x,y)-\mathring{w}(x,y)\right\|^2_{H^1(\Omega)}\leq Ce^{-bN}. \tag{14}$$

6 Parallelization techniques

We now consider parallelization techniques for problems with Dirichlet and mixed Neumann and Dirichlet boundary conditions separately, which result in different orders of complexity.

6.1 Dirichlet problems

For Dirichlet problems the SEF are fully non-conforming. Having mapped the elements to $\mathcal{E}$ we now rewrite the SEF representation from (5) as

$$\mathcal{F}_v = \left(\left\{ v_k^{i,j}(\xi,\eta) \right\}_{i,j\le N,k}, \left\{ v_k^{i,j}(\xi,\eta) \right\}_{i,j>N,k} \right). \tag{15}$$

We now define a quadratic form

$$\begin{aligned} &\mathcal{W}^N(\mathcal{F}_v) \\ &= \sum_{k=1}^{p}\sum_{i=1}^{I_k} \left| v_k^{i,1} \right|^2 + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k} \left\| v_k^{i,j}(\xi,\eta) \right\|_{2,\mathcal{Q}}^2 + \sum_{l=1}^{\mathcal{M}} \left\| v_{p+1}^{l}(\xi,\eta) \right\|_{2,\mathcal{E}}^2. \end{aligned} \tag{16}$$

It should be noted that $v_k^{i,1} = b_k$ for $1 \le i \le I_k$. Moreover, for $j \le N$ ξ is a linear function of τ_k and η is a linear function of θ_k which maps $\widetilde{\Omega}_k^{i,j}$ to $\mathcal{Q}$. Now by Theorem 3

$$\mathcal{W}^N(\mathcal{F}_v) \le C(\ln N)^2 \mathcal{V}^N(\mathcal{F}_v) \qquad \text{for all } \mathcal{F}_v \in \mathcal{P}^N. \tag{17}$$

At the same time there exists a constant K such that

$$\frac{1}{K}\mathcal{V}^N(\mathcal{F}_v) \le \mathcal{W}^N(\mathcal{F}_v) \qquad \text{for all } \mathcal{F}_v \in \mathcal{P}^N. \tag{18}$$

This follows immediately from the trace theorems for Sobolev spaces.

Hence we can conclude that the two quadratic forms $\mathcal{W}^N$ and $\mathcal{V}^N$ are spectrally equivalent and that there exists a constant K such that

$$\frac{1}{K}\mathcal{V}^N(\mathcal{F}_v) \le \mathcal{W}^N(\mathcal{F}_v) \le K(\ln N)^2 \mathcal{V}^N(\mathcal{F}_v) \qquad \text{for all } \mathcal{F}_v \in \mathcal{P}^N. \tag{19}$$

It is clear that if we use $\mathcal{W}^N$ as a preconditioner then the condition number of the preconditioned system is $O(\ln N)^2$.

Now $\mathcal{W}^N$ is a block diagonal matrix where each block corresponds to the H^2 norm of the SEF defined on a particular element which is mapped onto either the master square $\mathcal{Q}$ or the master triangle $\mathcal{T}$. Consider the bilinear form $\mathcal{R}^N\left(u^N, v^N\right)$ induced by the H^2 norm on $\mathcal{T}$ i.e.

$$\mathcal{R}^N\left(u^N, u^N\right) = \left\| u^N \right\|_{H^2(\mathcal{T})}^2.$$

Here u^N and v^N are polynomials of degree N in ξ and η respectively. Now u^N can be represented by the vector $\left(u_1, u_2, \ldots, u_{(N+1)^2}\right)^t$, where $u_i = u(\xi_i, \eta_i)$ for

$1 \leq i \leq (N+1)^2$. Here $\{(\xi_i, \eta_i)\}_{1 \leq i \leq (N+1)^2}$ are the quadrature points on $\mathcal{T}$. Now there is a matrix A such that

$$\mathcal{R}^N \left(u^N, v^N \right) = \sum_{i=1}^{(N+1)^2} \sum_{j=1}^{(N+1)^2} u_i A_{i,j} v_j.$$

The matrix A can be determined by its columns Ae_i where e_i is a unit vector with a 1 in its i^{th} place and 0 elsewhere.

Now using integration by parts Ae_i can be computed in $O\left(N^4\right)$ operations in exactly the same way as we have computed the residual vector in the normal equations. If we distribute the $(N+1)^2$ columns among the N_P processors the matrix A can be computed in time $O\left(N^5\right)$ since $N_P = O\left(N\right)$. Moreover, the L-U decomposition of A can be computed in $O\left(N^5\right)$ time and stored on every processor. Once this has been done the action of the inverse of the matrix on a vector can be computed in $O\left(N^4\right)$ time on a processor. The bounds we have provided for the triangle $\mathcal{T}$ also provide an upper bound for the square $\mathcal{Q}$.

Finally, since we would need to perform $O\left(N \ln N\right)$ iterations to obtain the solution to exponential accuracy and every iteration requires time $O\left(N^4\right)$, the time required to compute the solution would be $O\left(N^5 \ln N\right)$. It is shown in Section 6.2 that the *h-p* finite element method requires $O\left(N^6 \ln N\right)$ time to compute the solution to exponential accuracy on a parallel computer with N_P processors. Hence the proposed method is asymptotically faster than the *h-p* finite element method by a factor of N.

6.2 Mixed problems

We first write one more representation for SEF which excludes the corner most elements, i.e. corresponding to $j = 1$, as

$$\mathcal{F}_u^1 = \left(\left\{ u_k^{i,j} \left(\xi, \eta \right) \right\}_{i,2 \leq j \leq N,k}, \left\{ u_k^{i,j} \left(\xi, \eta \right) \right\}_{i,j > N,k} \right) \tag{20}$$

Now let U denote the values of $\mathcal{F}_u$ at the quadrature points, except that the common values at the vertices of the elements are counted only once. We may then write U as $U = \left[U^I, U^B \right]^t$, where $[.]^t$ denotes the transpose. Here U_B denotes the common values of $\mathcal{F}_u^1$ at the vertices of the elements, and U_I denotes the values of the remaining elements ordered as rows and concatenated in a consistent order of elements. Let $\mathcal{V}^N\left(\mathcal{F}_v\right)$ be as defined in (6). Recall that $\mathcal{P}_V^N$ denotes the space of functions $\mathcal{F}_v$ which are continuous at the vertices of the elements on which they are defined. Moreover let $v_k^{i,1} = b_k$ for $1 \leq i \leq I_k$. We shall let V denote the values of $\mathcal{F}_v^1$ at the quadrature points ordered as at the beginning of this section. Further, we

may write

$$V = \begin{bmatrix} V_I \\ V_B \end{bmatrix}.$$

Then there is a symmetric positive definite matrix A such that

$$\mathcal{V}^N\left(\mathcal{F}_v^1\right) = \begin{bmatrix} V_I^t & V_B^t \end{bmatrix} \begin{bmatrix} A_{II} & A_{IB} \\ A_{IB}^t & A_{BB} \end{bmatrix} \begin{bmatrix} V_I \\ V_B \end{bmatrix}. \tag{21}$$

When we solve the minimization problem we have to finally solve a system of equations of the form

$$Au = h, \tag{22}$$

where h can also be written as

$$h = \begin{bmatrix} h_I \\ h_B \end{bmatrix}. \tag{23}$$

Now to solve the system (22) we use the block L-U factorization of A, viz.

$$A = \begin{bmatrix} I & 0 \\ A_{IB}^t & I \end{bmatrix} \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1}A_{IB} \\ 0 & I \end{bmatrix}, \tag{24}$$

where the Schur complement matrix S is defined as

$$S = A_{BB} - (A_{IB})^t (A_{II})^{-1} A_{IB}. \tag{25}$$

and $(.)^{-1}$ denotes the inverse. Consequently, solving (22) based on the L-U factorization given in (24) reduces to solving the system of equations

$$SU_B = \widetilde{h}_B, \text{ where} \tag{26a}$$

$$\widetilde{h}_B = h_B - A_{IB}^t A_{II}^{-1} h_I. \tag{26b}$$

As we can see from the definition of S the feasibility of such a process depends on our being able to compute $A_{IB}V_B$, $A_{II}V_I$ and $A_{BB}V_B$ for any V_I, V_B efficiently, and this can always be done since AV can be computed inexpensively as has been described in Section 4. However, in addition to this it is imperative that we should be able to construct effective preconditioners for the matrix A_{II} so that the condition number of the preconditioned system is as small as possible. If we are able to do this then it will be possible to compute $A_{II}^{-1}V_I$ efficiently using the PCGM for any vector V_I. Now consider the space of functions $\mathcal{P}_0^N \subseteq \mathcal{P}_V^N$. In particular, this implies that $w_k^{i,1} = 0$ for all i and k. Let W denote the vector of the values of these functions at the LGL points, ordered as at the beginning of this section. Then W

has the form $\left[W_I^t \ 0 \right]^t$, and so

$$\mathcal{V}^N \left(\mathcal{F}_w^1 \right) = W_I^t A_{II} W_I. \tag{27}$$

Now using Theorem 5 we have the following result.

Let $\mathcal{F}_w^1$ belong to the space of functions $\mathcal{P}_0^N$. Then the following estimate

$$\begin{aligned} &\sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| w_k^{i,j} \left(\xi, \eta \right) \right\|_{2,\mathcal{Q}}^2 + \sum_{l=1}^{\mathcal{M}} \left\| w_{p+1}^{l} \left(\xi, \eta \right) \right\|_{2,\mathcal{E}}^2 \\ \leq & C \left(\ln N \right)^2 \mathcal{V}^N \left(\mathcal{F}_w^1 \right), \end{aligned} \tag{28}$$

hold for N large enough. In the above $w_k^{i,1} \left(\xi, \eta \right)$ is taken to be identically zero for $1 \leq k \leq p$ and $1 \leq i \leq I_k$.

We now define the quadratic form

$$\mathcal{U}^N \left(\mathcal{F}_w^1 \right) = \sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| w_k^{i,j} \left(\xi, \eta \right) \right\|_{2,\mathcal{Q}}^2 + \sum_{l=1}^{\mathcal{M}} \left\| w_{p+1}^{l} \left(\xi, \eta \right) \right\|_{2,\mathcal{E}}^2 \tag{29}$$

for all $\mathcal{F}_w^1 \in \mathcal{P}_0^N$. As in Section 6.1 we can conclude that the quadratic forms $\mathcal{U}^N$ and $\mathcal{V}^N$ are spectrally equivalent and that there exists a constant K such that

$$\frac{1}{K} \mathcal{V}^N \left(\mathcal{F}_w^1 \right) \leq \mathcal{U}^N \left(\mathcal{F}_w^1 \right) \leq K \left(\ln N \right)^2 \mathcal{V}^N \left(\mathcal{F}_w^1 \right) \quad \text{for all } \mathcal{F}_w^1 \in \mathcal{P}_0^N. \tag{30}$$

We can now use the quadratic form $\mathcal{U}^N$, which consists of a decoupled set of quadratic forms on each element as a preconditioner for A_{II}. We can do this by inverting the block diagonal matrix representation of $\mathcal{U}^N$. We can assemble this matrix by distributing the computation of its columns among the N_P processors and then compute the inverse of this matrix using L-U decomposition distributed among the N_P processors as has been described in Section 6.1.

Now from (30) we can conclude that if we were to compute $A_{II}^{-1} U_I$ using the PCGM then the condition number of the preconditioned matrix would be $O \left(\ln N \right)^2$. Hence, to compute $A_{II}^{-1} U_I$ to an accuracy of $O \left(e^{-bN} \right)$ would require $O \left(N \ln N \right)$ iterations of the PCGM.

We now return to the steps involved in solving the system of equations (22). As a first step we would need to solve the much smaller system of equations (26a-26b). Here the dimension of the vector U_B is $N_B = O \left(N \right)$, which is proportional to the number of elements into which we have divided the original domain Ω. Now to solve (26a) to an accuracy of $O \left(e^{-bN} \right)$ we need to be able to compute the residual

$$R_B = S V_B - \widetilde{h}_B \tag{31}$$

with the same accuracy and in an efficient manner. The bottleneck in computing R_B consists in computing $A_{II}^{-1} A_{IB} V_B$ to $O\left(e^{-bN}\right)$ and we have already seen that this can be done using $O\left(N \ln N\right)$ iterations of the PCGM for computing $A_{II}^{-1}\left(A_{IB} V_B\right)$ for a given vector V_B.

We now show that it possible to construct an accurate approximation to the Schur complement matrix S. Since S is a $N_B \times N_B$ matrix, where $N_B = O\left(N\right)$, its dimension is quite small as compared to the Schur complement matrix for the *h-p* finite element method. Let e_k be a column vector of dimension N_B with a 1 at the k^{th} place and 0 elsewhere. Let $S_k = S e_k$. Then $S = [S_1\, S_2 \ldots S_{N_B}]$. Now using Theorem 4 we can show that

$$\left\| S^{-1} \right\| \leq C N^4. \tag{32}$$

Moreover, it is easy to show that

$$\left\| A_{IB} \right\| \leq C N^5. \tag{33}$$

Now $S_k = A_{BB} e_k - A_{IB}^t A_{II}^{-1} A_{IB} e_k$. Suppose we compute an approximation $\widehat{S_k}$ to S_k using $O\left(N^{1/2} \ln N\right)$ iterations of the PCGM for computing $A_{II}^{-1} V_I$. Then using (32-33) we can show that the error between $\widehat{S_k}$ and S_k would be $O\left(e^{-bN^{1/2}}\right)$. If we were to do this for every $1 \leq k \leq N_B$ then the number of iterations required for computing $\widehat{S}$ would be $O\left(N^{3/2} \ln N\right)$. We would then have computed $\widehat{S}$, which is an approximate representation of S. In fact, these two matrices would not be just spectrally close but we would have

$$\left(1 - c e^{-bN^{1/2}}\right) S \leq \widehat{S} \leq \left(1 + c e^{-bN^{1/2}}\right) S \tag{34}$$

for N large enough where b and c are positive constants. Hence, if we were to solve (26a) using $\widehat{S}$ as a preconditioner for S then the error after $N^{1/2}$ steps would be $O\left(e^{-bN}\right)$. Since each step of this process requires $O\left(N \ln N\right)$ iterations for computing $A_{II}^{-1}\left(A_{IB} V_B\right)$ we would have performed $O\left(N^{3/2} \ln N\right)$ steps of the PCGM for computing $A_{II}^{-1}\left(A_{IB} V_B\right)$, where V_B would vary after every $O\left(N \ln N\right)$ steps during this process. This is the same as the number of iterations required to form $\widehat{S}$. Hence we can solve (26a) to exponential accuracy in N using only $O\left(N^{3/2} \ln N\right)$ iterations of the PCGM for computing $A_{II}^{-1} W_I$. Finally, it remains to compute U_I. Now from (22) we have

$$A_{II} U_I = h_I - A_{IB} U_B. \tag{35}$$

Clearly we can solve the above system of equations for U_I to an accuracy of $O\left(e^{-bN}\right)$ using $O\left(N \ln N\right)$ iterations of the PCGM to compute $A_{II}^{-1} V_I$ for a given vector V_I. Thus the overall complexity of the method we have described would

require $O\left(N^{3/2}\ln N\right)$ iterations of the PCGM to obtain a solution to the system of equations (22) to exponential accuracy. Moreover, the time required on a parallel computer with N_P processors would be $O\left(N^{5.5}\ln N\right)$. Hence the proposed method is asymptotically faster than the *h-p* FEM which would solve the same class of problem in $O\left(N^6\ln N\right)$ time.

We shall now briefly examine the complexity of the solution procedure for the *h-p* FEM. The set of CBV for the FEM consists of the values of the functions at the vertices and sides of the elements. In [9] it has been shown that we can construct an approximation $\widehat{S}$ to S such that the condition number χ of the preconditioned system satisfies

$$\chi \leq C\left(1+(\ln N)^2\right)$$

for a constant C. Thus to solve (26a) to an accuracy $O\left(e^{-bN}\right)$ will require $O\left(N\ln N\right)$ iterations of the PCGM using $\widehat{S}$ as a preconditioner. Now to compute the residual in the Schur complement system to an accuracy of $O\left(e^{-bN}\right)$ requires $O\left(N\right)$ iterations of the PCGM to compute $A_{II}^{-1}A_{IB}V_B$ [13]. Hence we would need to perform $O\left(N^2\ln N\right)$ iterations of the PCGM for computing $A_{II}^{-1}V_I$, where V_I will vary after every sequence of $O\left(N\ln N\right)$ steps. To solve (22) to an accuracy of $O\left(e^{-bN}\right)$ will thus require time $O\left(N^6\ln N\right)$ on a parallel computer with N_P processors.

We conclude this section by examining the structure of the Schur matrix S more closely. Let v_k and v_j be two vertices and let $d_{k,j}$ be the length of the shortest path joining v_k and v_j. Here the length of the path is the number of edges lying on the path. Let us suppose that we use m iterations of the PCGM to compute an approximation $\widehat{S_k}$ to $S_k = Se_k$. Here e_k is the vector of CBV which is 1 at v_k and 0 elsewhere. It is easy to see that the vector $\widehat{S_k}$ will be nonzero only at those vertices v_j whose distance from v_k is less than or equal to $m+3$. Now we have already shown that

$$\left\|S_k - \widehat{S_k}\right\| = O\left(e^{-bm/\ln N}\right).$$

Hence we can conclude that

$$|S_{k,j}| = O\left(e^{-bd_{k,j}/\ln N}\right).$$

Thus the entries of the Schur matrix S decay rapidly away from the diagonal.

Finally, we do not need to make the SEF continuous at the vertices of all the elements but only at a restricted set of vertices lying on the boundary of Ω. This follows from the remark made in Section 6.

6.3 Load balancing

The dimension of $u_k^{i,j}$ is $(N^j+1)^2$ for $2 \leq j \leq N$ where $\alpha j \leq N^j \leq N$, α is a positive constant. For the sake of simplicity we choose an upper bound i.e. $N^j = N$ for all $j \geq 2$ and map each $u_k^{i,j}(\xi,\eta)$ onto separate processors. By doing so we are able to achieve perfect load balancing among individual processors, but at the cost of making some of the processors do extra computational work which would not increase the accuracy of the numerical solution substantially. Alternatively, if we were to assign each $u_k^{i,j}(\xi,\eta)$ with varying N^j onto different processors this would cause an imbalance in the loads assigned to different processors and we would need to employ load balancing techniques. The numerical results with this upper bound confirm the theoretical estimates and demonstrate the robustness of the method.

7 Computational results

In this section we present the numerical results of the model problems which aim to verify the asymptotic error estimates and estimates of computational complexity and demonstrate the robustness of the method. In [15] we had considered only a sectoral domain for our numerical results and have shown that the presented method delivers exponential convergence. Here we extend that to a full domain and consider similar examples as of Pathria and Karniadakis [12], i.e. Laplace and Poisson's equations on an *L*-shaped domain with pure Dirichlet or mixed boundary conditions. Let N denotes the total number of elements in the domain. For the *L*-shaped domain we divide the sectoral neighborhood (of the corner with singularity) into 6 subsectors (not necessarily to be of equal refinement, though) and 6 large elements in the remaining domain. We also choose the geometric ratio as $\mu = 0.15$. Further, we define the relative error as $\|e\|_{H^1 R} = \|e\|_{H^1} / \|u\|_{H^1}$.

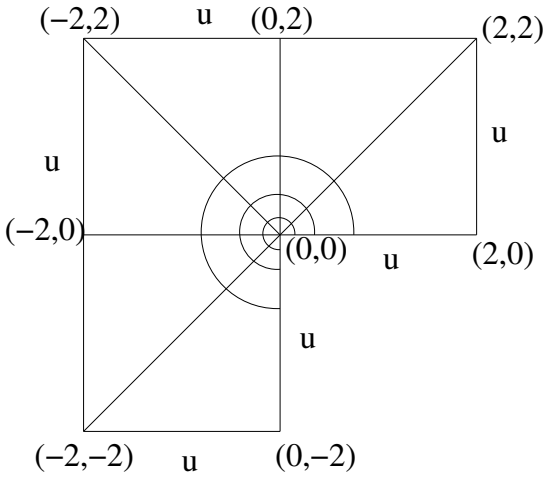

Fig. 3. Domain with pure Dirichlet boundary conditions

First we consider the Laplace problem with all the boundary conditions as Dirichlet, as shown in Fig. 3. Without any loss of generality we can choose our data such that the singularity arises only at the corner of $3\pi/2$ angle. However, for the domains with singularity at other corners we can apply the same technique everywhere. We chose our data such that the solution is of the form of the leading order singularity $r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$. Fig. 4 shows the error on the $\ln$ of relative error against N and the

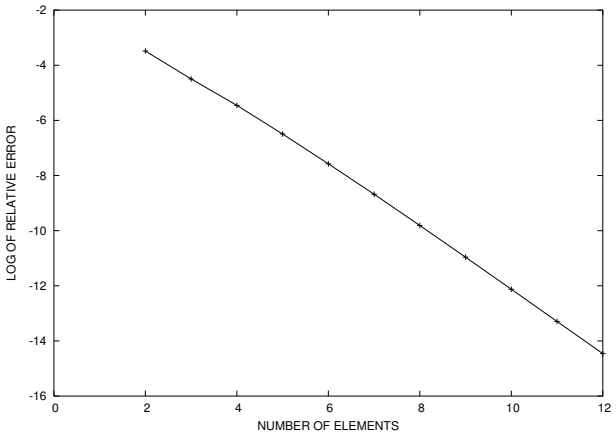

Fig. 4. Log of relative H^1 error vs. N for the Laplace problem on the domain of Fig. 3

relationship is almost linear.

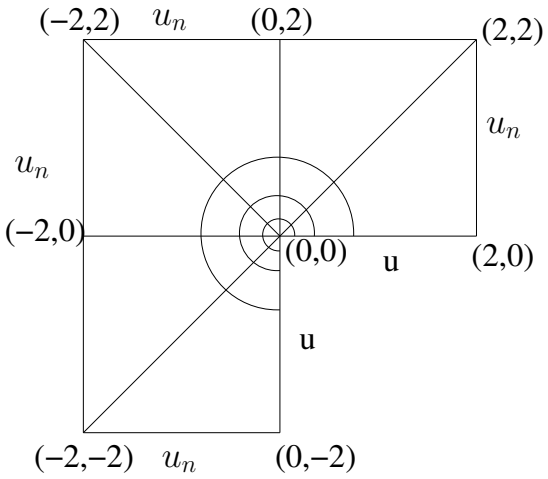

Fig. 5. Domain with Dirichlet boundary conditions at the corner with singularity and Neumann elsewhere

Next, we consider a similar domain as in Fig. 3, but, with mixed boundary conditions, as shown in Fig. 5. However, we specify Dirichlet conditions on the boundaries meeting at the corner. Once again we chose our data such that the solution is of the form of the leading order singularity $r^{\frac{2}{3}}\sin(\frac{2}{3}\theta)$ (since both the boundary conditions meeting at the corner are of Dirichlet type). Fig. 6 shows the error on the $\ln$ of relative error against N and the relationship is again almost linear. Following the remark 6 we didn't require to compute the Schur complement system for this problem.

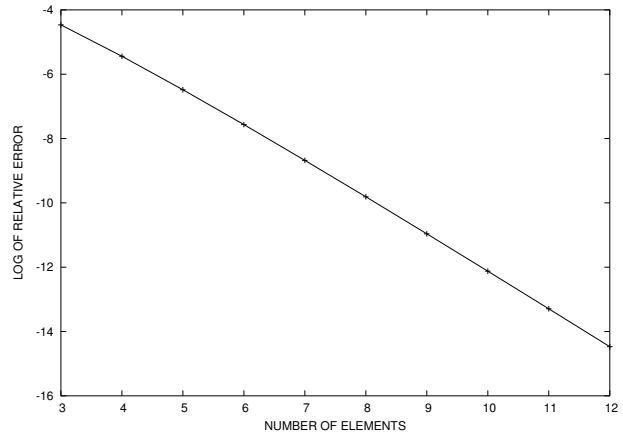

Fig. 6. Log of relative H^1 error vs. N for the Laplace problem on the domain of Fig. 5

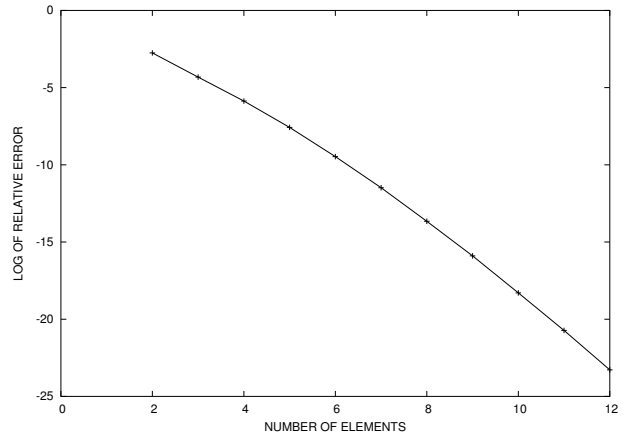

Fig. 7. Log of relative H^1 error vs. N for the Poisson problem on the domain of Fig. 3

Now we consider the Poisson problem with all the boundary conditions as Dirichlet, for the domain as shown in Fig. 3. Here we choose the data such that the exact solution is analytic and is of the form of $y(y-3x)/2$. Fig. 7 shows the error on the $\ln$ of relative error against N, which decays very rapidly.

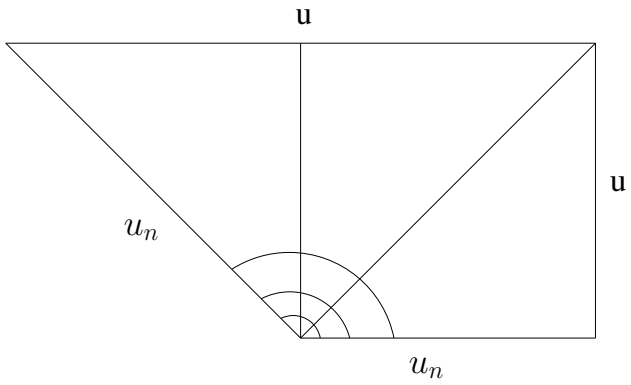

Fig. 8. Domain with Neumann boundary conditions at the corner and Dirichlet elsewhere

Now again we consider the Laplace problem with mixed boundary conditions, as

shown in Fig. 8. However, we specify Neumann conditions on the boundaries meeting at the corner where the sides form an angle of $\frac{3\pi}{4}$. Here we chose our data such that the exact solution has a singularity at the corner and is of the form of $r^{\frac{4}{3}}\cos(\frac{4}{3}\theta)$. Fig. 9 shows the $\ln$ of relative error against N, which confirms the ex-

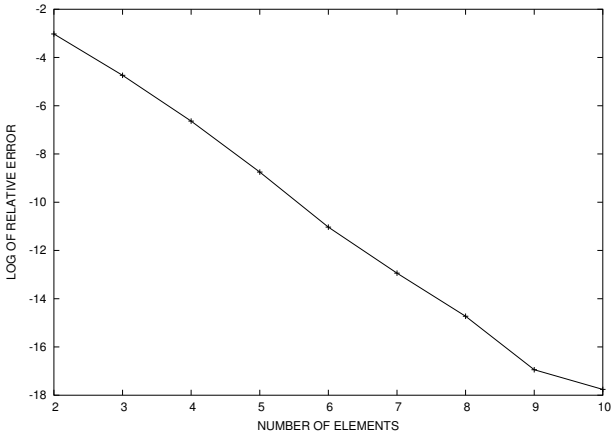

Fig. 9. Log of relative H^1 error vs. N for the Laplace problem on the domain of Fig. 8

ponential convergence. However, since both the boundaries meeting at the corner are Neumann, we need to solve the Schur complement system.

We now provide the results for the solution of the Schur complement matrix. Fig.

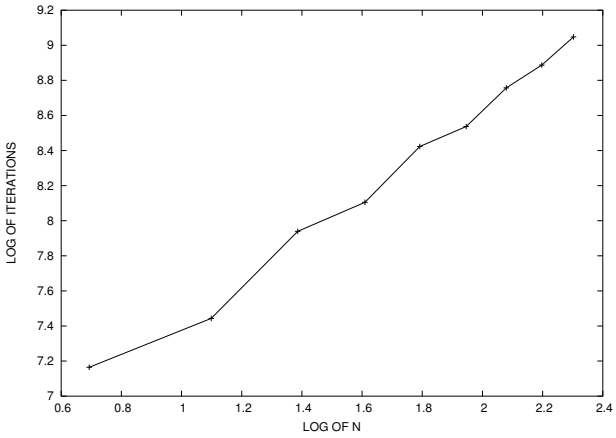

Fig. 10. Log of iterations vs. log N for the Schur complement matrix.

10 shows the $\ln$ of iterations against the $\ln$ of N. Here iterations denote the minimum number of iterations of the PCGM needed to solve the problem to maximum accuracy achievable, i.e. solving the Schur complement system to a higher accuracy will only increase the total number of iterations to obtain the solution. To check the asymptotic estimate that $Iterations = O(N^{3/2}\ln N)$, we fit a straight line to the given data points $N = 2$ to $N = 10$ and compute the slope. The slope obtained is m.

Table 1
Relationship between N_0 and m

N_0	2	4	6	8
Slope m	1.2022	1.2322	1.2494	1.2978

8 Conclusion

We have presented a least-squares collocation approach to solve elliptic boundary value problems with mixed boundary conditions on curvilinear polygons to exponential accuracy within the framework of spectral methods. The computational complexity of the proposed scheme is better than that of the standard *h-p* finite element method. The algorithm for the preconditioner, which is of block diagonal form and with optimal condition number, is quite simple and easy to implement with minimum extra effort. Moreover, the construction of the preconditioner for the Schur complement matrix is very simple and with low dimensions, unlike the case for finite element methods, and this would have significant impact for three dimensional problems. The residuals during the iterations of PCGM are computed efficiently without having to compute and store any *mass and stiffness* matrices. Except when computing the two scalars, which are required to update the approximate solution and the search direction during each step of PCGM, we need to exchange the data only with neighboring elements, and hence the inter-processor communication is small. Though the ideas in this paper deal with problems in two dimensions, they generalize to three dimensions. We intend to study these problems in our future work.

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