

Discrepancy estimates for point sets on the s-dimensional Sierpinski carpet

**L.L. Cristea, F. Pillichshammer, G. Pirsic,
K. Scheicher**

RICAM-Report 2004-13

Discrepancy estimates for point sets on the s -dimensional Sierpiński carpet

L.L. Cristea*, F. Pillichshammer†, G. Pirsic and K. Scheicher

Abstract

In a recent paper Cristea and Tichy introduced several types of discrepancies of point sets on the s -dimensional Sierpiński carpet and proved various relations between these discrepancies. In the present paper we prove a general lower bound for those discrepancies in terms of N , the cardinality of the point set, and we give a probabilistic proof for the existence of point sets with “small” discrepancy. Furthermore we consider a van der Corput type construction of point sets on C_s and determine the exact order of convergence of various notions of discrepancy. Finally, Carpet-Walsh functions are defined to prove an Erdős-Turán-Koksma inequality which we apply to digital point sets on the carpet.

Keywords: Discrepancy, uniform distribution, fractal sets, Sierpiński carpet.
MSC 2000: 11K06, 11K38, 28A80.

1 Introduction

The s -dimensional Sierpiński carpet C_s , $s \geq 2$, is a fractal set embedded in \mathbb{R}^s and can be obtained as follows.

Let $A_0 = [0, 1]^s$ be the closed unit-cube in \mathbb{R}^s . Divide A_0 into 3^s congruent cubes with side length $1/3$ and delete the open “central” cube. We denote

*The first author is supported by the Austrian Research Foundation (FWF), Project S 8308-N05.

†The second author is supported by the Austrian Research Foundation (FWF), Project S 8305 and Project P17022-N12.

the resulting set by A_1 , i.e.,

$$A_1 = A_0 \setminus \left(\frac{1}{3}, \frac{2}{3} \right)^s.$$

The set A_1 is the union of $3^s - 1$ cubes with side length $1/3$. Repeat this operation for each of these $3^s - 1$ cubes successively to obtain A_2, A_3, \dots . For $n \in \mathbb{N}_0$, A_n is the union of $(3^s - 1)^n$ cubes with side length 3^{-n} , called *elementary cubes of level n* . We will often call such cubes simply n -cubes. We have

$$A_n = A_{n-1} \setminus \left(\bigcup_{k_1, \dots, k_s=0}^{3^{n-1}-1} \prod_{i=1}^s \left(\frac{k_i}{3^{n-1}} + \frac{1}{3^n}, \frac{k_i}{3^{n-1}} + \frac{2}{3^n} \right) \right).$$

Now the s -dimensional Sierpiński carpet C_s is defined as the intersection of the sets A_n , $n \in \mathbb{N}_0$, i.e.,

$$C_s := \bigcap_{n=0}^{\infty} A_n.$$

(See Figure 1 for an illustration of $\bigcap_{n=0}^4 A_n$.)

It is well known that C_s is a fractal set which verifies the open set condition and has Hausdorff dimension $\alpha_s = \frac{\log(3^s-1)}{\log 3}$. Furthermore it can be shown that $0 < \mathcal{H}^{\alpha_s}(C_s) < \infty$, where \mathcal{H}^{α_s} is the α_s -dimensional Hausdorff measure on C_s . Hence we may define the normalized Hausdorff measure μ on C_s by $\mu(A) = \mathcal{H}^{\alpha_s}(A)/\mathcal{H}^{\alpha_s}(C_s)$ for any Borel set $A \subseteq C_s$. For a survey on properties of fractal sets we refer to [2, 7, 8, 9].

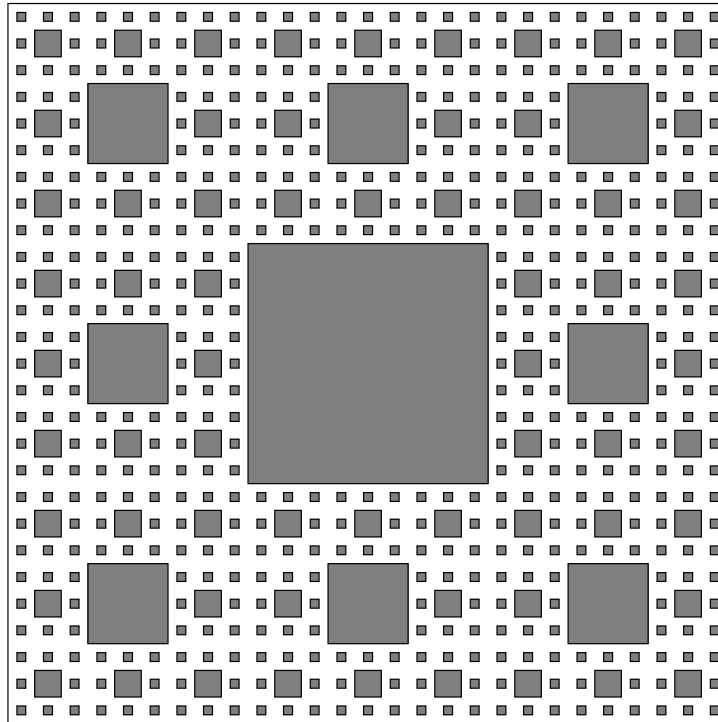
The notion of discrepancy is intimately related to that of uniform distribution: let X be a compact metric space with a Borel probability measure ν . A sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ is said to be ν -uniformly distributed if for any Borel set $B \subseteq X$ with $\nu(\partial B) = 0$

$$\lim_{N \rightarrow \infty} \frac{A_N(\omega, B)}{N} = \nu(B),$$

where $A_N(\omega, B)$ denotes the number of indices n , $0 \leq n \leq N - 1$, such that \mathbf{x}_n is contained in B . For more details we refer to [15], see also [6, 13].

Let \mathcal{D} be a system of Borel sets $B \subseteq X$ such that $\nu(\partial B) = 0$ for each $B \in \mathcal{D}$. Then the discrepancy of a point set $P_N = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ with respect to \mathcal{D} is defined by

$$D_N^{\mathcal{D}}(P_N) := \sup_{B \in \mathcal{D}} \left| \frac{A_N(P_N, B)}{N} - \nu(B) \right|.$$

Figure 1: Dimension $s = 2$, $\bigcap_{n=0}^4 A_n$.

For an infinite sequence ω the discrepancy $D_N^{\mathcal{D}}(\omega)$ is defined as the discrepancy of the first N points of the sequence. The discrepancy of a given sequence in X is a quantitative measure for the irregularity of distribution of the sequence in X .

Discrepancies of sequences on fractal sets have already been studied for example for the planar Sierpiński gasket [10] (see also [6]), and also for the Sierpiński gasket in higher dimension, see [14]. In [1] an \mathcal{L}_2 discrepancy of point sets on self-similar fractals that fulfill the open set condition was studied.

As in [5] we consider the case $X = C_s$, $s \geq 2$, endowed with the Euclidean metric. The existence of μ -uniformly distributed sequences of points on the fractal C_s follows from [15, Chapter 3, Theorem 2.2]. In [4] and [5] the authors introduce different discrepancies on C_s by choosing different systems \mathcal{D} of Borel sets in C_s .

The first discrepancy introduced on C_s is the so-called *elementary discrepancy* $D_N^{\mathcal{E}}$ which is defined by the system \mathcal{E} of all elementary cubes of level n , for all $n \in \mathbb{N}_0$ (intersected with C_s). Each n -cube ($n \geq 1$) is considered together with “half” of its boundary, namely with all of its lower-dimensional faces that are closer to the origin (in the Euclidean metric) than any of the parallel faces of the same dimension in the n -cube. There are situations when the n -cubes are considered together with more than “half” of their boundary: if a face of the n -cube is contained in a face of a “deleted” m -cube ($m \leq n$), then the n -cube is taken together with this face.

Furthermore, \mathcal{D} is considered to be the system \mathcal{S} of all sets which are intersections of C_s with intervals of the form

$$J = \prod_{i=1}^s [a_i, b_i)$$

such that all vertices of J belong to C_s . (If $b_i = 1$ for some i , $1 \leq i \leq s$, then $[a_i, b_i]$ occurs instead of $[a_i, b_i)$ in the above product.) The related discrepancy is called *carpet discrepancy* and will be denoted by $D_N^{\mathcal{S}}$.

In the present paper we also define a *carpet star discrepancy* $D_N^{\mathcal{S}^*}$ where the system \mathcal{S}^* consists of all sets of \mathcal{S} with $(a_1, \dots, a_s) = (0, \dots, 0)$. Note that the carpet star discrepancy is similar to the corner discrepancy defined in [5] except that for the corner discrepancy a slightly larger class of intervals is considered.

We have the following relations between the kinds of discrepancy introduced above:

Proposition 1 *Let $s \geq 2$ and let $P_N = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be a point set in C_s .*

1. *We have*

$$D_N^{\mathcal{E}}(P_N) \leq D_N^{\mathcal{S}}(P_N) \leq c(s) (D_N^{\mathcal{E}}(P_N))^{1 - \frac{s-1}{\alpha_s}},$$

where $c(s) > 0$ only depends on the dimension s and where $\alpha_s = \frac{\log(3^s - 1)}{\log 3}$ denotes the Hausdorff dimension of C_s .

2. *We have*

$$D_N^{\mathcal{S}^*}(P_N) \leq D_N^{\mathcal{S}}(P_N) \leq 2^s D_N^{\mathcal{S}^*}(P_N).$$

Proof. The proof of the first statement was given in [5]. The left-hand inequality in the second statement follows from the definitions of the discrepancies, and the right-hand inequality can be proven by using the inclusion-exclusion principle. We refer to [5, Proposition 3] for the detailed proof of an analogous result. \square

For more types of discrepancies on C_s and relations between those discrepancies we refer to [4, 5].

We end this Introduction with a brief outline of the paper. In Section 2 we prove a lower bound for the carpet star discrepancy of sequences in C_s and for any $s \geq 2$ we prove the existence of a point set with carpet star discrepancy of order $(\log N/N)^{1/2}$. In Section 3 we give a van der Corput type construction of a sequence on C_s and we determine the exact order of convergence of the elementary and the carpet (star) discrepancy. In order to do this we make use of the fact that C_s can be defined as the attractor (or invariant set) of an IFS (Iterated Functions System). Finally, in the last section, Walsh functions on the carpet are defined and applied in the proof of an Erdős-Turán-Koksma inequality. Also, IFS-digital sequences are introduced and a carpet star discrepancy bound is given for IFS-digital point sets.

2 General discrepancy bounds

In this section we prove bounds for the carpet star discrepancy $D_N^{S^*}$ of sequences on C_s . Bounds for other types of discrepancy can be obtained by applying the estimates from Proposition 1.

In [10] Grabner and Tichy use W. Schmidt's theorem on irregularities of distribution to prove a lower bound for a certain discrepancy of sequences on the planar Sierpiński gasket. Here we use their technique to prove a lower bound for the carpet star discrepancy of sequences in C_s .

Theorem 1 *Let $\omega = (\mathbf{x}_n)_{n \geq 0}$ be a sequence in C_s , $s \geq 2$. Then the inequality*

$$D_N^{S^*}(\omega) \geq c \frac{\log N}{N}$$

holds for infinitely many N , where $c > 0$ denotes an absolute constant.

Proof. We consider intervals of the form

$$I(x) := ([0, x] \times [0, 1)^{s-1}) \cap C_s.$$

Let $F(x) := \mu(I(x))$, where μ denotes the normalized Hausdorff measure on C_s . Then for any x we have

$$D_N^{\mathcal{S}^*}(\omega) \geq \left| \frac{A_N(\omega, I(x))}{N} - F(x) \right|. \quad (1)$$

For any $\mathbf{x} = (x_1, \dots, x_s) \in C_s$ let $k_1(\mathbf{x}) := x_1$. Then it is clear that

$$A_N(\omega, I(x)) = A_N((k_1(\mathbf{x}_n))_{n \geq 0}, [0, x]).$$

Since F is strictly increasing and continuous we find that $k_1(\mathbf{x}_n) \in [0, x]$ if and only if $F(k_1(\mathbf{x}_n)) \in [0, F(x)]$. Hence from (1) we obtain

$$D_N^{\mathcal{S}^*}(\omega) \geq \left| \frac{A_N((F(k_1(\mathbf{x}_n)))_{n \geq 0}, [0, F(x)])}{N} - F(x) \right|. \quad (2)$$

It follows by a result of Schmidt [20] (see also [6, 15]) that there exists a constant $c > 0$ such that

$$D_N^{\mathcal{S}^*}(\omega) \geq \sup_{0 \leq y \leq 1} \left| \frac{A_N((F(k_1(\mathbf{x}_n)))_{n \geq 0}, [0, y])}{N} - y \right| \geq c \frac{\log N}{N}$$

for infinitely many N . (Kuipers and Niederreiter [15] showed that one may choose $c = 1/(66 \log 4)$.) \square

In the following theorem we give a probabilistic proof for the existence of point sets on C_s with “small” carpet star discrepancy. The idea of the proof of this result was first given by Beck in the early 1980s, see [3]. A similar approach was later used by Grabner and Tichy [10] to prove the existence of point sets on the planar Sierpiński gasket with small discrepancy, and by Heinrich et al. [11] to prove the existence of point sets in $[0, 1)^s$ with small star discrepancy. Here we mainly follow [11].

Theorem 2 *For every integer $s > 1$ there exists a constant $c(s) > 0$ such that for every positive integer $N > 1$ there exists a point set P_N in C_s consisting of N points such that*

$$D_N^{\mathcal{S}^*}(P_N) \leq c(s) \sqrt{\frac{\log N}{N}}.$$

For the proof of Theorem 2 we need the following lemma.

Lemma 1 *Let Γ_{3^k} be the equidistant grid on $[0, 1]^s$ with mesh-size $1/3^k$. Then for any point set P consisting of N points in C_s we have*

$$D_N^{S^*}(P) \leq \max_{\mathbf{x} \in \Gamma_{3^k}} \left| \frac{A_N(P, [\mathbf{0}, \mathbf{x}])}{N} - \mu([\mathbf{0}, \mathbf{x}]) \right| + \delta(s, k),$$

where $\delta(s, k) := s \left(\frac{3^s - 1}{3^s - 1} \right)^k$ and $[\mathbf{0}, \mathbf{x}] := C_s \cap \prod_{i=1}^s [0, x_i]$ for any $\mathbf{x} = (x_1, \dots, x_s) \in [0, 1]^s$.

Remark 1 1. Note that some points of Γ_{3^k} are not contained in C_s , e.g. $(4/9, 4/9) \in \Gamma_9 \setminus C_2$.

2. The cardinality of Γ_{3^k} is $(3^k + 1)^s$. Hence Lemma 1 shows that the supremum in the definition of the carpet star discrepancy can be replaced by a maximum extended over the finite set Γ_{3^k} with a maximal “error” of at most $\delta(s, k)$.

Proof. From the definition of $D_N^{S^*}$ we find that for any $\eta > 0$ there exists an $\mathbf{x}^* \in C_s$ such that the inequality

$$D_N^{S^*} \leq \left| \frac{A_N(P, [\mathbf{0}, \mathbf{x}^*])}{N} - \mu([\mathbf{0}, \mathbf{x}^*]) \right| + \eta \quad (3)$$

holds. Let now $\mathbf{x}, \mathbf{y} \in \Gamma_{3^k}$, $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$ such that

$$x_i \leq x_i^* < x_i + \frac{1}{3^k} =: y_i,$$

(or

$$x_i < x_i^* \leq x_i + \frac{1}{3^k} =: y_i,$$

should $x_i^* = 1$ hold for some i) for $1 \leq i \leq s$. Then we have

$$\mu([\mathbf{0}, \mathbf{y}]) - \mu([\mathbf{0}, \mathbf{x}]) \leq [(3^k)^s - (3^k - 1)^s] \frac{1}{(3^s - 1)^k} \leq s(3^k)^{s-1} \frac{1}{(3^s - 1)^k} = \delta(s, k).$$

On the other hand,

$$-\delta(s, k) + \mu([\mathbf{0}, \mathbf{y}]) \leq \mu([\mathbf{0}, \mathbf{x}]) \leq \mu([\mathbf{0}, \mathbf{x}^*]) \leq \mu([\mathbf{0}, \mathbf{y}]) \leq \mu([\mathbf{0}, \mathbf{x}]) + \delta(s, k)$$

and therefore

$$\begin{aligned} \mu([\mathbf{0}, \mathbf{y})) - \frac{A_N(P, [\mathbf{0}, \mathbf{y}))}{N} - \delta(s, k) &\leq \mu([\mathbf{0}, \mathbf{x}^*)) - \frac{A_N(P, [\mathbf{0}, \mathbf{x}^*))}{N} \\ &\leq \mu([\mathbf{0}, \mathbf{x})) - \frac{A_N(P, [\mathbf{0}, \mathbf{x}))}{N} + \delta(s, k). \end{aligned}$$

From here and from (3) the result follows (since $\eta > 0$ can be arbitrarily small). \square

Now we can give the proof of Theorem 2.

Proof. Let $\mathbf{z}_0, \dots, \mathbf{z}_{N-1}$ be independent, μ -uniformly distributed random variables on C_s . For $\mathbf{x} \in C_s$ let

$$\xi_{\mathbf{x}}^{(i)} = \mathbf{1}_{[\mathbf{0}, \mathbf{x})}(\mathbf{z}_i) - \mu([\mathbf{0}, \mathbf{x})),$$

$0 \leq i \leq N-1$. Then we have $\mathbb{E}[\xi_{\mathbf{x}}^{(i)}] = 0$ and $|\xi_{\mathbf{x}}^{(i)}| \leq 1$ for every $0 \leq i \leq N-1$. From Hoeffding's inequality (see for example [18]) it follows that for each $\mathbf{x} \in C_s$ we have

$$\mathbb{P} \left[\left| \frac{1}{N} \sum_{i=0}^{N-1} \xi_{\mathbf{x}}^{(i)} \right| \geq \delta(s, k) \right] \leq 2 e^{-\frac{\delta(s, k)^2 N}{2}}.$$

Hence with Lemma 1 and Remark 1 we obtain

$$\begin{aligned} \mathbb{P}[D_N^{S^*} \leq 2\delta(s, k)] &\geq \mathbb{P} \left[\max_{\mathbf{x} \in \Gamma_{3^k}} \left| \frac{A_N(P, [\mathbf{0}, \mathbf{x}))}{N} - \mu([\mathbf{0}, \mathbf{x})) \right| \leq \delta(s, k) \right] \\ &\geq 1 - (3^k + 1)^s 2 e^{-\frac{\delta(s, k)^2 N}{2}}. \end{aligned}$$

Inserting the value of $\delta(s, k)$ we get

$$\mathbb{P} \left[D_N^{S^*} \leq 2s \left(\frac{3^{s-1}}{3^s - 1} \right)^k \right] \geq 1 - (3^k + 1)^s 2 e^{-\frac{s^2 N}{2} \frac{3^{s-1}}{3^s - 1} 2^k}.$$

The right hand side of the above inequality is strictly positive iff

$$0 > \log 2 + s \log(3^k + 1) - \frac{s^2 N}{2} \left(\frac{3^{s-1}}{3^s - 1} \right)^{2k}.$$

This is true for $k < x_0 = x_0(s, N)$, where x_0 satisfies

$$0 = \log 2 + s \log(3^{x_0} + 1) - \frac{s^2 N}{2} \left(\frac{3^{s-1}}{3^s - 1} \right)^{2x_0}.$$

This is equivalent to

$$\left(\frac{3^{s-1}}{3^s - 1} \right)^{2x_0} = \frac{2}{s^2 N} (s \log(3^{x_0} + 1) + \log 2). \quad (4)$$

It follows that

$$\left(\frac{3^s - 1}{3^{s-1}} \right)^{2x_0} = \frac{s^2 N}{2 (s \log(3^{x_0} + 1) + \log 2)} < \frac{s^2 N}{4 \log 2}$$

and hence

$$x_0 < \log_{\frac{3^s - 1}{3^{s-1}}} \sqrt{\frac{s^2 N}{4 \log 2}}.$$

Now we insert this back into (4) and obtain

$$\left(\frac{3^{s-1}}{3^s - 1} \right)^{2x_0} < \frac{2}{s^2 N} \left(s \log \left(3^{\log_{\frac{3^s - 1}{3^{s-1}}} \sqrt{\frac{s^2 N}{4 \log 2}}} + 1 \right) + \log 2 \right) =: R(s, N)$$

and further (note that $\frac{3^{s-1}}{3^s - 1} < 1$)

$$x_0 > \log_{\frac{3^{s-1}}{3^s - 1}} \sqrt{R(s, N)}.$$

We have shown that if $k \leq \log_{\frac{3^{s-1}}{3^s - 1}} \sqrt{R(s, N)}$, then there exists a point set P_N in C_s consisting of N points such that

$$D_N^{\mathcal{S}^*}(P_N) \leq 2s \left(\frac{3^{s-1}}{3^s - 1} \right)^k.$$

Choosing for $k = \left\lfloor \log_{\frac{3^{s-1}}{3^s - 1}} \sqrt{R(s, N)} \right\rfloor$, we obtain that there exists a point set P_N in C_s consisting of N points such that

$$D_N^{\mathcal{S}^*}(P_N) \leq 2s \left(\frac{3^{s-1}}{3^s - 1} \right)^{\left\lfloor \log_{\frac{3^{s-1}}{3^s - 1}} \sqrt{R(s, N)} \right\rfloor} \leq 2s \frac{3^s - 1}{3^{s-1}} \sqrt{R(s, N)} \leq 6s \sqrt{R(s, N)}.$$

Since (note that $\log(3)/\log(\frac{3^s-1}{3^{s-1}}) = 1/(\alpha_s - s + 1)$)

$$\begin{aligned} R(s, N) &= \frac{2}{s^2 N} \left(s \log \left(3^{\frac{\log 3^s - 1}{3^{s-1}}} \sqrt{\frac{s^2 N}{4 \log 2}} + 1 \right) + \log 2 \right) \\ &= \frac{2}{s^2 N} \left(s \log \left(\left(\sqrt{\frac{s^2 N}{4 \log 2}} \right)^{\frac{1}{\alpha_s - s + 1}} + 1 \right) + \log 2 \right) \\ &\leq \frac{2}{s^2 N} \left((s + 1) \log 2 + \frac{1}{2(\alpha_s - s + 1)} \log \left(\frac{s^2 N}{4 \log 2} \right) \right), \end{aligned}$$

the result follows. \square

3 A van der Corput type construction

Constructions of analogs of the van der Corput sequence on fractals have already been done for the (planar) Sierpiński gasket [10] and for the s -dimensional Sierpiński carpet C_s , for $s \geq 2$ [4]. In both cases the authors state (without explicit proofs) that the elementary discrepancy of the corresponding sequence is of order $\mathcal{O}(1/N)$. In the present section we give the detailed proof for the Sierpiński carpet C_s and we give the exact order of convergence for the carpet (star) discrepancy of the van der Corput sequence on C_s .

Let us define the van der Corput sequence on C_s starting from the approach of this fractal by means of IFS. An Iterated Functions System, in short IFS, is a (finite) family of contractions. In particular, the families of similarities of the same ratio $r < 1$ are IFS's. This often occurs when one constructs so-called *self-similar fractals*. It is known that, if $\{f_1, f_2, \dots, f_m\}$ is a family of contractions on a closed set $D \subset \mathbb{R}^s$, then there exists a unique non-empty compact set F that is invariant for the f_i , $i = 1, \dots, m$, i.e., which satisfies $F = \cup_{i=1}^m f_i(F)$. This unique set F is called the invariant set or the attractor of the IFS $\{f_1, f_2, \dots, f_m\}$. For more details about IFS's we refer e.g. to [2] and [8].

In fact, C_s can be obtained as the attractor of an IFS consisting of $3^s - 1$ affine contractions on $[0, 1]^s$,

$$f_m(y_1, \dots, y_s) = \left(\frac{1}{3}y_1 + \frac{\alpha_1}{3}, \frac{1}{3}y_2 + \frac{\alpha_2}{3}, \dots, \frac{1}{3}y_s + \frac{\alpha_s}{3} \right),$$

where

$$m \in \{0, \dots, 3^s - 1\} \setminus \left\{ \frac{3^s - 1}{2} \right\}, \quad m = \sum_{i=1}^s \alpha_i \cdot 3^{s-i}, \alpha_i \in \{0, 1, 2\}.$$

We have chosen the indices of the functions f_m such that the following condition is satisfied: if $\mathbf{p} \in C_s$ is a point lying in the interior of a 1-cube R and \mathbf{p} has Cartesian coordinates y_i , $1 \leq i \leq s$, with the ternary digit expansions

$$y_i = \frac{\epsilon_1^i}{3} + \frac{\epsilon_2^i}{3^2} + \frac{\epsilon_3^i}{3^3} + \dots,$$

where $\epsilon_l^i \in \{0, 1, 2\}$ for $1 \leq i \leq s$ and $l = 1, 2, \dots$, then, for $m = \sum_{i=1}^s \epsilon_1^i \cdot 3^{s-i} \in \{0, \dots, 3^s - 1\} \setminus \left\{ \frac{3^s - 1}{2} \right\}$ we have $f_m([0, 1]^s) = R$. This IFS provides addresses for all points of the fractal set C_s . For details regarding IFS-addresses we refer e.g. to [2]. The special case of the Sierpiński carpet has already been approached in [4]. In the following we will use the addresses provided by the IFS mentioned above in order to define a particular sequence $\omega = (\mathbf{x}_n)_{n \geq 0}$ on C_s .

For this purpose we expand every integer $n \geq 0$ in the base $3^s - 1$:

$$n = \sum_{l=0}^L \alpha_{l+1}(n) \cdot (3^s - 1)^l, \quad \alpha_{L+1}(n) \neq 0$$

and define

$$\delta_l(n) = \begin{cases} \alpha_l(n) & \text{if } \alpha_l(n) \neq \frac{3^s - 1}{2}, \\ 3^s - 1 & \text{if } \alpha_l(n) = \frac{3^s - 1}{2}, \end{cases}$$

for any $l \geq 1$.

We define \mathbf{x}_n as the point encoded (in the sense of IFS-addresses) by

$$(\delta_1(n), \delta_2(n), \dots, \delta_{L+1}(n), 0^\infty).$$

The sequence ω of points on C_s which is defined in this way is a C_s -analogue of the van der Corput sequence in the unit interval in \mathbb{R} (see e.g. [13]) and we call it van der Corput sequence on C_s .

Theorem 3 *Let ω be the van der Corput sequence on C_s . Then for any $N \geq 1$*

1. for the elementary discrepancy we have

$$\frac{1}{N} \leq D_N^{\mathcal{E}}(\omega) \leq c_1(s) \frac{1}{N},$$

2. for the carpet (star) discrepancy we have

$$c_2(s) \frac{1}{N^{1-\frac{s-1}{\alpha_s}}} \leq D_N^{\mathcal{S}^*}(\omega) \leq D_N^{\mathcal{S}}(\omega) \leq c_3(s) \frac{1}{N^{1-\frac{s-1}{\alpha_s}}}.$$

Here $c_i(s) > 0$, $1 \leq i \leq 3$, depend only on s .

Proof. 1. The left-hand inequality is trivial. For the right-hand inequality let us denote, for $k \in \mathbb{N}_0$, by \mathcal{E}_k the set of elementary intervals of level k and let

$$D_N^{(k)}(\omega) := \sup_{E \in \mathcal{E}_k} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_E(\mathbf{x}_n) - \mu(E) \right|,$$

where $\omega = (\mathbf{x}_n)_{n \geq 0}$. For simplicity we write $b_s := 3^s - 1$, the number of $(l+1)$ -cubes contained in an l -cube for every $l \geq 0$.

In order to find an upper bound for $D_N^{(k)}(\omega)$ let us consider the following cases:

(a) Assume that k is such that $N \leq b_s^k$. Then every k -cube contains at most one point of the set $\{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$, and we obtain

$$D_N^{(k)}(\omega) = \max \left\{ \left| \frac{1}{N} - \frac{1}{b_s^k} \right|, \left| 0 - \frac{1}{b_s^k} \right| \right\} \leq \frac{b_s - 1}{b_s^k} \leq \frac{b_s - 1}{N}.$$

Thus, for any k with $N \leq b_s^k$, we have

$$D_N^{(k)}(\omega) \leq \frac{b_s - 1}{N}.$$

(b) Assume that k is such that $N > b_s^k$. As every k -cube corresponds (e.g. by its IFS-address) to a residual class a modulo b_s^k , $0 \leq a < b_s^k$, for a given elementary cube $E \in \mathcal{E}_k$ we have

$$\sum_{n=0}^{N-1} \mathbf{1}_E(\mathbf{x}_n) = \#\{n < N : n \equiv a \pmod{b_s^k}\} = \left\lfloor \frac{N - a}{b_s^k} \right\rfloor + 1.$$

Therefore we obtain

$$\begin{aligned} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_E(\mathbf{x}_n) - \mu(E) \right| &= \left| \frac{1}{N} \left\lfloor \frac{N-a}{b_s^k} \right\rfloor + \frac{1}{N} - \frac{1}{b_s^k} \right| \\ &\leq \max \left\{ \frac{a}{N \cdot b_s^k}, \left| \frac{1}{N} - \frac{a}{N \cdot b_s^k} \right| \right\} \leq \frac{1}{N}. \end{aligned}$$

Thus we have shown that the inequality

$$D_N^{(k)}(\omega) \leq \frac{b_s - 1}{N} = \frac{3^s - 2}{N}$$

holds for every $k \in \mathbb{N}_0$. The result follows (with $c_1(s) = 3^s - 2$).

2. The upper bound follows from 1. and Proposition 1.

To prove the lower bound let $N \geq 1$ and let k be the unique integer with $(3^s - 1)^{k-1} \leq N < (3^s - 1)^k$. Then the points $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ of the van der Corput sequence are the vertices of elementary cubes of level k that are closest to the origin. Hence the interval

$$E_{k+1} := \left[0, \frac{3^{k+1} - 1}{3^{k+1}} \right)^s \cap C_s$$

contains all the points $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$, i.e., $A_N(\omega, E_{k+1})/N = 1$. We obtain

$$D_N^{S^*} \geq 1 - \mu(E_{k+1}) = \mu(C_s \setminus E_{k+1}).$$

The set $C_s \setminus E_{k+1}$ is a disjoint union of $(3^{k+1})^s - (3^{k+1} - 1)^s$ elementary cubes of level $k+1$ each of which has normalized Hausdorff measure $1/(3^s - 1)^{k+1}$. Thus

$$\begin{aligned} \mu(C_s \setminus E_{k+1}) &= \frac{1}{(3^s - 1)^{k+1}} ((3^{k+1})^s - (3^{k+1} - 1)^s) \\ &\geq \frac{1}{(3^s - 1)^{k+1}} s (3^{k+1} - 1)^{s-1} \geq c(s) \frac{(3^k)^{s-1}}{(3^s - 1)^{k-1}} \\ &\geq c(s) \frac{(3^{\log_{3^s-1} N})^{s-1}}{N} = c(s) \frac{1}{N^{1-\frac{s-1}{\alpha_s}}}, \end{aligned}$$

which is the desired result. □

4 Walsh functions on the carpet and applications

In this section we aim to give a version on the Sierpiński carpet of the well-known Erdős-Turán-Koksma (ETK) inequality (see e.g. [12]) for the carpet star discrepancy. The crucial point in the proof of such inequalities is the Fourier expansion of indicator functions, with respect to some orthonormal \mathcal{L}_2 -basis. Here, we choose a kind of Walsh function system as such a basis. Also, we define a way to employ “digital” methods and apply the ETK-inequality to give a discrepancy bound for digital point sets on the carpet.

As in the previous section, we will make use of the IFS-address of a point. We want to make sure that each point gets a unique address. For a given point \mathbf{p}_0 we proceed as follows: first assign digits to the elementary cubes of level 1. Then determine in which of them the point lies to obtain the first digit; note that the elementary cubes partition the carpet so that there is only one such cube. Then repeat the process with the points $\mathbf{p}_{n+1} = 3\mathbf{p}_n \bmod [0, 1]^s$, $n \geq 0$ to get the next digits. (We assume the same assignment of digits to n -cubes in each level, but actually the introduction of permutations at any point of the process seems to be a merely technical but otherwise not too difficult generalization.)

If, say, the l -th IFS-digit is nonzero but all subsequent digits are zero, we call l the IFS-length. The IFS-address of an elementary cube shall be the common initial IFS-digit vector of all points inside that cube.

First we define the Walsh function system on the Sierpiński carpet.

Definition 1 Let $b_s := 3^s - 1$. If $k = \sum_{i=1}^n k_i b_s^{i-1}$ is the b_s -adic expansion of $k \in \mathbb{N}_0$, then the k -th Walsh function on the Sierpiński carpet is constant on all n -cubes and for all $\mathbf{x} \in C_s$ in an n -cube with IFS-address (x_1, \dots, x_n) it attains the value

$${}_C \text{wal}_k(\mathbf{x}) := \exp(2\pi i \cdot (k_1 x_1 + \dots + k_n x_n) / b_s).$$

(In particular, ${}_C \text{wal}_0$ is constant 1 on the whole carpet.)

Earlier definitions of Walsh functions have been defined on the unit cube $[0, 1]^s$, using the b -adic expansion for an arbitrary $b > 1$, also Cantor digit expansions have been considered. (See, e.g. [16, 19] for an introduction and further references.)

We state some properties of the Walsh function system.

Proposition 2 1. $\mathcal{W}_{C_s} := \{c_s \text{wal}_k\}_{k \geq 0}$ is a complete orthonormal basis on the space of square-integrable functions on the Sierpiński carpet with respect to the normalized Hausdorff measure, $\mathcal{L}_2(C_s, \mu)$.

2. If we define an abelian group on C_s by introducing as group operation \oplus , the digit-wise addition modulo b_s of the IFS-addresses of points, then \mathcal{W}_{C_s} is the dual or character group of C_s . In particular

$$\begin{aligned} c_s \text{wal}_k(\mathbf{x}) \oplus c_s \text{wal}_k(\mathbf{y}) &= c_s \text{wal}_k(\mathbf{x} \oplus \mathbf{y}) \\ c_s \text{wal}_k(\mathbf{x}) \oplus c_s \text{wal}_l(\mathbf{x}) &= c_s \text{wal}_{k \oplus l}(\mathbf{x}) \\ c_s \text{wal}_k(\ominus \mathbf{x}) &= \overline{c_s \text{wal}_k(\mathbf{x})} \\ &= c_s \text{wal}_{\ominus k}(\mathbf{x}), \end{aligned}$$

for all $k, l \in \mathbb{N}_0$, $\mathbf{x}, \mathbf{y} \in C_s$. The operation \oplus on the indices is the digit-wise addition of the b_s -adic expansions; $\ominus \mathbf{x}$ is the inverse of \mathbf{x} in the group.

3. Let m be the length of the b_s -adic expansion of $k \in \mathbb{N}$ (i.e., $m = \lfloor \log_{b_s}(k) \rfloor + 1$), let \mathcal{E}_n be any n -cube of C_s and \mathbf{x} an arbitrary point of the carpet inside that n -cube, then

$$\int_{\mathcal{E}_n} c_s \text{wal}_k d\mu = \begin{cases} 0 & \text{if } n < m, \\ b_s^{-n} \cdot c_s \text{wal}_k(\mathbf{x}) & \text{if } n \geq m. \end{cases}$$

4. The sum $\sum_{k=0}^{b_s^n - 1} c_s \text{wal}_k$ evaluates to the indicator function of $C_s \cap [0, b_s^{-n})^s$.

Proof. Defining a homeomorphism of the s -dimensional carpet to $[0, 1)$ by the bijective map using the IFS-address of a point and the induced topology on the carpet, the properties follow immediately from those of the generalized (univariate) Walsh function system in base b_s . For example, 3 follows from the fact that b -adic intervals of $[0, 1)$ get mapped to elementary cubes of the Sierpiński carpet. \square

As the next step we will need the Walsh-Fourier expansion of the indicator (sometimes called characteristic) functions of half-open intervals with a corner point in the origin. (These are the sets in S^* in the definition of the carpet star discrepancy $D_N^{S^*}$.)

Lemma 2 For $\mathbf{a} \in [0, 1]^s$, let

$$I_{\mathbf{a}} = I_{(a_1, \dots, a_s)} := C_s \cap \prod_{i=1}^s [0, a_i],$$

and $\mathbf{1}_{I_{\mathbf{a}}}$ the corresponding indicator function. For $0 < k \in \mathbb{N}$, let l be the length of the b_s -adic expansion of k .

Then $\hat{\mathbf{1}}_{\mathbf{a}}(0) = \mu(I_{\mathbf{a}})$ and for $k > 0$ the modulus of the k -th carpet Walsh-Fourier coefficient is bounded by

$$|\hat{\mathbf{1}}_{\mathbf{a}}(k)| := \left| \int_{C_s} \mathbf{1}_{I_{\mathbf{a}}} \cdot \overline{C_s \text{wal}_k} d\mu \right| \leq \mu(I_{\mathbf{a}}) - \mu(I_{(a_1(l-1), \dots, a_s(l-1))}) \leq \delta(s, l),$$

where, as in Lemma 1, $\delta(s, l) = s(3^{l(s-1)})/(3^s - 1)^l$ and by $x(m)$ for $x \in [0, 1]$, $m \in \mathbb{N}$ we denote the truncation of x to m ternary digits, i.e., $x(m) := \lfloor 3^m x \rfloor / 3^m$.

Furthermore, if \mathbf{a} lies on the 3-adic grid $\Gamma_{3^L} = [0, 1]^s \cap 3^{-L}\mathbb{N}_0^s$ for some $L \in \mathbb{N}_0$ then $\hat{\mathbf{1}}_{\mathbf{a}}(k)$ vanishes for $k > b_s^L$. In this case the Walsh-Fourier expansion is finite.

Proof. By Proposition 2(3), we can disregard the integral over $I_{(a_1(l-1), \dots, a_s(l-1))}$ since this interval is a union of level $(l-1)$ elementary cubes, where the integral vanishes. So

$$\int_{C_s} \mathbf{1}_{I_{\mathbf{a}}} \cdot \overline{C_s \text{wal}_k} d\mu = \int_{C_s} \mathbf{1}_{I_{\mathbf{a}} \setminus I_{(a_1(l-1), \dots, a_s(l-1))}} \cdot \overline{C_s \text{wal}_k} d\mu.$$

The bound follows immediately from application of the triangle inequality on the above equation. For the upper bound $\delta(s, l)$ cf. the proof of Lemma 1.

An immediate consequence of the above is that for $\mathbf{a} \in 3^{-L}\mathbb{N}_0^s$ for some $L \in \mathbb{N}_0$ clearly the bound becomes zero for $l > L$ and so the coefficient does as well. \square

We are now ready to prove

Theorem 4 (*ETK-inequality on the Sierpiński carpet*)

Let P be a point set of cardinality N on C_s . For any $L, s \in \mathbb{N}$,

$$\begin{aligned} D_N^{\mathcal{S}^*}(P) &\leq \delta(s, L) + \sum_{k=1}^{b_s^L-1} \delta(s, \lfloor \log_{b_s}(b_s k) \rfloor) |S_N(k)| \\ &\leq sb_s^{(s-1)/\alpha_s} \left(\frac{1}{N^{1-\frac{(s-1)}{\alpha_s}}} + \sum_{k=1}^{b_s^L-1} \frac{s}{k^{1-\frac{(s-1)}{\alpha_s}}} |S_N(k)| \right) \end{aligned}$$

where $S_N(k) := \sum_{\mathbf{p} \in P} \mathbf{1}_{C_s \text{wal}_k}(\mathbf{p})$ and $\delta(s, L)$ as in Lemma 2. Furthermore, for $s \geq \log_3(L)$,

$$D_N^{\mathcal{S}^*}(P) \leq 9s \left(\frac{1}{N^{1/\alpha_s}} + 9s \sum_{k=1}^{b_s^L-1} \frac{1}{k^{1/\alpha_s}} |S_N(k)| \right).$$

Proof. First we perform a discretization using Lemma 1 of Section 2, i.e.,

$$D_N^{\mathcal{S}^*} \leq \delta(s, L) + \max_{\mathbf{x} \in \Gamma_{3^L}} \left| \frac{A_N(P, [\mathbf{0}, \mathbf{x}])}{N} - \mu([\mathbf{0}, \mathbf{x}]) \right| =: \delta(s, L) + \max_{\mathbf{x} \in \Gamma_{3^L}} |\Delta_N(P, \mathbf{x})|.$$

We now consider the second summand more closely.

Let $I_{\mathbf{x}}$ be the interval spanned between the origin and $\mathbf{x} \in \Gamma_{3^L}$. Let $\mathbf{1}_{\mathbf{x}}$ be its indicator function (i.e., assuming 1 on $I_{\mathbf{x}}$ and 0 otherwise). We can use the Carpet-Walsh series expansion of $\mathbf{1}_{\mathbf{x}}$ (which exists and converges by Proposition 2(1)) to calculate the local star discrepancy of the points \mathbf{p} of a point set P as follows.

$$\begin{aligned} |\Delta_N(P, \mathbf{x})| &= \left| \frac{1}{N} \sum_{\mathbf{p} \in P} (\mathbf{1}_{\mathbf{x}}(\mathbf{p}) - \mu(I_{\mathbf{x}})) \right| = \left| \frac{1}{N} \sum_{\mathbf{p} \in P} \sum_{k=1}^{b_s^L-1} \hat{\mathbf{1}}_{\mathbf{x}}(k)_{C_s \text{wal}_k}(\mathbf{p}) \right| \\ &= \left| \sum_{k=1}^{b_s^L-1} \hat{\mathbf{1}}_{\mathbf{x}}(k) \left(\frac{1}{N} \sum_{\mathbf{p} \in P} \mathbf{1}_{C_s \text{wal}_k}(\mathbf{p}) \right) \right| \leq \sum_{k=1}^{b_s^L-1} |\hat{\mathbf{1}}_{\mathbf{x}}(k)| \cdot |S_N(k)|. \end{aligned}$$

Now the first part of the result follows from the bound on $|\hat{\mathbf{1}}_{\mathbf{x}}(k)|$ of Lemma 2 and, for $b_s^{l-1} \leq k < b_s^l$,

$$\delta(s, l) = s \left(\frac{3^{s-1}}{3^{\alpha_s}} \right)^l = s \frac{b_s^{l(s-1)/\alpha_s}}{b_s^l} < s \frac{b_s^{(s-1)/\alpha_s}}{k^{1-\frac{(s-1)}{\alpha_s}}}.$$

For the second part note that

$$\delta(s, l) = s \left(\frac{3^{s-1}}{3^{\alpha_s}} \right)^l \leq 3s \frac{3^{L(s-\alpha_s)}}{k^{1/\alpha_s}}$$

and the fact that for $s > \log_3(L)$ we have $s(1-1/L) < \alpha_s$, thus $L(s-\alpha_s) < 1$.
 \square

Another way to obtain a sequence in the Sierpiński carpet is the so-called *digital method*. In the context of the current setting this would mean to proceed as follows.

Definition 2 Let an infinite matrix $C \in \mathbb{Z}_{b_s}^{\infty \times \infty}$ be given such that the upper left $m \times m$ sub-matrices are regular for all $m \in \mathbb{N}$. To obtain the n -th point of an *IFS-digital sequence*, consider the infinite b_s -adic digit vector \vec{d}_n of n as a vector in \mathbb{Z}_{b_s} . Then the IFS-digits of the n -th point shall be given by the coordinates of $C\vec{d}_n$. (For safety, C should be chosen such that in each column almost all entries are 0 to avoid ambiguous IFS-addresses.)

The above procedure corresponds to a b_s -adic digital $(0, 1)$ -sequence (see e.g. [17, Ch.4], [6, Sec. 3.1.2], for more information about digital sequences on the unit cube).

We can now give the carpet star discrepancy of finite point sets of such sequences.

Corollary 1 For $L, s \in \mathbb{N}$ let P_N be the set of the first $N = b_s^L$ points of an IFS-digital sequence $(\mathbf{x}_n)_{n \geq 0}$ on the Sierpiński carpet. Then

$$D_N^{S^*}(P_N) \leq \frac{s}{N^{1-\frac{s-1}{\alpha_s}}}.$$

Proof. This follows immediately from Theorem 4 if we show $S_N(k) = 0$ for $k > 0$. (This is a simple consequence of the character properties of the Walsh function system.) For $m \in \mathbb{N}$, let \vec{d}_m denote the infinite vector of the b_s -adic digits m_1, m_2, \dots of m . Then, with $\mathbf{x}_n = C\vec{d}_n$ and $0 \leq k < b_s^L$,

$$\begin{aligned} S_N(k) &= \sum_{n=0}^{N-1} C_s \text{wal}_k(\mathbf{x}_n) = \sum_{n=0}^{N-1} \exp(2\pi i (\vec{d}_k | C\vec{d}_n) / b_s) \\ &= \sum_{n_1, \dots, n_L=0}^{b_s-1} \prod_{j=1}^L \exp\left(2\pi i \frac{(\vec{d}_k | \vec{c}_j) n_j}{b_s}\right) = \prod_{j=1}^L \sum_{n=0}^{b_s-1} \exp\left(2\pi i \frac{(\vec{d}_k | \vec{c}_j) n}{b_s}\right), \end{aligned}$$

where \vec{c}_j is the j -th column vector of C and $(\vec{x}|\vec{y})$ denotes the scalar product. The product can be nonzero only if $(\vec{d}_k|\vec{c}_j) = 0$ for all j . But by the definition of C , the vectors $\vec{c}_j(L)$ (i.e., truncated to length L) are linearly independent in $\mathbb{Z}_{b_s}^L$, so this can only happen for $\vec{d}_k = \mathbf{0}$, i.e., $k = 0$.

□

Remark 2 By choosing C to be the identity matrix we obtain the van der Corput sequence of Section 3 and get the same upper bound order of $D_N^{S^*}$ for $N = b_s^L$ points.

References

- [1] Albrecher H, Matoušek J, and Tichy R F (2000) Discrepancy of point sequences on fractal sets. *Publicationes Mathematicae Debrecen*, **56/3-4**: 233–249
- [2] Barnsley M F (1993) *Fractals Everywhere*, Boston: Academic Press
- [3] Beck J (1984) Some upper bounds in the theory of irregularities of distribution. *Acta Arithmetica*, **44** No. 2: 115–130
- [4] Cristea L L (2002) *Geometric Properties of Fractals*, TU Graz: PhD thesis
- [5] Cristea L L, Tichy R F (2003) Discrepancies of point sequences on the Sierpiński carpet. *Math Slovaca* **53** No. 4: 351–367
- [6] Drmota M, Tichy R F (1997) *Sequences, Discrepancies and Applications* volume 1651 of *Lecture Notes in Mathematics*, Berlin: Springer
- [7] Falconer K J (1985) *The Geometry of Fractal Sets*, Cambridge: Cambridge University Press
- [8] Falconer K J (1990) *Fractal Geometry*, Chichester: John Wiley & Sons
- [9] Falconer K J (1997) *Techniques in Fractal Geometry*, Chichester: John Wiley & Sons
- [10] Grabner P J, Tichy R F (1998) Equidistribution and Brownian Motion on the Sierpiński Gasket. *Monatsh Math*, **125**: 147–164

- [11] Heinrich S, Novak E, Wasilkowski G, Woźniakowski H (2001) The inverse of the star-discrepancy depends linearly on the dimension. *Acta Arith*, **96**: 279–302
- [12] Hellekalek P (1994) General Discrepancy estimates: the Walsh function system. *Acta Arithmetica*, **67**, no.3: 209–218
- [13] Hlawka E (1984) *The Theory of Uniform Distribution*, (Translated from the German by Henry Orde with a foreword by S K Zaremba), *Topics and Texts in Math*, Vol 1, Reading: A B Academic Publishers (Eastern Press)
- [14] Klinger B (1998) *Discrepancy of Point Sequences and Numerical Integration*, TU Graz: PhD thesis
- [15] Kuipers L , Niederreiter H (1974) *Uniform Distribution of Sequences*, New York: John Wiley
- [16] Larcher G , Piršic G (1999) Base change problems for generalized Walsh series and multivariate numerical integration. *Pacific J Math* **189**, no 1: 75–105
- [17] Niederreiter H (1992) Random number generation and quasi-Monte Carlo methods, *CBMS-NSF Regional Conference Series in Applied Mathematics*, 63, Philadelphia: SIAM
- [18] Pollard D (1984) *Convergence of Stochastic Processes*, New York: Springer
- [19] Schipp F, Wade W R, Simon P (1990) *Walsh series An introduction to dyadic harmonic analysis*, Bristol: Adam Hilger, Ltd
- [20] Schmidt W M (1972) Irregularities of distribution, VII. *Acta Arith*, **21**: 45–50

Authors' Addresses:

Ligia L Cristea, Institut für Mathematik C, TU Graz, Steyrergasse 30 / III, A - 8010 Graz, Austria (Department of Mathematics, Politehnica University

of Timisoara, Romania). Email: cristea@finanz.math.tu-graz.ac.at

Friedrich Pillichshammer, Institut für Analysis, Universität Linz, Altenbergstraße 69, A-4040 Linz, Austria. Email: friedrich.pillichshammer@jku.at

Gottlieb Pirsic, Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria. Email: gottlieb.pirsic@oeaw.ac.at

Klaus Scheicher, Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria. Email: klaus.scheicher@oeaw.ac.at