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RICAM-Report 2004-09

Global Uniqueness and Hölder Stability for Recovering a Nonlinear Source Term in a Parabolic Equation ‡

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Abstract. Consider the semilinear parabolic equation

$$-u_t(x,t) + u_{xx} + q(u) = f(x,t),$$

with the initial condition

$$u(x,0) = u_0(x),$$

Dirichlet boundary conditions

$$u(0,t) = \varphi_0(t), \qquad u(1,t) = \varphi_1(t)$$

and a sufficiently regular source term $q(\cdot)$, which is assumed to be known *a priori* on the range of $u_0(x)$. We investigate the inverse problem of determining the function $q(\cdot)$ outside this range from measurements of the Neumann boundary data

$$u_x(0,t) = \psi_0(t), \qquad u_x(1,t) = \psi_1(t).$$

Via the method of Carleman estimates, we derive global uniqueness of a solution (u,q) to this inverse problem and Hölder stability of the functions u and q with respect to errors in the Neumann data ψ_0, ψ_1, the initial condition u_0 and the a priori knowledge of the function q (on the range of u_0). These results are illustrated by numerical tests. The results of this paper can be extended to more general nonlinear parabolic equations.

1. Introduction

For some diffusion processes, e.g., heat transfer problem, linear equations are a sufficiently accurate model for the underlying physical system. However, in many technical and industrial applications, in particular for large ranges of temperatures, nonlinear effects, which may be due to temperature dependence of material parameters or radiation, have to be taken into account. Nonlinear heat transfer laws appear, e.g., in the modelling of cooling processes for steel or glass in liquids and gases, e.g., in the continuous casting of steel [13]. Nonlinear diffusion equations also arise in furnace reactions (see, e.g., [40]).

‡ supported by the the Austrian National Science Foundation FWF under grant SFB F013/08

Prominent examples for inverse problems in diffusion processes are backwards or sideways heat equation, and a variety of parameter identification problems (for an account of some important inverse problems in diffusion see the Proceedings [9], or Beck, Blackwell and Clair [2] and Alifanov [1] for an overview over inverse heat conduction problems). Note, that most of these inverse problems are ill-posed, i.e. their solution depends unstably on the data, and thus have to be solved via regularization techniques.

Stable identification of space or/and time dependent parameters or source terms leads to (nonlinear) inverse problems typically governed by linear parabolic equations, see, e.g., [11]. An analysis and stable numerical treatment of unknown parameters depending on the the physical state, i.e., identification of nonlinearities, see e.g. [32, 33], is more involved, since already the underlying equation is nonlinear (nonlinear parameters in boundary conditions have been treated for instance in [39]).

An important issue is also the availability of data: While for some applications, e.g., inverse problems in groundwater filtration, see, e.g., [14], it is reasonable to assume distributed measurements (measurements of the state u on the whole domain), in many cases measurements will be possible only at the boundary. Thus, identification from a single set of or possibly multiple boundary measurements is of special interest.

In this paper we consider a simple model problem (1.1) for a nonlinear diffusion process (we think of heat transfer and thus call u the "temperature"), and show that under reasonable assumptions the nonlinearity, in our case a nonlinear source term, can be uniquely and stably identified over a wide range of temperatures by a single experiment in a simple setup from one set of boundary measurements. We point out possible extensions of our result to more general equations later in this section. Note, that a parameter q depending on the physical state u can be determined only on the range of states, which are actually reached; in general, this range is unknown and depends on the parameter q. However, in an experimental setup, we have in mind, this range is known *a priori*. We give a short description of the experimental setup:

The experiment is started at low temperature, where material parameters are usually known (or in some cases even constant), with the initial temperature decreasing from the left to the right. Then we start heating the left boundary, while the right one is kept cooler than the left (for example at constant temperature) and measure the heat flux over the boundaries.

Heating on the left boundary is continued until the temperature range, on which we want to determine $q(u)$, is exhausted. In order to guarantee stable reconstruction of the nonlinear parameter, the temperature on the left boundary has to be increased even slightly more (see Theorem 2 below).

By the method of Carleman estimates, we show that with such an experiment, $q(u)$ in (1.1) can be uniquely and stably determined and illustrate this by numerical tests. With this setup in mind, we now state our problem and assumptions in detail:

Let $\Omega := (0, 1)$, $Q_T := \Omega \times (0, T)$ for some $T > 0$. We consider the quasilinear parabolic equation

$$-u_t + u_{xx} + q(u) + f(x, t) = 0, \qquad (x, t) \in Q_T \tag{1.1}$$

with sufficiently regular functions q and f, initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega \tag{1.2}$$

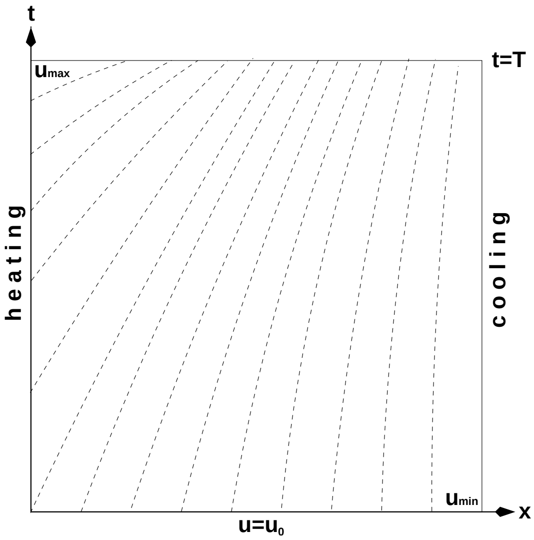

Figure 1. Isolines in a typical experiment

and Dirichlet boundary conditions

$$u(0,t) = \varphi_0(t), \qquad u(1,t) = \varphi_1(t), \qquad t \in (0,T). \tag{1.3}$$

The Neumann boundary data

$$u_x(0,t) = \psi_0(t), \quad u_x(1,t) = \psi_1(t), \qquad t \in (0,T) \tag{1.4}$$

(given by measurements) are to be used to identify the function $q(u)$. Motivated by the experimental setup as described above, we make the following assumptions on the functions q, φ_i and solutions u of (1.1)-(1.3)

(A) There exists a constant $\gamma > 0$, such that

$$\varphi_0'(t) \geq \gamma, \quad \varphi_1'(t) \geq 0, \quad \text{for} \quad t \in (0,T),\ i = 0, 1,$$

(B) $u \in C^{4+\theta, 2+\theta/2}(\overline{Q_T})$ for some $0 < \theta < 1$, and

(C) there exists a constant $\gamma_1 > 0$ such that $u_x(x,t) < -\gamma_1$ on Q_T.

Assumption (A) simply requires to control the temperature at the boundary in an appropriate way. Condition (C), which is a key ingredient in the proof of our results, is reasonable, if the temperature difference between the two boundaries is kept sufficiently large. Sufficient conditions for (C) can be derived by differentiating (1.1)-(1.3) with respect to x and using maximum principles (see [12, 34]), e.g., $\psi_0(t)$, $\psi_1(t)$, $u_0'(x) \leq -C$ and $f(x,t) = 0$ would be sufficient, if $|q(u)|$ is bounded, which is again reasonable to assume. The conditions on the measurements ψ_i can be checked during the experiment. Note that under assumptions (A) and (C), $u(x,t)$ takes values in

$$D := [\varphi_1(0), \varphi_0(T)]$$

only. Hence we may assume q to be a function on D only. We further assume that q is sufficiently smooth, i.e.,

(D) $q \in C^{2+\theta}(D)$.

For existence and regularity of solutions u to (1.1)-(1.3) (assumption (B)), and the necessary compatibility conditions for boundary and initial conditions, we refer to Ladyzhenskaya, Solonikov and Ural'ceva [34] or Friedman [12].

In the sequel we will assume that

(E) $\varphi_1(T) \leq \varphi_0(0)$,

which is satisfied, if the temperature on the right boundary $\varphi_1(t)$ is kept constant. Otherwise, this assumption can be made without loss of generality, since one always can find a finite number n and time horizons $0 < \tau = T_0 < T_1 < \ldots < T_n = T$ such that on each subdomain $Q_i := (0,1) \times [T_i - \tau, T_{i+1}]$, $i = 0, \ldots, n-1$ the condition

$$\varphi_1(T_{i+1}) \leq \varphi_0(T_i)$$

is satisfied. Theorems 1 and 2 below then can be applied iteratively to the subdomains $Q_i^\tau := (0,1) \times [T_i - \tau, T_{i+1}]$. Now consider the following

Inverse Problem 1
Assume that

$$u_x(0,t) = \psi_0(t), \quad u_x(1,t) = \psi_1(t), \qquad t \in (0,T)$$

are given and the function $q(y)$ is known for $y \in D_0$, where

$$D_0 := [\varphi_1(0), \varphi_0(0)]$$

Determine the functions u and q satisfying (1.1)-(1.3).

The assumption that q is known on D_0 is essential for our proofs, but not too restrictive, since D_0 can be an arbitrarily small interval and the temperature is assumed to be low there (see the remarks on the experimental setup above).

In this paper we investigate uniqueness and stability of solutions to this Inverse Problem under assumptions (A)-(E). Obviously, by (A) and (C), and since q is assumed to be known on D_0, it is sufficient to determine q on

$$D_1 := \{y : \varphi_0(0) < y \leq \varphi_0(T)\}.$$

Inverse coefficient and source problems for parabolic equations are well studied in the literature. Uniqueness results for 1-D inverse problems for quasilinear parabolic equations with single boundary measurements have previously been derived by Muzylev [37] (for a piecewise analytic coefficient $q(u)$ in a parabolic equation), Pilant and Rundell [38]. Kügler [32] and Kügler and Engl [33] investigated uniqueness and stability via regularization for quasilinear parabolic and elliptic equations. Note that there is a close connection between stability estimates and convergence rates for regularization methods for inverse problems (see, e.g., [8, 6, 22]).

The method of Carleman estimates for inverse problems (introduced by Klibanov [24], Bukhgeim [3], Bukhgeim and Klibanov [4]) has been extensively used for studying uniqueness of inverse coefficient problems for partial differential equations by single boundary measurements. Klibanov [28], for instance, proved global uniqueness for a general nonlinear 1-D parabolic equation (under the assumption that the solution is measured at $n+1$ interior points, where n is the number of coefficients to be determined). The method of Carleman estimates was also applied for the proof of global uniqueness in a multidimensional inverse problem for nonlinear elliptic equations [26, 29]. However, the unknown coefficient there does not depend on the solution of the equation. In other publications, Carleman estimates were applied to inverse problems for linear parabolic equations, see, e.g., Imanuvilov and Yamamoto [19], Imanuvilov, Isakov and Yamamoto [18], Klibanov [25, 27, 29], Lin and Wang [36] and references cited there. Recently, Klibanov [30] proved global uniqueness of a nonlinear source term in a multidimensional parabolic problem.

For an overview of the method of Carleman estimates applied to inverse coefficient problems, see the recently published monograph [31]; on inverse source problems see, e.g., the book by Isakov [20].

In Engl, Scherzer, Yamamoto [10], controllability arguments have been used to identify source terms in linear parabolic and hyperbolic equations. The question of controllability of linear and quasilinear parabolic equations by boundary measurements has further been considered in Chae, Imanuvilov and Kim [5] and Imanuvilov [16, 17].

Stability for an inverse source problem for a linear parabolic equation has previously been investigated by Imanuvilov and Yamamoto [19], also by the method of Carleman estimates.

More results concerning uniqueness and stability of multidimensional inverse problems for nonlinear parabolic or elliptic equations are available for data obtained by multiple measurements, e.g., in case of availability of (part of) the Dirichlet-to-Neumann map. A series of publications started from the paper of Isakov [21], in which a linearization method was used; see also [22, 23] and the references cited there.

This paper investigates uniqueness and stability of a source term in a nonlinear parabolic pde by a single measurement of the Neumann boundary data. For proving the main theorems below, we will also apply Carleman estimates. However, the results cited above are not applicable to our problem, due to the nonlinearity of the equation or the available data. In particular, also for practicability of the experiment we have in mind, we do not assume $u(x, \theta)$ to be known for some $\theta > 0$, which is a key assumption in some of the previous results on uniqueness (e.g. in [20]) and also in the stability result [19]. Here, we assume only single boundary measurements and knowledge of the initial condition.

With the aim of simplifying the presentation, we are not concerned here with minimal regularity assumptions or the maximal class of problems, to which our results are applicable. We only mention, that with minor modification of the proofs, our results below can also be applied to

$$-u_t + (a_2(x)u_x)_x + a_1(x)u_x + a_0(x)u + q(u) = f$$

or

$$-u_t + u_{xx} + q(u)u_x = f$$

In Klibanov [30] more general nonlinear parabolic equations (in 3-D) of the form

$$u_t = F(\Delta u, \nabla u, u, \vec{x}, t, q(u))$$

are investigated and uniqueness in the corresponding inverse problems is shown.

Our first result is concerned with global uniqueness of solutions $(u(x, t), q(u(x, t)))$ to the Inverse Problem 1:

Theorem 1 *Let u_i, $i = 1,2$ denote the solutions of (1.1)-(1.3) with q replaced by q_i, $i = 1,2$. Suppose that q_i, u_i satisfy (A)-(D) and $q_1(y) = q_2(y)$ for $y \in D_0$. If*

$$u_{1x}(0,t) = u_{2x}(0,t), \quad u_{1x}(1,t) = u_{2x}(1,t) \qquad \text{for} \quad t \in (0,T),$$

then $u_1 = u_2$ in Q_T and $q_1 = q_2$ in D.

Theorem 2 is concerned with stability of solutions to the above inverse problem with respect to error in the Neumann data (1.4), the initial condition (1.2) and the *a priori* knowledge of the nonlinear parameter $q(y)$ for $y \in D_0$:

Theorem 2 *Let u_i, $i = 1,2$ denote the solutions of (1.1),(1.3) with q replaced by q_i, $i = 1,2$ and initial condition*

$$u_i(x,0) = u_{i,0}(x), \qquad x \in [0,1].$$

Suppose that q_i, u_i satisfy (A)-(D), $\|q_i\|_{C^{2+\theta}} \le M$ and the following error bounds hold

$$\|q_1|_{D_0} - q_2|_{D_0}\|_{L_2(D_0)},\ \|u_{01} - u_{02}\|_{L_2(0,1)} \le \varepsilon \tag{1.5}$$

$$\|u_{1x}(0,\cdot) - u_{2x}(0,\cdot)\|_{L_2(0,T)},\ \|u_{1x}(1,\cdot) - u_{2x}(1,\cdot)\|_{L_2(0,T)} \le \varepsilon. \tag{1.6}$$

Then for any sufficiently small $\zeta > 0$ there exist a constant $C > 0$ and a number $\eta = \eta(\zeta) > 0$ independent of u_i, q_i and ε such that

$$\|q_1 - q_2\|_{H^2(D(\zeta))} \le C\,\varepsilon^{\eta},$$

where $D(\zeta) := [\varphi_1(0), \varphi_0(T) - \zeta] \subset D$.

The stability result of Theorem 2 is "almost global" in the sense that for any sufficiently small $\zeta > 0$ the conclusion holds true on domains $[\varphi_1(0), \varphi_0(T) - \zeta]$, but the corresponding Hölder exponents $\eta(\zeta)$ might approach 0 as $\zeta \to 0$. Thus in order to stably identify q on $[\varphi_1(0), \varphi_0(T)]$, we have to continue our experiment and measurements a bit beyond T.

The outline of the paper is as follows: In Section 2, we introduce a new variable and transform (1.1) to a nonlinear equation over a domain with curvilinear boundaries, where the coefficient q only depends on the new variable. By another transformation, we derive a corresponding integro-differential inequality. The main result of Section 3 is the proof of a new pointwise Carleman estimate. For a general outline of the method of Carleman estimates for Cauchy problems we refer to, e.g., Hörmander [15] or Lavrent'ev, Romanov and Shishatskii [35]. In constructing the Carleman weight function we take into account the special features of the problem under consideration. The main difficulty with the Carleman estimate in our case is that only Dirichlet data are available on a part of the curvilinear boundary, which is not a level set of the Carleman weight function, whereas in previous works, Dirichlet and Neumann data were available on such a part of the boundary. The crucial point in our proof is positivity of certain integrals over this part of the boundary (see Lemma 2). Based on the estimates of Section 3, we prove the uniqueness result and the Hölder estimate in Section 4. We conclude with numerical test examples illustrating the theoretical results in Section 5.

2. The Transformed Problem

Employing the monotonicity (see assumption (C)) of a solution u, we now transform (1.1) - (1.3) to a problem, where the unknown function q does no longer depend on the solution u, but on a new spatial variable:

Let $v\,(y,t)$ be defined by

$$u\,(v\,(y,t)\,,t) = y, \tag{2.1}$$

which is possible since by assumption (C)

$$u_x(x,t) < -\gamma \quad \text{for} \quad (x,t) \in Q_T.$$

The variable y serves as a new spatial variable and corresponds to the isolines of the function $u(x,t)$. Application of the chain rule yields

$$u_x = \frac{1}{v_y} < -\gamma, \qquad u_{xx} = -v_{yy}\cdot\frac{1}{v_y^3}, \qquad u_t = -\frac{v_t}{v_y}. \tag{2.2}$$

Substitution into (1.1) - (1.3) yields,

$$v_y^2 \cdot v_t = v_{yy} - q\,(y)\,v_y^3 - f\,(v\,(y,t)\,,t)\cdot v_y^3 \qquad (y,t) \in \overline{G}, \tag{2.3}$$

$$v(y,0) = v_0(y) \qquad y \in D_0, \tag{2.4}$$

$$v(\varphi_1(t),t) = 1, \quad v(\varphi_0(t),t) = 0 \qquad t \in [0,T]. \tag{2.5}$$

where $G = \{\varphi_1(t) < y < \varphi_0(t),\ \ t \in (0,T)\}$ has curvilinear boundaries and the function $v_0(y)$ is the inverse of $u_0(x)$, i.e. $u_0(v_0(y)) = y$. The Neumann data (1.4) are transformed to

$$v_y(\varphi_1(t),t) = \frac{1}{\psi_1(t)}, \quad v_y(\varphi_0(t),t) = \frac{1}{\psi_0(t)}, \qquad t \in (0,T). \tag{2.6}$$

Because of (C), we have $\psi_i(t) < -\gamma$ and hence the boundary values for v are well-defined.

In a next step, we will differentiate (2.3) with respect to t in order to eliminate q from the equation. Before we do that, we describe in more detail the domain where (2.3)-(2.5) are valid.

2.1. The Carleman Weight Function

For the proof of the Carleman estimate in Section 3, certain subdomains of G, which are bounded at the top by the level set of a Carleman weight function, will play a role. Let $0 < \delta < T$ be sufficiently small and define

$$s(y,t) := \alpha\cdot(y - \varphi_1(0)) + \beta\cdot t + \frac{1}{2}, \qquad \alpha,\beta > 0, \tag{2.7}$$

where $\alpha = \alpha(\delta)$, $\beta = \beta(\delta)$ are defined by

$$\alpha = \frac{\delta}{2}[(\varphi_0(T-\delta) - \varphi_1(0))\cdot T - (\varphi_1(T) - \varphi_1(0))\cdot(T-\delta)]^{-1}, \tag{2.8}$$

$$\beta = \frac{1}{2}(\varphi_0(T-\delta) - \varphi_1(T)) \tag{2.9}$$

$$\times\,[(\varphi_0(T-\delta) - \varphi_1(0))\cdot T - (\varphi_1(T) - \varphi_1(0))\cdot(T-\delta)]^{-1}. \tag{2.10}$$

By (A), (C) , (E) and $0 < \delta < T$ it follows that $\alpha,\beta > 0$ (see also Figure 2). Furthermore, define the domains $G_\delta \subset G$ via

$$G_\delta := \{(y,t) \in G \,:\, s(y,t) < 1\}. \tag{2.11}$$

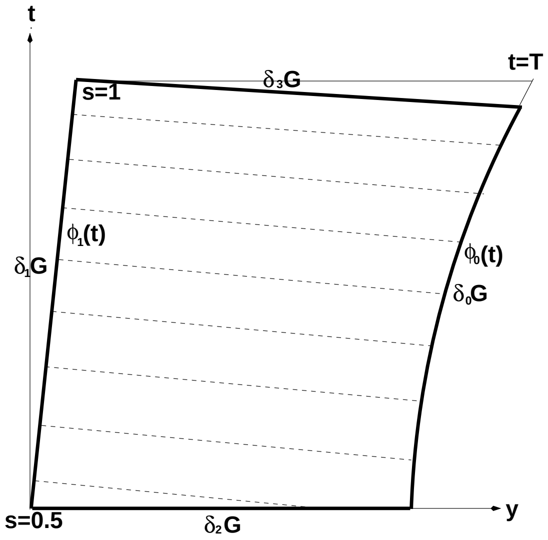

Figure 2. Domain and Isolines of $s(x, y)$

By (2.7) - (2.11), $s(y,t) \in (1/2, 1)$ on G_δ.

In the domain $G_\delta \subset G$, we consider the Carleman Weight Function (CWF) (see [12] and Figure 2)

$$\mathcal{C}_{\lambda,\nu}(y,t) = \exp\left(\lambda s^{-\nu}\right). \tag{2.12}$$

The boundary of the domain G_δ consist of four parts, $\partial G_\delta = \bigcup_{i=0}^{3} \partial_i G_\delta$,

$$\begin{aligned}
\partial_0 G_\delta &= \{(y,t) : y = \varphi_0(t),\ t \in [0, T-\delta]\} \\
\partial_1 G_\delta &= \{(y,t) : y = \varphi_1(t),\ t \in [0, T]\} \\
\partial_2 G_\delta &= \{(y,t) : t = 0,\ y \in (\varphi_1(0), \varphi_0(0))\} \\
\partial_3 G_\delta &= \{(y,t) : s(y,t) = 1,\ y \in (\varphi_1(T), \varphi_0(T-\delta))\}
\end{aligned}$$

2.2. An Integro Differential Inequality

Suppose, there exist two pairs of functions (u_1, q_1) and (u_2, q_2) satisfying (1.1) - (1.4) and assumptions (A) - (D). Let v_1, v_2 be defined via (2.1) and denote $\tilde{q} = q_1 - q_2$ and $\tilde{v} = v_1 - v_2$. Then

$$\begin{aligned}
L\tilde{v} &= v_{1y}^2 \cdot \tilde{v}_t - \tilde{v}_{yy} - c\,(y,t) \cdot \tilde{v}_y - d\,(y,t) \cdot \tilde{v} && (2.1) \\
&= \tilde{q}\,(y)\, v_{1y}^3, && (y,t) \in G,
\end{aligned}$$

$$\tilde{v} = \tilde{v}_y = 0, \qquad (y,t) \in \partial_0 G \cup \partial_1 G, \tag{2.2}$$

$$\tilde{v} = \tilde{v}_t = 0, \qquad (y,t) \in \partial_2 G. \tag{2.3}$$

In the case of inexact Neumann boundary data, which we consider for the stability result, $\tilde{v}_y = 0$ and $\tilde{v}_t$ has to be replaced by $|\tilde{v}_y|, |\tilde{v}_t| \le \varepsilon$ in (2.2), (2.3). By the regularity assumption (B), the coefficients $c\,(y,t)\,, d\,(y,t)$ in (2.2) are in $C^1\left(\overline{G}\right)$. By the Implicit

Function Theorem we have $v \in C^{4+\theta,2+\theta/2}\left(\overline{G}\right)$, because of (C). Furthermore, by (2.2) and (C), $v_{1y} \neq 0$ in $\overline{G}$. Hence, we can choose a positive constant M_1 such that

$$M_1 \geq \max\left[\left\|\frac{1}{v_{1y}}\right\|_{C(\overline{G})}, \quad \|v_{1y}\|_{C^{2,1}(\overline{G})}, \quad \|c\|_{C^1(\overline{G})}, \quad \|d\|_{C^1(\overline{G})}\right]. \tag{2.4}$$

Note that by (2.2), $\tilde{v}\,(\varphi_i(t), t) = 0$ for $i = 0, 1$. Differentiation yields with (2.2)

$$\tilde{v}_t(y,t) = 0, \qquad (y,t) \in \partial_0 G \cup \partial_1 G. \tag{2.5}$$

Dividing (2.2) by v_{1y}^3, we obtain

$$\tilde{q}\,(y) = \frac{L\tilde{v}}{v_{1y}^3}.$$

and consequently

$$\frac{\partial}{\partial t}\left[\frac{L\tilde{v}}{v_{1y}^3}\right] = 0, \qquad \text{in} \quad G.$$

The latter yields

$$[L\tilde{v}]_t - \frac{\left[v_{1y}^3\right]_t}{v_{1y}^3} \cdot L\tilde{v} = 0. \tag{2.6}$$

Now we derive an integro-differential equation for the function

$$w := \tilde{v}_t - \frac{\left[v_{1y}^3\right]_t}{v_{1y}^3} \cdot \tilde{v}: \tag{2.7}$$

(2.7), (2.2) and (2.3) imply

$$\tilde{v}\,(y,t) = \int_{k(y)}^{t} K\,(y,t,\tau)\, w\,(y,\tau)\, d\tau, \quad \text{for} \quad (y,t) \in G, \tag{2.8}$$

with

$$K\,(y,t,\tau) = \frac{v_{1y}^3\,(y,t)}{v_{1y}^3\,(y,\tau)},$$

and

$$k(y) = \begin{cases} 0, & y \in (0, \varphi\,(0)) \\ \tilde{\varphi}_0(y), & y \in (\varphi\,(0)\,, \varphi_0\,(T))\,. \end{cases}$$

Using (2.8) and (2.7) in (2.6) yields

$$Lw + \int_{k(y)}^{t} K_1\,(y,t,\tau)\, w_y\,(y,\tau)\, d\tau + \int_{k(y)}^{t} K_2\,(y,t,\tau)\, w\,(y,\tau)\, d\tau = 0, \tag{2.9}$$

for $(y,t) \in G$ with kernels K_1, $K_2 \in C(\overline{G})$. By (2.2), (2.3), (2.5) and (2.7),

$$w = 0,\ w_y = 0, \qquad (y,t) \in \partial_1 G \tag{2.10}$$

$$w = 0, \qquad (y,t) \in \partial_0 G, \tag{2.11}$$

$$w = 0,\ w_t = 0, \qquad (y,t) \in \partial_2 G. \tag{2.12}$$

The second condition in (2.10) follows from (2.2) and (2.2) by

$$\begin{aligned}\frac{d}{dt}[\tilde{v}_y(\varphi_1(\cdot),\cdot)] &= \tilde{v}_{ty} + \varphi_1' \cdot \tilde{v}_{yy}\\ &= \tilde{v}_{ty} + \varphi_1' \cdot [\tilde{v}_{1y}^2 \cdot \tilde{v}_t - c \cdot \tilde{v}_y - d \cdot \tilde{v} - v_{1y}^3 \cdot \tilde{q}]\\ &= \tilde{v}_{ty},\end{aligned}$$

and since $q(y)$ is assumed to be known on $\partial_1 G$ and consequently $\tilde{q} = 0$ there. (2.12) follows similarly. In the case of error in the Neumann data, only bounds for the left-hand sides in (2.10)-(2.12) are be available.

Instead of (2.9) we will only work with the following integro-differential inequality, which is implied by (2.9):

$$\left|v_{1y}^2 \cdot w_t - w_{yy}\right| \le M_2 \left(|w_y| + |w| + \int\limits_{k(y)}^{t} \left(|w_y\,(y,\tau)| + |w\,(y,\tau)|\right) d\tau \right) \quad (2.13)$$

in G, where the constant M_2 depends on M_1 of (2.4). Below we will work with the inequality (2.13) only, supplied by conditions (2.10)-(2.12). The main difficulty in the the following calculations is that only Dirichlet data are available on the right curvilinear boundary $\partial_0 G$, see (2.11).

3. A Pointwise Carleman Estimate

We now establish a pointwise Carleman estimate for the parabolic operator appearing in the left-hand side of

$$L_0 := a \cdot \frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2}$$

for coefficients $a = a(y,t) \in C^1$. The additional difficulty and novelty in this estimate, as compared with the one in [12], Chapter 4, Section 1 are:

(i) the part $\partial_0 G_\delta$ of the boundary of G_δ is curvilinear and only Dirichlet data is given on this part, see (2.11), whereas in the conventional case, both Dirichlet and Neumann boundary conditions would be given on such a part of the boundary, which is not a level set of the Carleman weight function.

(ii) the proof of non-negativity of certain integrals over the curvilinear boundary $\partial_0 G$ (see Lemma 2).

Lemma 1 below represents a modification of Lemma 2 in [12, Chapter 4, Section 1], which is needed to take into account the above differences to the conventional case. We present the proof in details in the appendix, since certain terms will be needed explicitly for Lemma 2 and the proof of the uniqueness result.

Lemma 1 *Let $s(y,t)$, α, β, δ and G_δ be defined as above (2.7)-(2.11). Additionally, let*

$$a(y,t) \in C^1(\overline{G}_\delta), \qquad a \ge M_3 > 0, \qquad \|a\|_{C^1} \le M_4. \quad (3.1)$$

Then there exists a constant

$$\nu_0 = \nu_0\,(M_3, M_4, \delta) > 1,$$

such that for all $\nu \geq \nu_0$, for all $\lambda \geq 2$ and for all functions $u \in C^{2,1}(\overline{G}_\delta)$ the following pointwise Carleman estimate is valid:

$$\begin{aligned}&(a \cdot u_t - u_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu}\\ &\geq C\left(\lambda\nu^2 s^{-\nu-2} \cdot u_y^2 + \lambda^3\nu^4 s^{-3\nu-4} u^2\right) \cdot \mathcal{C}^2_{\lambda,\nu} + \frac{\partial U}{\partial y} + \frac{\partial V}{\partial t},\end{aligned}$$

where the positive constant C depends on α, β, ν_0, and the functions U and V satisfy the estimate

$$\begin{aligned}&|U| + |V|\\ &\leq \tilde{C}\left[\lambda\nu\alpha s^{-\nu-1} \cdot \left(u_y^2 + u_t^2\right) + \lambda^3\nu^3\alpha^3 s^{-3\nu-3} \cdot u^2\right] \cdot \mathcal{C}^2_{\lambda,\nu}(y,t).\end{aligned} \tag{3.2}$$

with $\tilde{C}$ depending on ν_0, α, β.

The function U and V are defined by (A.18), (A.19) in the proof of Lemma 1, which is very technical and given in the appendix. Lemma 1 is valid only on subdomains $G_\delta \subsetneq G$, in particular, C will tend to zero when δ does.

The main theorems of Section 1 will now essentially be proved by integrating the estimate of Lemma 1 over the domain G_δ. The terms involving U and V will appear in integrals over parts of the boundary of G_δ. As already indicated in the beginning of this section, in contrast to [[35], Chapter 4, Section 1] only Dirichlet data are available on an essential part of the boundary, where the Carleman weight function is not minimal. The following Lemma establishes (almost) positivity of certain integrals over this part of the boundary:

Lemma 2 *Let U, V as in Lemma 1 (see (A.18), (A.19) in the appendix) and let $\varphi_0(t)$, G_δ and $\partial_0 G_\delta$ as above, in particular $\varphi_0'(t) \geq \gamma$ by (D). Denote the outward directed unit normal vector on $\partial_0 G_\delta$ by*

$$n := \frac{(1, -\varphi_0'(t))}{\sqrt{1 + [\varphi_0'(t)]^2}}.$$

If $h(t) := u(\varphi_0(t), t)$ satisfies

$$\|h\|_{H^1(0,T-\delta)} \leq \varepsilon \tag{3.3}$$

then the following estimate holds with a positive constant C independent of λ:

$$\begin{aligned}&\int_{\partial_0 G_\delta} [U\ \cos(n,y) + V\ \cos(n,t)]\, ds\\ \geq \tilde{C}\lambda\nu\alpha &\int_{\partial_0 G_\delta} [s^{-\nu-1}\, u_y^2\, \mathcal{C}^2_{\lambda,\nu}]\, ds - \varepsilon \cdot C\lambda^3\nu^3\alpha^3 \int_{\partial_0 G_\delta} [s^{-3\nu-3}\,(|u_y| + |u_t| + \varepsilon)\, \mathcal{C}^2_{\lambda,\nu}]\, ds\end{aligned}$$

Proof.

$$\begin{aligned}&U \cdot \cos(n,y)\\ &= \left[2\lambda\nu\alpha s^{-\nu-1} \cdot u_y^2 - 2a \cdot u_t \cdot u_y + \lambda^3\nu^3\alpha^3 s^{-3\nu-3} \cdot r(u)\right] \cdot \exp(2\lambda s^{-\nu}) \cdot c(t)\end{aligned}$$

where $c(t) := \frac{1}{\sqrt{1+[\varphi'(t)]^2}}$ and $r(u) \sim \varepsilon$ by (3.3). Using $h' = u_t + \varphi_0' \cdot u_y$ we obtain

$$\begin{aligned}u_y \cdot u_t &= u_y \cdot (\varphi' \cdot u_y + h')\\ &= \varphi_0' \cdot u_y^2 + u_y \cdot h'\end{aligned}$$

Together with (3.3) this yields for ν_0 large enough and $\nu \geq \nu_0$

$$\begin{aligned} & U \cdot \cos(n, y) \\ & \geq c(t) \cdot [\lambda\nu\alpha s^{-\nu-1} \cdot u_y^2 - C\lambda^4\nu^3\alpha^3 s^{-3\nu-3}(|u_y| + |u_t| + \varepsilon) \cdot \varepsilon] \cdot \mathcal{C}_{\lambda,\nu}^2 \end{aligned}$$

and

$$|V \cdot \cos(n,t)| \leq |V| \leq C\,[u_y^2 + \lambda^2\nu^2\alpha^2 s^{-2\nu-2}(|u_y| + \varepsilon) \cdot \varepsilon] \cdot \mathcal{C}_{\lambda,\nu}^2.$$

Summarizing, we obtain

$$\begin{aligned} & |U \cdot \cos(n, y) + V \cdot \cos(n, t)| \\ & \geq \tilde{C}\lambda\nu\alpha s^{-\nu-1} \cdot u_y^2 - \varepsilon C\lambda^4\nu^3\alpha^3 s^{-3\nu-3}(|u_y| + |u_t| + \varepsilon) \cdot \mathcal{C}_{\lambda,\nu}^2 \end{aligned}$$

with an appropriate positive constant C independent of λ and ν, as long as $\nu \geq \nu_0$. ■

Before we prove Theorem 2, we use regularity of functions u and q (see assumption (B),(D)) to lift the error bounds (1.5), (1.6) to higher Sobolev norms:

Lemma 3 *Let the assumptions of Theorem 2 hold. Then*

$$\begin{aligned} & \|q_1 - q_2\|_{H^2(D_0)},\ \|u_{01} - u_{02}\|_{H^2(0,1)} \leq C\varepsilon^\gamma \\ & \|u_{1x}(0,\cdot) - u_{2x}(0,\cdot)\|_{H^2(0,T)},\ \|u_{1x}(1,\cdot) - u_{2x}(1,\cdot)\|_{H^2(0,T)} \leq C\varepsilon^\gamma. \end{aligned}$$

Proof. $\tilde{q} \in C^{2+\theta}(D)$ implies $\tilde{q} \in H^{2+\theta'}(D)$ for $0 < \theta' < \theta$. Applying the interpolation inequality, we get

$$\|\tilde{q}\|_{H^{2+\theta^*}(D)} \leq C\|\tilde{q}\|_{H^{2+\theta'}(D)}^{1/p}\|\tilde{q}\|_{L^2(D)}^{1/p'}$$

for $0 < \theta^* < \theta'$ and $p = (2+\theta')/(2+\theta^*)$, $p' = p/(p-1)$. The other estimates follow in the same way. ■

Since we will not explicitly refer to the Hölder exponents of the above estimates, we may assume (for ease of notation) that

$$\|q_1 - q_2\|_{H^1(D_0)},\ \|w(\cdot,0)\|_{L_2(D_0)} \leq \varepsilon \tag{3.4}$$

$$\|w(\varphi_1(\cdot),\cdot)\|_{H^1(0,T)},\ \|w_y(\varphi_1(\cdot),\cdot)\|_{H^1(0,T)} \leq \varepsilon \tag{3.5}$$

$$\|w(\varphi_0(\cdot),\cdot)\|_{H^1(0,T)}, \leq \varepsilon \tag{3.6}$$

holds. The estimates for w follow directly by applying the transformations of Section 2 ($u \to v \to w$).

With similar arguments as above and due to the regularity assumption (B) on solutions u, any L_2 estimate on u can be lifted to an estimate in $W_2^{2,1}$:

Lemma 4 *There exist* $0 < l_1, l_2 < 1$ *and a constant* C *such that*

$$\|u\|_{H^{2,1}(Q_T)} \leq C\|u\|_{C^{2+\theta,1+\theta/2}(\overline{Q_T})}^{l_1}\|u\|_{L_2(Q_T)}^{l_2},$$

for any $u \in C^{2+\theta,1+\theta/2}(\overline{Q_T})$.

4. Proof of the Main Theorems

Proof of Theorem 2. The differential inequality (2.13) implies

$$\int_{G_\delta} (a \cdot w_t - w_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu} dy\, dt \le M_5 \int_{G_\delta} (w_y^2 + w^2) \cdot \mathcal{C}^2_{\lambda,\nu} dy\, dt$$

$$+M_6 \int_{G_\delta} \int_{\tilde{\varphi}_0(y)}^{t} [w_y^2(y,\tau) + w^2(y,\tau)] d\tau \cdot \mathcal{C}^2_{\lambda,\nu}(y,t)\, dy\, dt$$

where the constant M_5, M_6 depend on M_1, M_2 and δ. The last term can be estimated by interchanging the order of time integrals,

$$\int_a^b \int_a^t f(\tau) d\tau\, g(t)\, dt = \int_a^b \int_\tau^b g(t)\, dt\, f(\tau) d\tau,$$

which we apply with $f = w_y^2 + w^2$ and $g = \mathcal{C}^2_{\lambda,\nu}$. Observe that by (2.8), $\beta \ge 0$ and hence $\mathcal{C}_{\lambda,\nu}(y,t)$ is decreasing in t. Thus

$$\mathcal{C}_{\lambda,\nu}(y,t) \ge \mathcal{C}_{\lambda,\nu}(y,\tau), \qquad \text{for} \quad t \le \tau$$

and consequently

$$\int_\tau^T \mathcal{C}^2_{\lambda,\nu}(y,t)\, dt \le T\, \mathcal{C}^2_{\lambda,\nu}(y,\tau).$$

This implies in turn that

$$\int_{G_\delta} (a \cdot w_t - w_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu}\, dy\, dt \le M_7 \int_{G_\delta} (w_y^2 + w^2) \cdot \mathcal{C}^2_{\lambda,\nu}\, dy\, dt \tag{4.1}$$

In a second step, we estimate the left-hand side of (4.1) from below. By Lemma 1,

$$\int_{G_\delta} (a \cdot w_t - w_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu} dy\, dt$$

$$\ge C \int_{G_\delta} [(\lambda \nu^2 s^{-\nu-2} \cdot w_y^2 + \lambda^3 \nu^4 s^{-3\nu-4} \cdot w^2) \cdot \mathcal{C}^2_{\lambda,\nu}]\, dy\, dt$$

$$+ \int_{\partial G_\delta} [U \cos(n,y) + V \cos(n,t)]\, ds,$$

for all $\nu \ge \nu_0$ with ν_0 sufficiently large, for all $\lambda \ge 2$ and a positive constant C not depending on ν or λ.

By (3.4), (3.5) and Lemma 1,

$$|U|, |V| \le C\varepsilon \qquad \text{on} \quad \partial_1 G_\delta \cup \partial_2 G_\delta.$$

On $\partial_0 G_\delta$, we use Lemma 2, which yields,

$$\int_{\partial_0 G_\delta} U\ \cos(n,y) + V\, cos(n,t)\, ds$$

$$\geq -\varepsilon \cdot C\lambda^4\nu^3\alpha^3 \int\limits_{\partial_0 G_\delta} s^{-3\nu-3}\left(|w_y| + |w_t| + \varepsilon\right)\mathcal{C}^2_{\lambda,\nu}\, ds$$
$$\geq -\varepsilon \cdot \tilde{C}\lambda^4\nu^3\alpha^3 \int\limits_{\partial_0 G_\delta} s^{-3\nu-3}\,\mathcal{C}^2_{\lambda,\nu}\, ds.$$

$\|u\|_{H^{2,1}}$ is bounded on $\partial_3 G_\delta$by assumption and $s = 1$. Thus

$$\int\limits_{\partial_3 G_\delta} P\,\cos(n,y) + Q\, cos(n,t)\, ds \leq C\lambda^3\nu^3\exp(2\lambda).$$

Summarizing, we get, by fixing $\nu = \nu_0$ large enough, that

$$\int\limits_{\partial G_\delta} U\,\cos(n,y) + V\,\cos(n,t)\, ds \geq -C\,\lambda^3[\exp(2\lambda) + \varepsilon \cdot \exp(\lambda 2^{\nu+1})]$$

Together with (4.1) and $G_{\delta,\xi} := \{(y,t) \in G_\delta : s(y, t \leq 1 - \xi)\}$, this yields

$$\exp(2\lambda(1-\xi)^{-\nu}) \int\limits_{G(\delta,\xi)} [\lambda u_y^2 + \lambda^3 u^2]\, dy\, dt \leq C\lambda^3\left[\exp(2\lambda) + \varepsilon\exp(\lambda 2^{\nu+1})\right], \tag{4.2}$$

where the constant C does not depend on λ.

Now we proceed as in the proof of the Hölder estimate in [12], Section 4, by balancing the terms on the right-hand side, i.e., choosing λ such that $\exp(2\lambda) = \varepsilon \cdot \exp(\lambda 2^{\nu+1})$. This yields

$$\lambda = \frac{1}{2[2^\nu - 1]}\ln(\frac{1}{\varepsilon}) \tag{4.3}$$

Inserting in (4.2) and omitting the term w_y^2, we get

$$\int\limits_{G_{\delta,\xi}} w^2 dy\, dt \leq 2C\lambda\exp(-2\lambda[(1-\xi)^{-\nu} - 1])$$
$$\leq \tilde{C} \cdot \ln(\frac{1}{\varepsilon}) \cdot \varepsilon^{\eta_0},$$

where $\eta_0 = \frac{(1-\xi)^{-\nu}-1}{2^\nu - 1}$. For $0 < \xi < \frac{1}{2}$, it follows that $0 < \eta_0 < 1$. Hence, for ε sufficiently small, and by Lemma 4,

$$\|w\|^2_{H^{2,1}(G_{\delta,\xi})} \leq C\varepsilon^{\tilde{\eta}},$$

for some $\tilde{\eta} > 0$. The estimate for $\tilde{v}$ now follows via (2.8) and implies $\|u_1 - u_2\|^2_{H^{2,1}(G_{\delta,\xi})} \leq C\varepsilon^{\eta}$, and, via (2.2), $\|q_1 - q_2\|^2_{L_2(D(\xi))} \leq C\varepsilon^{\tilde{\eta}}$. ■

Proof of Theorem 1 . Applying Theorem 2 with $\varepsilon = 0$ we get $u_1(x,t) = u_2(x,t)$ on $G_{\delta,\xi}$ for arbitrary positive δ, ξ (small enough). Hence by continuity of solutions u_i it follows that $u_1(x,t) = u_2(x,t)$ on G and similarly $q_1(y) = q_2(y)$ on D. ■

5. Numerical Tests

For the sake of brevity, we consider here only the case of error in the Neumann data (1.4). Error in the initial condition u_0 and in the *a priori* knowledge of the parameter $q(y)$ for $y \in D_0$ can be treated in a similar way. Let ε denote the noise level of our measurements, i.e.,

$$\|\psi_i^\varepsilon - \psi_i^0\|_{L_2(0,T)} \leq \varepsilon,$$

with the error free data $\psi_i^0(t) = u_x(x_i, t; q^0)$, $x_i = 0, 1$ and exact solution solution q^0. The existence of attainable, error-free data ψ_i^0 has to be assumed. Furthermore, let $\psi_i^0(t) \leq -\gamma_1 < 0$ for $(t \in [0, T])$. For measured (noisy) data ψ_0^ε, ψ_1^ε, the existence of a parameter q^ε, such that ψ_0^ε, ψ_1^ε are Neumann boundary values of a solution $u^\varepsilon(q^\varepsilon)$ to (1.1)-(1.3) corresponding to the parameter q^ε, cannot be assumed in general. In order to be able to apply Theorems 1, 2, we restrict the set of admissible parameters to the compact set

$$K_M := \{q \in C^{2+\theta}(D) : \|q\|_{C^{2+\theta}} \leq M, \quad q(y) = q^0(y) \text{ on } D_0\}, \tag{5.1}$$

which regularizes the ill-posed problem (see Tikhonov and Arsenin[41]), and minimize the least squares functional

$$f(q, \psi^\varepsilon) := \|u_x(0, \cdot; q) - \psi_0^\varepsilon\|^2_{L_2(0,T)} + \|u_x(1, \cdot; q) - \psi_1^\varepsilon\|^2_{L_2(0,T)}. \tag{5.2}$$

Due to the restriction to a compact set $\mathcal{K}_\mathcal{M}$, no further regularization is needed. Instead of minimizing (5.2), we require only quasi-minimizers q^ε, i.e.,

$$f(q^\varepsilon, \psi^\varepsilon) \leq \inf_{q \in K_M} f(q, \psi^\varepsilon) + C\,\varepsilon^2. \tag{5.3}$$

Let $q^{\varepsilon_k} \in K_M$ denote a sequence of quasi-minimizers of (5.2), then the stability estimate of Theorem 2 implies

$$\|q^{\varepsilon_k} - q^0\|_{H^2(D(\zeta))} \leq C_M\,\varepsilon^\eta$$

where $D(\zeta) := \{y : \varphi_1(0) \leq y \leq T - \zeta$.

5.1. The Test Problem

Let $Q := (0, 1) \times (0, 1)$. We discretize the parabolic equation (1.1)-(1.3) by a finite difference method on a uniform 100×100 grid. Minimization of the least-squares functional (5.2) is performed by a descent algorithm, in our case Landweber iteration (see [7]). The iteration is stopped after 100 loops. The right and left boundary conditions are chosen such that $u \in [0, 2]$. Thus $q(u)$ is defined and identifiable only on the interval $[0, 2]$. In our tests, q is discretized on a uniform grid with 100 nodes.

Some data noise in all of our examples is due to numerical errors, which are rather small when the solution u is smooth (Examples 1-4) but not negligible, if the solution u does not satisfy assumption (B), in particular, if the compatibility conditions are not satisfied (see comparison of Example 5 and 6). At the end of Example 2, we will also investigate the influence of (added) data noise.

In the first two examples, we try to identify different (smooth) functions q, where (u, q) satisfy conditions (A) - (D). We compare the reduction of the output error and the error in the parameter. The numerically observed Hölder exponents, which we compare to those predicted by Theorem 2, are the slopes in the log-log plots (error in u_x vs. error in q) and are represented by the following approximations below:

$$\eta_{i,j} = \frac{\log(e_i) - \log(e_j)}{\log(f_i) - \log(f_j)},$$

where $e_i = \|q_i - q\|^2$ is the error in the parameter and $f_i = \|u_x(q_i)(0,\cdot) - u_x(q)(0,\cdot)\|^2 + \|u_x(q_i)(1,\cdot) - u_x(q)(1,\cdot)\|^2$ denotes the residual.

Example 1 *Let* $q(u) = u^2$, $u_0(x) = 1 - x$, $\varphi_0(t) = 1 + t$, $\varphi_1(t) = 0$ *and* $f(x,t) = x - 1 + (1-x)^2 \cdot (1+t^2)$. *The corresponding solution to (1.1) - (1.3) is* $u(x,t) = (1-x)\cdot(1+t)$. *We assume the parameter* $q(u)$ *to be known for* $u \in (0,1)$. *The iteration is started with*

$$q_0(u) = \begin{cases} u^2, & u \in [0,1] \\ 1, & u \in (1,2] \end{cases}$$

Table 1. Residual, error and Hölder rate; left: Example 1, right: Example 2

f_i	e_i	$\eta_{i,50}$	iteration	f_i	e_i	$\eta_{i,50}$
0.238474	1.127917		1	0.157570	0.477502	
0.011876	0.293605		10	0.003831	0.047151	
0.005128	0.197503		20	0.002078	0.023137	
0.003477	0.153841		30	0.001274	0.013552	
0.002518	0.124354		40	0.000883	0.009683	
0.001910	0.103638		50	0.000681	0.007976	
0.001500	0.088368	0.659656	60	0.000562	0.006991	0.687285
0.001211	0.076722	0.659892	70	0.000482	0.006272	0.693555
0.001002	0.067591	0.662144	80	0.000421	0.005686	0.703555
0.000848	0.060271	0.667042	90	0.000373	0.005188	0.715020
0.000732	0.054291	0.674391	100	0.000334	0.004759	0.726070

As we expected from the theory (see Theorem 2), numerical Hölder exponents $\eta_{i,j}$, listed in the left part of Table 1, can be observed also numerically. Figure 3(a) plots the initial guess, the true parameter q^0 and the reconstruction q_{100} after 100 iterations. The parameter is identified not very well at high temperature, which is mainly due to the slow performance of the minimization algorithm used and the low sensitivity of the problem on values of q there. This behaviour is also indicated by the estimate of Theorem 2 and the subsequent remark.

In Example 2 we test the performance on more oscillating functions q^0:

Example 2 *Let* $q(u) = sin(\pi \cdot u)$, $u_0(x) = 1 - x$, $\varphi_0(t) = 1 + t$, $\varphi_1(t) = 0$ *and* $f(x,t) = x - 1 + \sin(\pi\cdot(1-x)\cdot(1+t))$. *The corresponding solution to (1.1) - (1.3) is* $u(x,t) = (1-x)\cdot(1+t)$. *We assume the parameter* $q(u)$ *to be known for* $u \in (0,1)$. *The iteration is started with*

$$q_0(u) = \begin{cases} \sin(\pi u), & u \in [0,1] \\ 0, & u \in (1,2] \end{cases}$$

Again, the theoretically predicted Hölder rate is also observed numerically.

Adding 1% artificial data noise in Example 2, the iteration can be stopped after less than 40 iterations (due to a discrepancy principle, see [7], or similar due to (5.3)), with a reconstruction of q comparable to that without noise. For more strongly oscillating parameters, we have to perform a higher number to reach the stopping criterion (5.3).

Table 2 and Figure 3(c),(d) show the corresponding results for $q(u) = sin(2\pi \cdot u)$ and $q(u) = sin(5\pi \cdot u)$ in the above example:

Table 2. Residual, error and Hölder rate; Example 2; left: $q(u) = sin(2\pi \cdot u)$, right: $q(u) = sin(5\pi \cdot u)$; the number of iterations for the latter example has to be increased in order to get good approximations.

f_i	e_i	$\eta_{i,50}$	iteration	f_i	e_i	$\eta_{i,50}$
0.091398	0.499491		1	0.035974	0.490008	
0.001716	0.025625		50	0.017557	0.320115	
0.000763	0.010497		100	0.011346	0.234746	
0.000489	0.007484		150	0.007730	0.175079	
0.000359	0.005947		200	0.005453	0.127945	
0.000282	0.004965		250	0.003933	0.091493	
0.000232	0.004297	0.740335	300	0.002888	0.065183	1.097758
0.000197	0.003828	0.725940	350	0.002164	0.047773	1.087352
0.000172	0.003484	0.712032	400	0.001659	0.037206	1.042551
0.000152	0.003217	0.701634	450	0.001308	0.031091	0.980245
0.000137	0.002999	0.695169	500	0.001061	0.027318	0.922719

The fact that the observed rates η_i for Example 2 with $q(u) = sin(5\pi \cdot u)$ on the right side of Table 2 are large is surprising at the first glimpse. Note, however, that the residual f_{500} and the error e_{500} are much larger than for the case $q(u) = sin(2\pi \cdot u)$, which means that the iterates are still relatively far away from the solution. In fact, if the iteration is continued, the observed rates are approximately the same as for $q(u) = sin(2\pi \cdot u)$, for instance, $\eta_{2000,1000} = 0.728$.

The above examples confirm the theoretical results of Section 1, in particular, Hölder rates concerning parameter convergence with respect to data noise are observed also numerically. The Landweber iteration method used to minimize (5.2) produces smooth iterates q_k, and the results are very stable with respect to the number of iterations, in particular the restriction to the compact set K_M does not become active in our numerical examples.

The following two examples are concerned with identifying parameters q, which do not satisfy the regularity assumption (D), whereas the solution u still satisfies (B).

Example 3 *Set* $q(u) = max(u - 3/2, 0)$, $u_0(x) = 1 - x$, $\varphi_0(t) = 1 + t$, $\varphi_1(t) = 0$ *and* $f(x,t) = x - 1 + max((1-x)\cdot(1+t) - 3/2, 0)$. *The corresponding solution to (1.1) - (1.3) is* $u(x,t) = (1-x)\cdot(1+t)$. *We assume the parameter* $q(u)$ *to be known for* $u \in (0,1)$. *The iteration is started with*

$$q_0(u) = \begin{cases} 0, & u \in [0,1] \\ u-1, & u \in (1,2] \end{cases}$$

Table 3. Residual, error and Hölder rate; left: Example 3, right: Example 4.

f_i	e_i	$\eta_{i,50}$	iteration	f_i	e_i	$\eta_{i,50}$
0.079624	0.282840		1	0.028191	0.203091	
0.001911	0.045573		10	0.007425	0.131571	
0.001114	0.035972		20	0.005616	0.118141	
0.000853	0.029988		30	0.004802	0.109577	
0.000660	0.025272		40	0.004219	0.103910	
0.000514	0.021502		50	0.003794	0.099994	
0.000404	0.018499	0.625561	60	0.003478	0.097090	0.338932
0.000322	0.016114	0.615853	70	0.003239	0.094815	0.336334
0.000261	0.014218	0.608901	80	0.003053	0.092927	0.337719
0.000216	0.012705	0.605831	90	0.002906	0.091308	0.340894
0.000183	0.011489	0.607363	100	0.002785	0.089892	0.344677
			400	0.001778	0.075214	0.397804
			500	0.001675	0.073570	0.388723

The oscillations in the parameter around the kink are an expected feature in approximation by smooth functions. The relatively large error at high temperature is again in accordance to Theorem 2, where convergence rates are guaranteed only on subdomains $D_\xi \subset D$.

Example 4 *Set* $q(u) = H(u - 3/2)$, *with* $H(\cdot)$ *denoting the Heavyside function. Additionally, let* $u_0(x) = 1-x$, $\varphi_0(t) = 1+t$, $\varphi_1(t) = 0$ *and* $f(x,t) = x-1+H((1-x)\cdot(1+t)-1/2)$. *The corresponding solution to (1.1) - (1.3) is* $u(x,t) = (1-x)\cdot(1+t)$. *We assume the parameter* $q(u)$ *to be known for* $u \in (0,1)$. *The iteration is started with*

$$q_0(u) = \begin{cases} 0, & u \in [0,1] \\ u-1, & u \in (1,2]) \end{cases}$$

As expected, identification of discontinuous parameters is harder than in the smooth case. Additionally, the observed Hölder rate is smaller and a higher number of iterations is needed to get a good approximation.

The next series of numerical test cases is concerned with examples, where the regularity assumption (B) on the solution u, in particular the compatibility conditions, or the condition on *a priori* information of the parameter are violated. The data for these examples are calculated numerically (to avoid an inverse crime on a 500×500 grid). In these examples, the numerical (data) error is relatively high, since the compatibility conditions are not satisfied.

Example 5 *Let* $q(u) = \sin(\pi \cdot u)$, $u_0(x) = 1-x$, $\varphi_0(t) = 1+t$, $\varphi_1(t) = 0$ *and* $f(x,t) = 0$. *Note that the compatibility conditions are not satisfied, in particular* $u \notin C^{4+\theta,2+\theta/2}(\overline{Q})$. *The iteration is started with*

$$q_0(u) = \begin{cases} \sin(\pi u), & u \in [0,1] \\ 0, & u \in (1,2] \end{cases}$$

The residuals f_i as well as the errors in the parameter are about 5 times larger than in Example 2, which can be explained by the fact that the true solution u has a weak singularity at the edges $x = 0, 1$, $t = 0$. The iteration becomes almost stationary after about 30 iterations, where it would have been stopped (due to (5.3) respectively

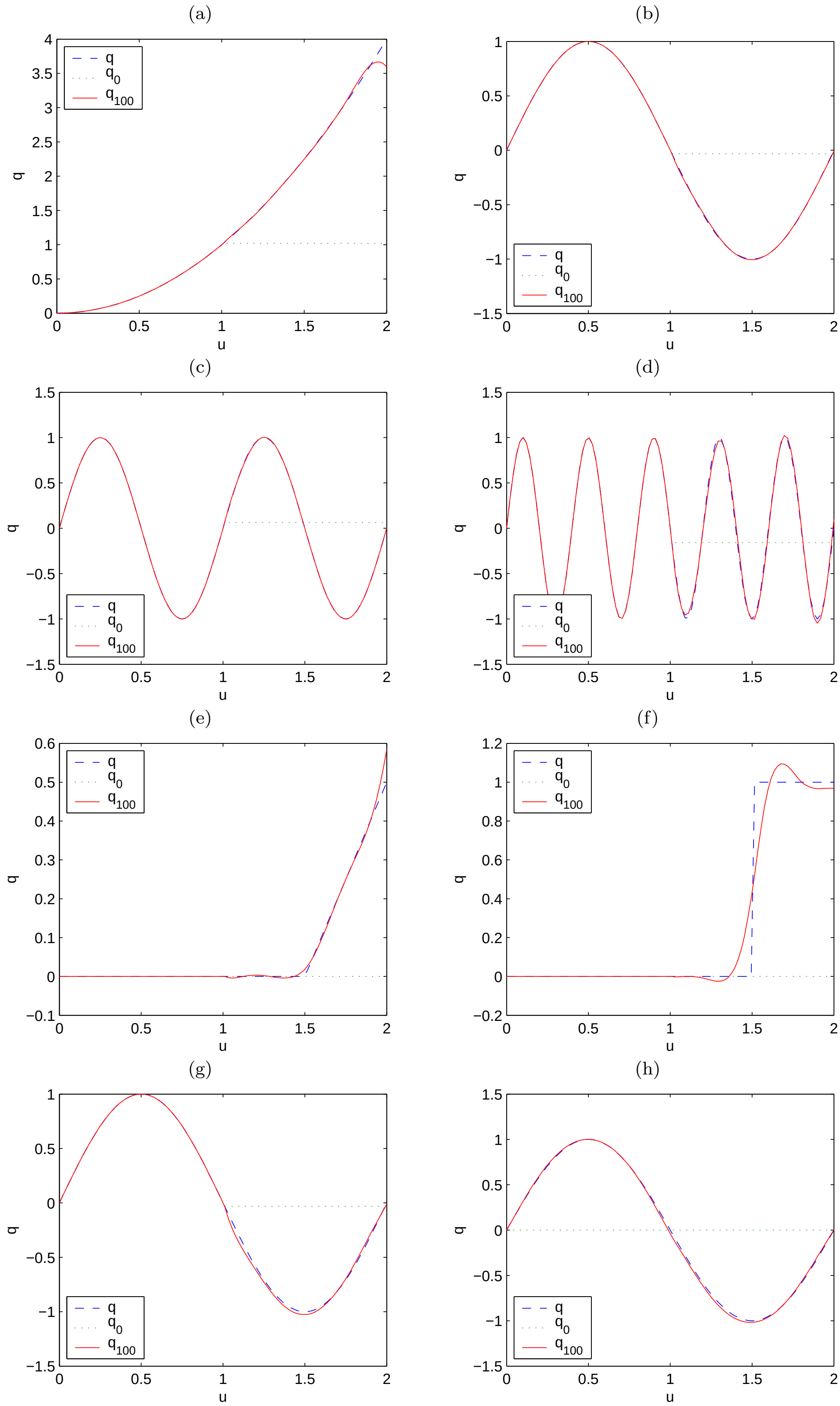

Figure 3. Initial guess, true parameter and reconstruction after 100 iterations; (a) Example 1; (b) Example 2; (c) and (d): Example 2 with $q(u) = sin(2\pi \cdot u)$ and $q(u) = sin(5\pi \cdot u)$; (e) and (f) corresponding to Examples 3, 4 with non-smooth parameters; Examples 5 (g) and 6 (h) with violation of the regularity assumption on u respectively the *a priori* knowledge of parameters q.

Table 4. Residual, error and Hölder rate; left: Example 5, right: Example 6

f_i	e_i	$\eta_{i,50}$	iteration	f_i	e_i	$\eta_{i,50}$
0.153879	0.477502		1	0.279313	0.703562	
0.004271	0.059052		10	0.017265	0.092826	
0.003017	0.036199		20	0.003359	0.021781	
0.002467	0.025534		30	0.001457	0.020092	
0.002248	0.021747		40	0.001174	0.019128	
0.002162	0.020620		50	0.001089	0.018361	
0.002127	0.020339	0.841669	60	0.001054	0.017996	0.615239
0.002112	0.020267	0.737597	70	0.001037	0.017884	0.535413
0.002104	0.020219	0.722490	80	0.001027	0.017903	0.430561
0.002099	0.020156	0.772016	90	0.001021	0.017996	0.310299
0.002095	0.020088	0.838390	100	0.001017	0.018102	0.206074

a discrepancy principle). In view of the large residuals f_i, it is surprising that still a good Hölder rate is observed numerically.

Example 6 *Let $q(u)$, $u_0(x)$, $\varphi_i(t)$, $f(x,t)$ as in Example 5 and set $q_0(u) = 0$ (violating the condition on the a-priori knowledge on q).*

The result using no *a priori* information on q seems to be better than in the previous example, with respect to residual and error in the parameter. Note, that omitting the *a priori* information on q yields additional freedom in q (q is not fixed on the interval D_0) and thus the numerical error (especially at t=0) can be reduced more effectively. The fact that the observed Hölder rates $\eta_{i,j}$ are decreasing (and lower than above) is not surprising, since a comparison with Example 5 suggests that the noise level should be around $\delta^2 \approx 0.002$, which means, the iteration would have been stopped already after about 24 iterations (using a discrepancy principle, or (5.3)) with $f_{25} \approx 0.0021$ and $e_{24} \approx 0.020$. Thus in view of possible errors also in the *a priori* knowledge of q, it seems advantageous to minimize (5.2) over the set

$$K_M := \{q \in C^{2+\theta}(D) : \|q\|_{C^{2+\theta}} \leq M, \quad \|q|_{D_0} - q^0|_{D_0}\|_{L^2} \leq \epsilon\},$$

or add a penalty term $C\|q|_{D_0} - q^0|_{D_0}\|^2$ to the functional (5.2).

6. Conclusion

Under reasonable smoothness and monotonicity assumptions, a nonlinear parameter $q(u)$ in the parabolic equation

$$-u_t + u_{xx} + q(u) = f$$

can be identified uniquely and stably (with Hölder rates) from a single set of boundary measurements (of the Neumann data). For heat transfer problems, the conditions (A)-(D) can be realized in a simple experimental setup. As indicated in Section 1, the results generalize to slightly more general problems, in particular to the important case $-u_t + (a(x)u_x)_x + q(u) = f$. Numerical tests confirm the theoretical results and indicate, that the assumption on *a priori* information of q can possibly be relaxed. Also non-smooth parameters seem to be identifiable in a Hölder-stable way, although the theory does not apply.

Appendix

Proof of Lemma 1
Throughout the following proofs, C, $\tilde{C}$ will denote positive constants, which do not depend on λ and ν, as soon as $\lambda \geq \lambda_0$, $\nu \geq \nu_0$, which is particulary important since the lower bound ν_0 will be increased subsequently.

Denote

$$w := u \cdot \exp(\lambda s^{-\nu}) = u \cdot \mathcal{C}_{\lambda,\nu},$$

hence, $u = w \cdot \exp(-\lambda s^{-\nu})$, and

$$u_y = [w_y + \lambda\nu\alpha s^{-\nu-1} \cdot w] \cdot \exp(-\lambda s^{-\nu}), \tag{A.1}$$

$$u_{yy} = [w_{yy} + 2\lambda\nu\alpha s^{-\nu-1} \cdot w_y + \lambda^2\nu^2\alpha^2 s^{-2\nu-2} c_1 \cdot w] \cdot \exp(-\lambda s^{-\nu}), \tag{A.2}$$

$$u_t = [w_t + \lambda\nu\beta s^{-\nu-1} \cdot w] \cdot \exp(-\lambda s^{-\nu}), \tag{A.3}$$

with $c_1 = 1 - \frac{\nu+1}{\lambda\nu} s^\nu$. Observe, that for $\nu \geq 2$ by (2.11) $c_1 \in (\frac{1}{4}, 1)$, in particular, for large ν, $c_1 \sim 1$.

Now (A.1)-(A.3) imply that for all $\lambda, \nu \geq 2$, the left-hand side of (3.2) is

$$\begin{aligned}&(a \cdot u_t - u_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu}\\ &= \{a \cdot w_t - w_{yy} - 2\lambda\nu\alpha s^{-\nu-1} \cdot w_y - c_2\lambda^2\nu^2\alpha^2 s^{-2\nu-2} \cdot w\}^2,\end{aligned} \tag{A.4}$$

with $c_2 = 1 - \frac{\nu+1}{\lambda\nu} s^\nu - \frac{1}{\lambda\nu}\frac{\beta}{\alpha^2} a s^{\nu+1}$. Again for $\nu \geq \nu_0$, ν_0 sufficiently large, $c_2 \sim 1$.
Denote

$$z_1 = a \cdot w_t \tag{A.5}$$

$$z_2 = -w_{yy} \tag{A.6}$$

$$z_3 = -2\lambda\nu\alpha s^{-\nu-1} \cdot w_y \tag{A.7}$$

$$z_4 = -c_2\lambda^2\nu^2\alpha^2 s^{-2\nu-2} \cdot w. \tag{A.8}$$

Then (A.4) can be rewritten as

$$\begin{aligned}&(a \cdot u_t - u_{yy})^2 \cdot \mathcal{C}^2_{\lambda,\nu}(y,t) = (z_1 + z_2 + z_3 + z_4)^2\\ &\geq z_1^2 + 2z_1z_2 + z_3^2 + 2z_1z_3 + 2z_1z_4 + 2z_2z_3 + 2z_3z_4.\end{aligned} \tag{A.9}$$

Now, we estimate all terms in the last line of (A.9) from below.

Step 1. Estimate $2z_1z_2$:

$$\begin{aligned}2z_1z_2 =& -2a \cdot w_t \cdot w_{yy}\\ =& \frac{\partial}{\partial y}(-2a \cdot w_t \cdot w_y) + 2a \cdot w_{ty} \cdot w_y + 2a_y \cdot w_t \cdot w_y\\ =& 2a_y \cdot w_t \cdot w_y - a_t \cdot w_y^2 + \frac{\partial}{\partial y}(-2a \cdot w_t \cdot w_y) + \frac{\partial}{\partial t}(a \cdot w_y^2)\end{aligned}$$

Thus, with (A.5) and (3.1),

$$\begin{aligned}&2z_1z_2\\ &\geq 2z_1 \cdot \frac{a_y}{a} \cdot w_y - M_4 \cdot w_y^2 + \frac{\partial}{\partial y}(-2a \cdot w_t \cdot w_y) + \frac{\partial}{\partial t}(a \cdot w_y^2).\end{aligned} \tag{A.10}$$

Step 2. Estimate $z_1^2 + z_3^2 + 2z_1z_2 + 2z_1z_3$: Using (A.5), (A.10), we obtain

$$\begin{aligned}
&z_1^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 \\
&\geq z_1^2 + z_3^2 + 2z_1\left[z_3 + \frac{a_y}{a}\cdot w_y\right] - M_4\cdot w_y^2 \\
&\quad + \frac{\partial}{\partial y}\left(-2a\cdot w_t\cdot w_y\right) + \frac{\partial}{\partial t}\left(a\cdot w_y\right).
\end{aligned}$$

Application of the Cauchy-Schwarz inequality to the third term on the right-hand side leads to

$$\begin{aligned}
&z_1^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 \\
&\geq -2z_3\cdot\frac{a_y}{a}\cdot w_y - [(\frac{a_y}{a})^2 + M_4]\cdot w_y^2 \\
&\quad + \frac{\partial}{\partial y}\left(-2a\cdot w_t\cdot w_y\right) + \frac{\partial}{\partial t}\left(a\cdot w_y^2\right)
\end{aligned}$$

Hence, using (A.7) and choosing a sufficiently large $\nu_0 > 1$, we obtain for all $\nu \geq \nu_0, \lambda \geq 2$ and some constant C depending on M_3, M_4, ν_0, α that

$$\begin{aligned}
&z_1^2 + z_3^2 + 2z_1z_2 + 2z_1z_3 \\
&\geq -[4\frac{M_4}{M_3}\lambda\nu\alpha s^{-\nu-1} + (\frac{M_4}{M_3})^2 + M_4]\cdot w_y^2 \\
&\quad + \frac{\partial}{\partial y}\left(-2a\cdot w_t\cdot w_y\right) + \frac{\partial}{\partial t}\left(a\cdot w_y^2\right) \\
&\geq -C\lambda\nu\alpha s^{-\nu-1}\cdot w_y^2 + \frac{\partial}{\partial y}\left(-2a\cdot w_t\cdot w_y\right) + \frac{\partial}{\partial t}\left(a\cdot w_y^2\right).
\end{aligned} \tag{A.11}$$

Step 3. Estimate $2z_2z_3$: By (A.6), (A.7)

$$\begin{aligned}
2z_2z_3 &= 4\lambda\nu\alpha s^{-\nu-1}\cdot w_y\cdot w_{yy} \\
&= 2\lambda\nu\left(\nu+1\right)\alpha^2 s^{-\nu-2}\cdot w_y^2 + \frac{\partial}{\partial y}\left(2\lambda\nu\alpha s^{-\nu-1}\cdot w_y^2\right).
\end{aligned} \tag{A.12}$$

It is essential for the whole estimate that w_y^2 enters both terms in the last line with positive, large parameters λ, ν.

Step 4. Estimate $2z_3z_4$: Inserting (A.7), and (A.8) yields

$$\begin{aligned}
2z_3z_4 &= 4\lambda^3\nu^3\alpha^3 c_2 s^{-3\nu-3}\cdot w_y\cdot w \\
&= 2\lambda^3\nu^3\alpha^3 c_3\cdot w^2 + \frac{\partial}{\partial y}\left(2\lambda^3\nu^3\alpha^3 c_2 s^{-3\nu-3}\cdot w^2\right),
\end{aligned}$$

where

$$\begin{aligned}
c_3 &= -\frac{\partial}{\partial y}(c_2 s^{-3\nu-3}) = (3\nu+3)\alpha c_2 s^{-1} - c_{2y} \\
&= (\nu+1)\alpha s^{-3\nu-4}\left[3 - \frac{2\nu+3}{\lambda\nu}s^\nu - \frac{2}{\lambda\nu}\frac{\beta}{\alpha^2}as^{\nu+1} + \frac{\beta}{\alpha^3}\frac{1}{\lambda\nu(\nu+1)}a_y s^{\nu+2}\right]
\end{aligned}$$

Note that for ν_0 large enough, $c_3 \sim 3(\nu+1)\alpha s^{-3\nu-4}$. Thus, with some positive constant C depending on α, β, M_3, M_4, ν_0,

$$2z_3z_4 \geq C\,\lambda^3\nu^4\alpha^4 s^{-3\nu-4}\cdot w^2 + \frac{\partial}{\partial y}\left(2\lambda^3\nu^3\alpha^3 s^{-3\nu-3}c_2\cdot w^2\right). \tag{A.13}$$

Again it is important that w^2 enters both terms on the right hand side with large parameters λ, ν.

Step 5. Estimate $2z_1z_4$:

$$2z_1z_4 = -2\lambda^2\nu^2\alpha^2\, a\, c_2\, s^{-2\nu-2}\cdot w_t\cdot w$$
$$= 2\lambda^2\nu^2\alpha^2 c_4\cdot w^2 + \frac{\partial}{\partial t}\left(-\lambda^2\nu^2\alpha^2\, a\, c_2\, s^{-2\nu-2}\cdot w^2\right)$$

with

$$c_4 = -\frac{\partial}{\partial y}[a\, c_2\, s^{-2\nu-2}] = (\nu+1)\,\alpha\, a\, s^{-2\nu-3}\times$$
$$\times\,[2-\frac{\nu+2}{\lambda\nu}s^{\nu}-\frac{1}{\lambda\nu}\frac{\beta}{\alpha^2}as^{\nu+1}+\frac{1}{\lambda\nu(\nu+1)}\frac{\beta}{\alpha^3}a_y s^{\nu+2}$$
$$-\frac{1}{\nu+1}\frac{1}{\alpha}\frac{a_y}{a}s(1-\frac{\nu+1}{\lambda\nu}-\frac{1}{\lambda\nu}\frac{\beta}{\alpha^2}s^{\nu+1})]$$

Note that for ν large enough, $c_4 \sim 2\nu\,\alpha\, a\, s^{-2\nu-3}$. Hence, with a constant C depending on ν_0, α, β, M_3, M_4

$$2z_1z_4 \geq C\lambda^2\nu^3\alpha^3 s^{-2\nu-3}\cdot w^2 + \frac{\partial}{\partial t}\left(-\lambda^2\nu^2\alpha^2\, a\, c_2\, s^{-2\nu-2}\cdot w^2\right). \quad \text{(A.14)}$$

Summing up (A.11) - (A.14), keeping in mind that $\frac{1}{2}\leq s<1$ in G_δ and using (A.9), we obtain

$$(z_1+z_2+z_3+z_4)^2$$
$$\geq C\,\lambda\nu^2\alpha^2 s^{-\nu-2}\cdot w_y^2 + C\,\lambda^3\nu^4\alpha^4\, s^{-3\nu-4}\cdot w^2 + \frac{\partial\tilde{U}}{\partial y}+\frac{\partial\tilde{V}}{\partial t}, \quad \text{(A.15)}$$

where $\tilde{U}$, $\tilde{V}$ are defined by

$$\tilde{U} = 2\lambda\nu\alpha s^{-\nu-1}\cdot w_y^2 - 2a\cdot w_t\cdot w_y + 2\lambda^3\nu^3\alpha^3\, c_2\, s^{-3\nu-3}\cdot w^2, \quad \text{(A.16)}$$
$$\tilde{V} = a\cdot w_y^2 - \lambda^2\nu^2\alpha^2\, a\, c_2\, s^{-2\nu-2}\cdot w^2; \quad \text{(A.17)}$$

(A.15) is valid for all $\lambda\geq 2$ and $\nu\geq\nu_0$, where ν_0 has to be chosen sufficiently large. C denotes some positive constant depending on α, β, M_3, M_4 and ν_0. Now we have to back-substitute for w in (A.15) - (A.17) using

$$w_y = [u_y - \lambda\nu\alpha s^{-\nu-1}u]\cdot\exp(\lambda s^{-\nu}),$$
$$w_t = [u_t - \lambda\nu\beta s^{-\nu-1}u]\cdot\exp(\lambda s^{-\nu}),$$

and consequently,

$$w_y^2 = [u_y - \lambda\nu\alpha s^{-\nu-1}\cdot u]^2\cdot\mathcal{C}_{\lambda,\nu}^2$$
$$= [u_y^2 + \lambda^2\nu^2\alpha^2 s^{-2\nu-2}\cdot u^2 - \lambda\nu(\nu+1)\alpha^2 s^{-\nu-2}\cdot u^2$$
$$+\frac{\partial}{\partial y}(-\lambda\nu\alpha s^{-\nu-1}\cdot u^2)]\cdot\mathcal{C}_{\lambda,\nu}^2$$
$$\geq [u_y^2 + \lambda\nu^2\alpha^2 s^{-2\nu-2}\cdot u^2]\cdot\mathcal{C}_{\lambda,\nu}^2 + \frac{\partial}{\partial y}[-\lambda\nu\alpha s^{-\nu-1}\cdot u^2]\cdot\mathcal{C}_{\lambda,\nu}^2,$$

where the last inequality follows by $\lambda\geq 2$. Substituting into (A.15) yields

$$(z_1+z_2+z_3+z_4)^2$$
$$\geq C\,\left(\lambda\nu^2\alpha^2 s^{-2\nu-2}u_y^2 + \lambda^3\nu^4\alpha^4 s^{-3\nu-4}u^2\right)\cdot\mathcal{C}_{\lambda,\nu}^2 + \frac{\partial U}{\partial y}+\frac{\partial V}{\partial t}$$

with

$$\begin{aligned} U = &[2\lambda\nu\alpha s^{-\nu-1}u_y^2 - 2au_y \cdot u_t \\ &+(2a\lambda\nu\beta s^{-\nu-1} - 4\lambda^2\nu^2\alpha^2 s^{-2\nu-2})u_y u + 2a\lambda\nu\alpha s^{-\nu-1}u_t u \\ &+2\lambda^3\nu^3\alpha^3 s^{-3\nu-3}(c_2 + 1 - 1\frac{1}{\lambda\nu}\frac{\beta}{\alpha^2}as^{\nu+1}) \cdot u^2] \cdot \mathcal{C}_{\lambda,\nu}^2 \end{aligned} \tag{A.18}$$

and

$$V = [au_y^2 - 2\lambda\nu\alpha a s^{-\nu-1}u_y u + a\lambda^2\nu^2\alpha^2 s^{-2\nu-2}(1-c_2)\cdot u^2]\cdot\mathcal{C}_{\lambda,\nu}^2. \tag{A.19}$$

The estimate (3.2) follows directly.

References

[1] O. M. Alifanov. *Inverse Heat Transfer Problems.* Springer, New York, 1994.

[2] J. Beck, B. Blackwell, and C. S. Clair. *Inverse Heat Conductions.* Wiley, Sussex, 1985.

[3] A. L. Bukhgeim. Carleman estimates for Volterra operators and uniqueness of inverse problems. In M. M. Lavrent'ev, editor, *Non classical problems of mathematical physics*, pages 56–64, Novosibirsk, 1981. Computing Center of the Siberian Branch of the Russian Academy of Science. (in Russian).

[4] A. L. Bukhgeim and M. V. Klibanov. Global uniqueness of a class of multidimensional inverse problems. *Soviet Math. Dokl.*, 24:244–247, 1981.

[5] D. Chae, O. Yu Imanuvilov, and S. M. Kim. Exact controllability of semilinear parabolic equations with Neumann boundary conditions. *J. Dyn. Control Syst.*, 2:449–483, 1996.

[6] J. Cheng and M. Yamamoto. One new strategy for a priori choice of regularizing parameters in Tikhonov's regularization. *Inverse Problems*, 16:L31 – 38, 2000.

[7] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problem.* Kluwer Academic Publishers, 1996.

[8] H. W. Engl and P. Manselli. Stability estimates and regularization for an inverse heat conduction problem in semi-infinite and finit time intervalls. *Numer. Funct. Anal. Optim*, 10:517–540, 1989.

[9] H. W. Engl and W. Rundell, editors. *Inverse Problems in Diffusion Processes*, Philadelphia, 1994. SIAM.

[10] H. W. Engl, O. Scherzer, and M. Yamamoto. Uniqueness and stable determination of forcing terms in linear partial differential equations with overspecified boundary data. *Inverse Problems*, 10:1253–1276, 1994.

[11] H. W. Engl and J. Zou. A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction. *Inverse Problems*, 16:1907–1923, 2000.

[12] Avner Friedman. *Partial Differential Equations of Parabolic Type.* Prentice Hall, 1964.

[13] W. Grever, A. Binder, H. W. Engl, and K. Mörwald. Optimal cooling strategies in continuous casting of steel with variable casting speed. *Inverse Problems in Engineering*, 2:289–300, 1997.

[14] M. Hanke. A regularization levenberg-marquardt schem, with applications to inverse groundwater filtration problems. *Inverse Problems*, 13:79–95, 1997.

[15] L. Hörmander. *Linear Partial Differential Operators.* Springer, Berlin, 1964.

[16] O. Yu Imanuvilov. Boundary controllability of parabolic equations. *Russ. Math. Surv.*, 48:192–194, 1993.

[17] O. Yu Imanuvilov. Controllability of parabolic equations. *Sbornik Math.*, 186:879–900, 1995.

[18] O. Yu Imanuvilov, V. Isakov, and M. Yamamoto. An inverse problem for the dynamical Lamé system. *Comm. Pure Appl. Math.*, 56:1–17, 2003.

[19] O. Yu Imanuvilov and M. Yamamoto. Lipschitz stability in inverse parabolic problems by the Carleman estimate. *Inverse Problems*, 14:1229–1245, 1998.

[20] V. Isakov. *Inverse Source Problems.* Number 34 in Mathematical Surveys and Monographs. AMS, Providence, Rhode Island, 1980.

[21] V. Isakov. An uniqueness in inverse problems for semilinear parabolic equations. *Ach. Rat. Mach. Anal.*, 129:1–13, 1993.

[22] V. Isakov. *Inverse Problems for Partial Differential Equations.* Springer, New York, 1998.

[23] V. Isakov. Uniqueness of recovery of some systems of semilinear partial differential equations. *Inverse Problems*, 17:607–618, 2001.

[24] M. V. Klibanov. Uniqueness of solutions in the "large" of some multidimensional inverse problems. In M. M. Lavrent'ev, editor, *Non classical problems of mathematical physics*, pages 101–114, Novosibirsk, 1981. Computing Center of the Siberian Branch of the Russian Academy of Science. (in Russian).

[25] M. V. Klibanov. Inverse problems in the "large" and Carleman bounds. *Differential Equations*, 20:755–760, 1984.

[26] M. V. Klibanov. Uniqueness in the large of solutions of inverse problems for a class of differential equations. *Differential Equations*, 20:1390–1395, 1985.

[27] M. V. Klibanov. Uniqueness of solutions of two inverse problems for a Maxwellian system. *U.S.S.R Comput. Math. Math. Phys.*, 26(4):67–72, 1986.

[28] M. V. Klibanov. A class of inverse problems for nonlinear parabolic equations. *Siberian Math. J.*, 27:698–708, 1987.

[29] M. V. Klibanov. Inverse problems and Carleman estimates. *Inverse Problems*, 8:575–596, 1992.

[30] M. V. Klibanov. Global uniqueness of a multidimensional inverse problem for a nonlinear parabolic equation by a Carleman estimate. *Inverse Problems*, 20:1003–1032, 2004.

[31] M. V. Klibanov and A. Timonov. *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications*. Inverse and Ill-Posed Problems Series. VSP, Netherlands, 2004.

[32] P. Kügler. Identification of a temperature dependent heat conductivity from single boundary measurements. *SIAM J. Num. Anal.*, pages 1543–1563, 2003.

[33] P. Kügler and H. W. Engl. Identification of a temperature dependent heat conductivity by Tikhonov regularization. *J. Inv. Ill-posed Problems*, 10:67–90, 2002.

[34] O. A. Ladyshenskaya, V. A. Solonikov, and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*. AMS, Providence, Rhode Island, 1968.

[35] M. M. Lavrent'ev, V. G. Romanov, and S. P. Shishatskii. *Ill-posed Oroblems of Mathematical Physics*. AMS, Providence, Rhode Island, 1986.

[36] C. L. Lin and J. N. Wang. Uniqueness in inverse problems for an elasticity system with residual stress by a single measurement. *Inverse Problems*, 19:807–820, 2003.

[37] N. V. Muzylev. Uniqueness theorems for some problems of heat conduction. *U.S.S.R Comput. Math. Math. Phys.*, 20:120–134, 1980.

[38] M. Pilant and W. Rundell. An inverse problem for a nonlinear parabolic equation. *Commun. Part. Diff. Equat.*, 11:445–457, 1986.

[39] A. Rösch. Stability estimates for the identification of nonlinear heat transfer laws. *Inverse Problems*, 12:743–756, 1996.

[40] A. B. Tayler. *Mathematical Models in Applied Mechanics*. Oxford University Press, 1986.

[41] A. N. Tikhonov and V. Y. Arsenin. *Solution of Ill-Posed Problems*. Wiley, New York, 1977.