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Abstract In the CRR model we introduce a transaction cost structure which covers piecewise proportional, fixed and constant costs. For a general utility function we formulate the problem of maximizing the expected utility of terminal wealth as a Markov control problem. An existence result is given and optimal strategies can be described by solutions of the dynamic programming equation. For logarithmic utility we provide detailed solutions in the one-period case and provide examples for the multi-dimensional case and for complex cost structures. For a combination of fixed and proportional costs a fast multi-period algorithm is introduced.

Key words Portfolio optimization, transaction costs, CRR model, utility maximization, Markov control processes

1 Introduction

In this work our objective is the maximization of the expected utility of the terminal wealth in the multi-period CRR model. We use this simple model consisting of a binomial stock and a bond with constant interest rate since our emphasis lies on the introduction of a general transaction cost structure covering all fee structures typically observed on the market, cf. Example 2.1. So far in discrete time mainly optimization problems involving proportional costs have been considered, e.g. in [4], [13], [17], for pricing also combinations of constant and proportional costs are treated, e.g. in [18].

But the CRR model can be used to approximate the Black-Scholes model in which it is allowed to invest in one stock driven by a geometric Brownian motion and one risk-free money market. So we shall give a short overview of the corresponding results in that continuous time model to gain some insight in the impact of different types of transaction costs.

Without transaction costs for logarithmic or power utility the optimal trading strategy is given by a constant optimal risky fraction η^*, the fraction of wealth invested in the stock, see [10]. This constant fraction corresponds to the well known Merton line. Transaction costs are then usually introduced proportional to the size of the transaction (proportional), proportional to the portfolio value (fixed), by paying a constant amount (constant), or as a combination of these. Usually an infinite horizon criterion is considered, e.g. discounted consumption or the Kelly criterion.

For proportional costs it is optimal not to trade if the risky fraction stays in a certain interval around η^*, the no trading region. Reaching the boundaries, infinitesimal trading occurs in such a way that the risky fraction process just stays in that interval, see [6], [16], [1].

Because of the occurrence of the infinitesimal trading at the boundary, it seems reasonable to add a fee for each transaction to punish frequent trading. This was done in another line of papers which deal with constant (and proportional) costs. An investor has now to choose optimal stopping times and optimal transactions at these times. Thus methods of optimal impulse control have to be used, see e.g. [7], [9], [12]. When the risky fraction process reaches the boundary of the no-trading region, transactions will now be such big that it restarts at some curve between boundary and η^*.

An elegant approach is provided in [11] for purely fixed costs. They show that for the objective of maximizing the expected asymptotic growth rate (Kelly criterion) can be reduced to an optimal stopping problem for linear costs for the risky fraction process. In [8] a combination of fixed and proportional costs is considered using renewal theoretic arguments. A quite general cost structure is introduced in [3], but to get explicit results they simplify to fixed costs.

Most of the continuous time results cited above are for infinite time horizon criteria. Exemptions are [7] and [9]. But it is numerically difficult to compute the solutions for logarithmic and power utility. Anyway, if we trade in discrete time some of the properties of continuous time strategies break down. For example, since in continuous time optimal strategies in many cases can be described by first exit times from the no-trading-region, only the boundary of this region is important. But in discrete time we can very well jump into the interior of the stopping regions (selling and buying region). In particular this will make a difference when we consider costs which are only piecewise continuous in the transaction size.

Thus, aside from the fact that trading takes place in discrete time, the motivation to look at discrete time models is: (i) A hope for more explicit results for finite time horizon problems, (ii) that optimal strategies might have a different structure than in continuous time, (iii) the consideration of more realistic transaction costs, and (iv) convergence to continuous time models.

Our emphasis lies on the introduction of a cost structure covering all the cost structures mentioned above and also those actual cost structures in

(2.2) and (2.3), to give a rigorous existence result, and to provide examples which highlight the dependence of the trading regions on the cost structure.

We proceed as follows: In Section 2 we introduce the basic model where we define transaction costs in terms of the amount of money Δ which we invest in the stocks. To handle more complex costs (e.g. with different rates for different magnitudes of Δ) we introduce a new parameter a, the *type of trading*, which can only attain finitely many values. To every a corresponds a fee structure, and the investor is allowed to make a feasible choice of (a, Δ). The introduction of the types of trading will split semi continuous costs in continuous parts and allows to transform the feasibility conditions to compact feasible state-action sets thus simplifying the derivation of existence results for optimal strategies.

While we need the formulation in terms of Δ to introduce the transaction costs in a comprehensible way, it seems – motivated by the results in continuous time – much more convenient to express the optimal strategies in terms of the risky fraction, the fraction of the money invested in the stock. So we will reformulate the control problem in Section 3 to controls of the form (a, η) where η is the new risky fraction after the trade has occurred. For a piecewise affine cost structure there is a one-to-one relationship between (a, Δ) and (a, η). In Section 4 we give a multi-period existence result for a general utility function based on the solution of the dynamic programming equation (DPE). In Corollary 4.1 we specialize the result to logarithmic utility since we exploit the additive structure of the corresponding DPE in Sections 5 and 6.

In Section 5 we give very explicit results for the one-period problem in the case that only types of trading $-1, 0, +1$ are needed, corresponding to selling, holding and buying stocks. The examples in Section 6 illustrate our results. In Example 6.1 we consider a combination of constant and proportional costs showing in the two-period case that the structure of the trading regions gets more complicated than in the continuous time case. The main Example 6.2 provides the solution for the complex cost structure (2.2). Treating these costs requires the concept of the types of trading and the introduction of the piecewise affine costs.

For few periods the trading regions presented in these examples can be computed solving the DPE recursively (DP algorithm). But since in the multi-period case the value functions in the DPE are no longer concave or differentiable, one has to be very careful with the choice of maximization procedures, so the DP algorithm becomes very slow. In Example 6.3 we sketch a fast algorithm for a combination of fixed and proportional costs. For these the trading regions do not depend on the current wealth and hence the optimal strategies can be described by only four constants for each trading time, two to describe the boundaries of the trading regions and two for the optimal new risky fractions after selling or buying stocks.

The multidimensional case, general distributions and convergence to the continuous time model will be postponed to subsequent publications. For the latter not much can be expected for complex cost structures since explicit representations of the optimal strategies in discrete time are difficult to obtain as the examples in Section 6 show.

2 Trading under transaction costs

We consider the *CRR model* for $N \in I\!N$ periods: Let $r \geq 1$, $u > r > d > 0$, $p \in (0,1)$, and $Y_1, \ldots, Y_n$ i.i.d. random variables on a filtered probability space $(\Omega, \mathcal{F}_N, (\mathcal{F}_n)_{n=0,\ldots,N}, P)$ satisfying

$$p = P(Y_n = u) = 1 - P(Y_n = d), \quad n = 1, \ldots, N.$$

We assume that $(\mathcal{F}_n)_{n=0,\ldots,N}$ is the filtration generated by $(Y_n)_{n=1,\ldots,N}$. *Bond* prices $(B_n)_{n=0,\ldots,N}$ with interest rate $r - 1 \geq 0$ and *stock* prices $(S_n)_{n=0,\ldots,N}$ are given by initial prices $B_0 = S_0 = 1$ and

$$B_n = r^n, \quad S_n = \prod_{k=1}^{n} Y_k, \quad n = 1, \ldots, N.$$

To describe the trading of an investor it suffices to know his *initial wealth* $x > 0$, his *initial risky fraction* $\pi \in [0,1]$ (the fraction of his wealth invested in the stock), and his $\mathcal{F}_n$-measurable decisions Δ_n, $n = 0, \ldots, N-1$, where the *transaction* Δ_n is the additional amount of money invested in the stock at time n.

To describe the evolution of the portfolio in a Markovian way we need two processes when considering transaction costs. We use the *wealth process* $(X_n)_{n=0,\ldots,N}$ and the *risky fraction process* $(\pi_n)_{n=0,\ldots,N}$. These correspond to the portfolio value and the fraction of this value which is invested in the stock before the transaction takes place. If the investor chooses to trade he faces transaction costs C_n which have to be paid from the bank account (bond). We assume

$$C_n = \tilde{c}(X_n, \Delta_n), \quad n = 0, \ldots, N-1, \tag{2.1}$$

where $\tilde{c} : (0,\infty) \times [0,\infty) \to [0,\infty)$ and $\tilde{c}(x,\Delta)$ is continuous in x, lower semi continuous in Δ with finitely many bounded jumps, and $\tilde{c}(\cdot, 0) = 0$.

Example 2.1 In Table 1 we summarize some reasonable transaction cost structures. The constants satisfy $\gamma, \delta, \gamma_-, \gamma_+, \gamma_1, \gamma_2 \in (0,1)$, $\gamma_1 > \gamma_2$, and $C, d, C_1, C_2 > 0$, $C_1 < C_2$.

While combinations of the first three cost structures in Table 1 cover all examples treated typically in portfolio optimization, we would like to set up a model that also allows to deal with more complex costs like e.g. the following two fee structures which correspond to the rates of a German

Table 1 Types of Transaction Costs

$\tilde{c}(x, \Delta)$	description
$\gamma\,\lvert\Delta\rvert$	*proportional* (to the amount traded)
C if $\Delta \neq 0$	*constant*
$\delta\, x$	*fixed* (proportional to the portfolio value)
$\gamma_-\lvert\Delta\rvert$ if $\Delta < 0$, $\gamma_+\lvert\Delta\rvert$ if $\Delta > 0$	proportional with different rates
γ, d if $\lvert\Delta\rvert \le d$, $\gamma\,\lvert\Delta\rvert$ if $\lvert\Delta\rvert > d$	proportional with minimum cost $\gamma\, d$
$\gamma_1\lvert\Delta\rvert$ if $\lvert\Delta\rvert < d$, $\gamma_2\lvert\Delta\rvert$ if $\lvert\Delta\rvert \ge d$	proportional depending on amount traded
C_1 if $\lvert\Delta\rvert \le d$, C_2 if $\lvert\Delta\rvert > d$	constant depending on amount traded

direct bank: The *constant plus piecewise proportional costs*,

$$10.00 + \begin{matrix} 0.294\% \\ 0.280\% \\ 0.210\% \\ 0.140\% \\ 0.105\% \\ 0.070\% \end{matrix} \quad \text{of} \quad |\Delta| \quad \text{if} \quad \begin{matrix} 0.01 \le |\Delta| \le 12\,500.00 \\ 12\,500.01 \le |\Delta| \le 25\,000.00 \\ 25\,000.01 \le |\Delta| \le 37\,500.00 \\ 37\,500.01 \le |\Delta| \le 50\,000.00 \\ 50\,000.01 \le |\Delta| \le 125\,000.00 \\ 125\,000.01 \le |\Delta|\,. \end{matrix} \tag{2.2}$$

or the simpler *piecewise constant costs*

$$\begin{matrix} 14.99 & \text{if} & 0.01 \le \Delta \le 10000.00 \\ 29.99 & \text{if} & \Delta \ge 10000.01 \end{matrix} \tag{2.3}$$

Note that the costs in (2.2) and (2.3) can be defined lower semi continuous in Δ. Hence all the cost structures in Example 2.1 are covered by our definition of $\tilde{c}$ above. But for the reformulation of the control problem in Section 3 this definition is too general. So we restrict the possible cost structures in Definition 2.1 below to costs which are piecewise affine in Δ.

But first we have to introduce the basic feasibility requirements. At time n, $n = 0, \dots, N-1$, a transaction Δ_n is *feasible* if the new wealth is strictly positive, i.e.

$$X_n - \tilde{c}(X_n, \Delta_n) > 0, \tag{2.4}$$

and if no short selling occurs (neither in the stock nor in the bond), i.e.

$$\pi_n X_n + \Delta_n \ge 0, \quad (1-\pi_n)X_n - \Delta_n - \tilde{c}(X_n, \Delta_n) \ge 0. \tag{2.5}$$

Using feasible transactions Δ_n, $n = 0, \dots, N-1$, and initial values $X_0 = x > 0$, $\pi_0 = \pi \in [0,1]$ the portfolio evolves for $n = 0, \dots, N-1$ according to

$$X_{n+1} = (X_n - \tilde{c}(X_n, \Delta_n) - \Delta_n)r + (\pi_n X_n + \Delta_n)Y_{n+1}, \tag{2.6}$$

$$\pi_{n+1} = (\pi_n X_n + \Delta_n)Y_{n+1}/X_{n+1}. \tag{2.7}$$

Definition 2.1 *Transaction costs are called piecewise affine if for some* $n_{\mathcal{T}} \in \mathbb{N}$ *there exist* $-\infty = d_{-n_{\mathcal{T}}} < \ldots < d_0 = 0 < \ldots < d_{n_{\mathcal{T}}} = \infty$ *such that for all* $a \in \mathcal{A}_{\mathcal{T}} = \{-n_{\mathcal{T}}, \ldots, n_{\mathcal{T}}\}$

$$\tilde{c}(x, \Delta) = \alpha_a(x)x + \beta_a(x)\Delta \quad \textit{for} \quad \Delta \in I_a$$

where the intervals I_a *are given by*

$$I_a = \begin{cases} (d_a, d_{a+1}) & \textit{if } a < 0, \\ \{0\} & \textit{if } a = 0, \\ (d_{a-1}, d_a) & \textit{if } a > 0, \end{cases}$$

and $\alpha_0 = \beta_0 = 0$*, and for* $a \neq 0$ *we assume that* α_a *is non-negative and continuous and* β_a *is continuous with values in* $(-1, 0]$ *if* $a < 0$ *and values in* $[0, 1)$ *if* $a > 0$*. At* $\Delta = d_a$*,* $-n_{\mathcal{T}} < a < n_{\mathcal{T}}$*, the costs are given by the lower semi continuity requirement.*

We assume that from now on piecewise affine transaction costs in the notation of Definition 2.1 are given. These might look restrictive but actually they cover all the cases of Example 2.1. The intervals on which $\tilde{c}(x, \cdot)$ is continuous can be open, half open or closed. To get compact sets which are preferable for the optimization we introduce the type of trading. Thereby we also obtain continuous cost functions and thus later on a continuous gain function.

Definition 2.2 *(i) At time* n *a trade* (A_n, Δ_n) *is given by* $\mathcal{F}_n$*-measurable random variables, the type of trading* A_n *and the transaction* Δ_n *with values in* $\mathcal{A}_{\mathcal{T}}$ *and* $\mathbb{R}$*, respectively, and satisfying for all* $a \in \mathcal{A}_{\mathcal{T}}$ *that* $\{A_n = a\} \subseteq \{\Delta_n \in \bar{I}_a\}$*, i.e.* $A_n = a$ *implies* $\Delta_n \in \bar{I}_a$*.*

(ii) For a trade the transaction costs C_n *at time* n *are then defined by*

$$C_n = \sum_{a=-n_{\mathcal{T}}}^{n_{\mathcal{T}}} \left(\alpha_a(X_n)X_n + \beta_a(X_n)\Delta_n\right) \mathbf{1}_{\{A_n = a\}}. \tag{2.8}$$

So C_n *is given by* $\tilde{c}(X_n, \Delta_n)$ *if* $\Delta_n \in I_{A_n}$ *and at the boundaries by the continuous extension on the closed intervals* $\bar{I}_a$*,* $a \in \mathcal{A}_{\mathcal{T}}$*.*

(iii) A trade (A_n, Δ_n) *is feasible for* (X_n, π_n) *if it satisfies (2.4), (2.5) with* $\tilde{c}(X_n, \Delta_n)$ *replaced by* C_n *as given in (2.8). We say that* $(X_n, \pi_n)_{n=0,\ldots,N}$ *is controlled by the trading strategy* $K = (A_n, \Delta_n)_{n=0,\ldots,N-1}$ *and may indicate this by writing* X_n^K*,* π_n^K*,* $n = 0, \ldots, N$*. A trading strategy is admissible if* (A_n, Δ_n) *is feasible for* (X_n^K, π_n^K)*,* $n = 0, \ldots, N-1$*, for all initial values* $x > 0$*,* $\pi \in [0, 1]$*.*

Note that at the points where $\tilde{c}(x, \cdot)$ jumps we have now two possible types of trading, the additional one corresponding to the higher costs. But it can easily be shown that for a strictly increasing utility function the higher costs cannot be optimal at this point, so we get the same optimal strategies as before. Equations (2.6) and (2.7) yield

Lemma 2.1 *For any admissible trading strategy* $K = (A_n, \Delta_n)_{n=0,\dots,N-1}$

$$X_n^K > 0, \quad \pi_n^K \in [0,1], \quad n = 1, \dots, N,$$

for all initial values $x > 0$, $\pi \in [0,1]$.

Example 2.2 The first three cost structures in Table 1 can be combined to

$$\tilde{c}(x, \Delta) = \Gamma(x)x\, \mathbf{1}_{\mathbb{R}\setminus\{0\}}(\Delta) + \gamma\, |\Delta|, \tag{2.9}$$

where Γ is continuous and non-negative and γ a constant in $[0,1)$. E.g. choosing $\Gamma(x) = C/x$, $C > 0$, and $\gamma > 0$ yields a combination of constant and proportional costs. Using types of trading $\mathcal{A}_{\mathcal{T}} = \{-1, 0, 1\}$, we obtain for an admissible trading strategy

$$C_n = A_n^2 \Gamma(X_n) X_n + A_n\, \gamma\, \Delta_n$$

since $|\Delta_n| = A_n\, \Delta_n$ for $\Delta_n \in \bar{I}_{A_n}$. So $A_n = -1$ corresponds to selling stocks, $A_n = 1$ to buying, and $A_n = 0$ to holding the stocks. Unlike in (2.9) buying or selling 0 stocks paying fees $\Gamma(X_n)X_n$ is now possible.

3 Reformulation

Obviously (2.6) and (2.7) are not in the most favourable form to derive optimal strategies in a dynamic programming approach. Before introducing the optimization problem at the end of this section we hence transform the control model of Section 2 to a much more suitable formulation in terms of new risky fractions.

The wealth X_n and the risky fraction π_n describe the portfolio value and the fraction of this value which is invested in the stock before the transaction takes place. It is convenient to introduce also the wealth ξ_n and the risky fraction η_n after the transaction Δ_n has been executed and the fee C_n been paid. Using admissible trading strategies $(A_n, \Delta_n)_{n=0,\dots,N-1}$ and initial values $X_0 = x > 0$, $\pi_0 = \pi \in [0,1]$, the portfolio then evolves for $n = 0, \dots, N-1$ according to

$$\xi_n = X_n - C_n, \tag{3.1}$$

$$\eta_n \xi_n = \pi_n X_n + \Delta_n, \tag{3.2}$$

$$X_{n+1} = \xi_n(r + \eta_n(Y_{n+1} - r)), \tag{3.3}$$

$$\pi_{n+1} = \frac{\eta_n Y_{n+1}}{r + \eta_n(Y_{n+1} - r)}. \tag{3.4}$$

The multiplicative structure of (3.2) is a first indicator that a reformulation in terms of the new risky fractions η might be useful. Since for logarithmic or power utility the optimal strategies are typically of the form that one has to choose a constant risky fraction (corresponding to the Merton line, see Example 4.1) it is in fact much more convenient to formulate the control problem in terms of η. Moreover the compactness of the feasible action sets is easier to derive. This and the factorization of the gain function below make the use of the theory of Markov controlled processes fruitful.

So we will consider (A_n, η_n) as *action* of the investor at time n. Reformulating the control model in terms of η we have to make sure that we can replicate all admissible trading strategies. The feasibility conditions in (2.4),(2.5) translate to

$$\xi_n > 0, \quad \eta_n \xi_n \geq 0, \quad (1-\eta_n)\xi_n \geq 0. \tag{3.5}$$

Thus for feasible trades

$$\eta_n \in [0,1] \tag{3.6}$$

and hence $1 - \beta_a(x)\eta_n > 0$ for all $a \in \mathcal{A}_\mathcal{T}$, $x > 0$. So by solving (3.1), (3.2) for Δ_n using (2.8) we get $\Delta_n = f_\Delta(X_n, \pi_n, A_n, \eta_n)$, where

$$f_\Delta(x, \pi, a, \eta) = \frac{(1-\alpha_a(x))\eta - \pi}{1 + \beta_a(x)\eta}\, x, \tag{3.7}$$

and $\eta_n = f_\eta(X_n, \pi_n, A_n, \Delta_n)$, where

$$f_\eta(x, \pi, a, \Delta) = \frac{\pi\, x + \Delta}{x - \alpha_a(x)x - \beta_a(x)\Delta}. \tag{3.8}$$

By (2.8) and (3.7), $C_n = c(X_n, \pi_n, A_n, \eta_n)$ where

$$c(x, \pi, a, \eta) = \frac{\alpha_a(x) + \beta_a(x)(\eta - \pi)}{1 + \beta_a(x)\eta}\, x. \tag{3.9}$$

Substituting in (3.1) yields

$$\xi_n = \frac{1 - \alpha_{A_n}(X_n) + \beta_{A_n}(X_n)\pi_n}{1 + \beta_{A_n}(X_n)\eta_n}\, X_n. \tag{3.10}$$

Thus (3.10), (3.3) and (3.4) show that the evolution of the portfolio can be described in terms of the new risky fractions η instead of the transactions Δ. So we may think that at time n the investor has to choose an $\mathcal{F}_n$-measurable *action* (A_n, η_n) in the *action space*

$$\mathcal{A} = \mathcal{A}_\mathcal{T} \times [0,1].$$

We have to identify the actions which correspond to feasible trades. That this is possible in a unique way is a consequence of the following easy lemma.

Lemma 3.1 *Let $x > 0$, $\pi \in [0,1]$, $a \in \mathcal{A}_\mathcal{T}$. If $1 - \alpha_a(x) + \beta_a(x)\pi > 0$, then $f_\eta(x, \pi, a, \cdot)$ is strictly increasing on $\{\Delta \in \mathbb{R} \,|\, x - \alpha_a(x)x - \beta_a(x)\Delta > 0\}$.*

By (3.1), (3.5), (3.6) and (3.10) the feasibility condition (2.4) is equivalent to

$$1 - \alpha_{A_n}(X_n) + \beta_{A_n}(X_n)\pi_n > 0. \tag{3.11}$$

Only the fact that (3.11) depends neither on Δ_n nor on η_n allows us to obtain compact feasible action sets in Definition 3.1 below. The feasibility conditions in (2.5) are equivalent to

$$\Delta_n \in \bar{J}_{A_n}, \quad \text{where} \quad \bar{J}_a = [-\pi\, x, (1 - \alpha_a(x) - \pi)x/(1 + \beta_a(x))].$$

So by Definition 2.2 (A_n, Δ_n) is feasible if and only if (3.11) holds and

$$\{A_n = a\} \subseteq \{\Delta_n \in \bar{I}_a \cap \bar{J}_a\}, \quad a \in \mathcal{A}_{\mathcal{T}}.$$

Furthermore (3.11) is the condition in Lemma 3.1. Hence for the intervals describing feasible transactions Δ_n for given A_n it is by Lemma 3.1 sufficient to map the boundary points to obtain the corresponding intervals for η_n yielding for the new risky fractions the feasible action sets $f_\eta(X_n, \pi_n, A_n, \bar{I}_{A_n} \cap \bar{J}_{A_n})$ below. Note that we also use that $1 - \alpha_{A_n}(X_n) \geq \xi_n/X_n > 0$ for feasible actions as can be seen by (3.5) and (3.10).

Definition 3.1 *An action (A_n, η_n) is feasible if $(A_n, \eta_n) \in \mathcal{A}(X_n, \pi_n)$, where for all $x > 0$, $\pi \in [0, 1]$,*

$$\mathcal{A}(x, \pi) = \{(a, \eta) \in \mathcal{A} \,|\, a \in \mathcal{A}_{\mathcal{T}},\, \eta \in \mathcal{A}_a(x, \pi)\},$$

where $\mathcal{A}_0(x, \pi) = \{\pi\}$,

$$\mathcal{A}_a(x, \pi) = \begin{cases} \emptyset, & \text{if } 1 - \alpha_a(x) - \beta_a(x)\pi \leq 0, \\ f_\eta(x, \pi, a, \bar{I}_a \cap \bar{J}_a), & \text{if } 1 - \alpha_a(x) - \beta_a(x)\pi > 0, \end{cases}$$

for $a < 0$, and

$$\mathcal{A}_a(x, \pi) = \begin{cases} \emptyset, & \text{if } 1 - \alpha_a(x) \leq 0, \\ f_\eta(x, \pi, a, \bar{I}_a \cap \bar{J}_a), & \text{if } 1 - \alpha_a(x) > 0, \end{cases}$$

for $a > 0$.

A control strategy $K = (A_n, \eta_n)_{n=0,\dots,N-1}$ is admissible if $(A_n, \eta_n) \in \mathcal{A}(X_n^K, \pi_n^K)$ for all initial values $x > 0$, $\pi \in [0, 1]$ and $n = 0, \dots, N-1$.

Note that due to $f_\eta(x, \pi, a, \bar{J}_a) = [0, 1]$, we have $\{a\} \times \mathcal{A}_a(x, \pi) \subseteq \mathcal{A}$. The definition of the admissible trading strategies in Definition 2.2 (iii), and Definition 3.1, (3.7), (3.8), Lemma 3.1 imply

Proposition 3.1 *For an admissible control strategy $K = (A_n, \eta_n)_{n=0,\dots,N-1}$ the trading strategy defined by $\tilde{K} = (A_n, f_\Delta(X_n^K, \pi_n^K, A_n, \eta_n))_{n=0,\dots,N-1}$ is admissible and the processes controlled by these strategies coincide. Using f_η we can vice versa construct admissible control strategies from admissible trading strategies.*

So $(X_n, \pi_n)_{n=0,\dots,N}$ is a by $K = (A_n, \eta_n)_{n=0,\dots,N-1}$ Markovian controlled process, which may be indicated by writing $(X_n^K, \pi_n^K)_{n=0,\dots,N}$ as we have already done above. By Lemma 2.1 and Proposition 3.1 the controlled processes stay in the *state space*

$$\mathcal{X} = (0, \infty) \times [0, 1].$$

A *utility function* is an upper-semi-continuous function $U : (0, \infty) \to I\!R$ and our objective is maximizing the expected utility of terminal wealth

$$J_N^K(x, \pi) = \mathrm{E}_{x,\pi}[U(X_N^K)] = \mathrm{E}[U(X_N^K) \mid X_0^K = x,\ \pi_0^K = \pi], \tag{3.12}$$

$(x, \pi) \in \mathcal{X}$, over all admissible control strategies K, i.e. we want to find an *optimal* control strategy $K^* = (A_n^*, \eta_n^*)_{n=0,\dots,N-1}$ which satisfies

$$J_N^{K^*}(x, \pi) = J_N^*(x, \pi), \quad (x, \pi) \in \mathcal{X}, \tag{3.13}$$

where $J_N^* = \sup\{J_N^K \mid K \text{ admissible}\}$.

The feasibility conditions which lead to the simple structure of $\mathcal{A}$ may not be adequate for every utility function but they allow for the weak definition of a utility function above and they are appropriate for logarithmic utility which we use in Sections 5 and 6.

4 Existence

In Theorem 4.1 we give an existence result for the optimization problem (3.12), (3.13). That this is possible relies very much on the fact that the multifunction $(x, \pi) \mapsto \mathcal{A}(x, \pi)$ is compact valued and the utility function upper semi continuous which leads to the existence of measurable selectors which provide the optimal strategies. But we cannot employ corresponding selection theorems directly since e.g. for using [15, Proposition 10.2] the multifunction is not upper semi continuous if defined on $\mathcal{X}$ or equivalently the corresponding sets we need in [2, Proposition 7.33] are not closed.

But relying on our special structure, the combination of the types of trading with only finitely many values and the risky fractions out of a compact set, we can use these selection theorems for each type of trading $a \in \mathcal{A}_{\mathcal{T}}$ if we use multifunctions $(x, \pi) \mapsto \mathcal{A}_a(x, \pi)$ on a suitable domain $\mathcal{X}_a$. Comparing these we obtain an optimal strategy.

For the proof we need some notation. The set of *feasible state-action pairs* is

$$\mathcal{K} = \{(x, \pi, a, \eta) \mid (x, \pi) \in \mathcal{X},\ (a, \eta) \in \mathcal{A}(x, \pi)\}.$$

The transition from n to $n + 1$ using action (A_n, η_n) is given by

$$(X_{n+1}, \pi_{n+1}) = f(X_n, \pi_n, A_n, \eta_n, Y_{n+1})$$

with *transition law* $f : \mathcal{K} \times \{u, d\} \to \mathcal{X}$, $f = (f_x, f_\pi)$,

$$f_x(x, \pi, a, \eta, y) = \frac{1 - \alpha_a(x) + \beta_a(x)\pi}{1 + \beta_a(x)\eta} \, (r + \eta(y - r))x,$$
$$f_\pi(\eta, y) = \frac{\eta\, y}{r + \eta(y - r)},$$

cf. (3.3), (3.10), (3.4). To compute the optimal value we introduce

$$V_0^*(x, \pi) = U(x), \quad (x, \pi) \in \mathcal{X}, \tag{4.1}$$

and for $k = 1, \ldots, N$, Y distributed as Y_1, $(x, \pi, a, \eta) \in \mathcal{K}$

$$V_k(x, \pi, a, \eta) = \mathrm{E}[V_{k-1}^*(f_x(x, \pi, a, \eta, Y), f_\pi(\eta, Y))], \tag{4.2}$$
$$V_k^*(x, \pi) = \sup_{(a,\eta) \in \mathcal{A}(x,\pi)} V_k(x, \pi, a, \eta), \quad (x, \pi) \in \mathcal{X}, \tag{4.3}$$

and to decompose the problem for each $a \in \mathcal{A}_\mathcal{T}$

$$V_k^a(x, \pi) = \sup_{\eta \in \mathcal{A}_a(x,\pi)} V_k(x, \pi, a, \eta), \quad (x, \pi) \in \mathcal{X}.$$

Note that the time argument k in the value functions is the time to maturity. We always use k for time to maturity and n for the trading time.

Theorem 4.1 *(i) For all $k = 0, \ldots, N$, V_k^* is finite and upper semi continuous. Furthermore a selector $\varphi_k = (\varphi_k^1, \varphi_k^2)$ for the general forward DPE given by (4.2), (4.3) exists, i.e. φ_k is measurable, and for all $(x, \pi) \in \mathcal{X}$, $\varphi_k(x, \pi) \in \mathcal{A}(x, \pi)$, and*

$$V_k^*(x, \pi) = \mathrm{E}[V_{k-1}^*(f_x(x, \pi, \varphi_k^1(x, \pi), \varphi_k^2(x, \pi), Y), f_\pi(\varphi_k^2(x, \pi), Y))].$$

(ii) The control strategy $K^* = (A_n^*, \eta_n^*)_{n=0,\ldots,N-1}$ defined by $A_n^* = \varphi_{N-n}^1(X_n^*, \pi_n^*)$, $\eta_n^* = \varphi_{N-n}^2(X_n^*, \pi_n^*)$ where X_n^*, π_n^* are obtained using K^* up to time $n - 1$, is optimal and

$$J_N^*(x, \pi) = J_N^{K^*}(x, \pi) = V_N^*(x, \pi), \quad (x, \pi) \in \mathcal{X}.$$

Proof (i) For $a \in \mathcal{A}_\mathcal{T}$, let $\mathcal{X}_a = \{(x, \pi) \in \mathcal{X} \,|\, \mathcal{A}_a(x, \pi) \neq \emptyset\}$ and

$$\psi_a : \mathcal{X}_a \to \mathcal{A}, \quad \psi_a(x, \pi) = \mathcal{A}_a(x, \pi).$$

V_0^* is finite and upper-semi-continuous. Suppose that for some $k \in \{1, \ldots, N\}$ the same is true for V_{k-1}^*. Then $V_{k-1}^*(f_x(\cdot, \cdot, a, \cdot, y), f_\pi(\cdot, y))$ is upper semi continuous, since f_x, f_π are continuous. So $V_k(\cdot, \cdot, a, \cdot)$ is upper semi continuous and finite as weighted sum of these functions.

By the representation in Definition 3.1 the multifunction ψ_a is compact-valued for every $a \in \mathcal{A}_\mathcal{T}$. Therefore Proposition 7.33 in [2] for $V_k(\cdot, \cdot, a, \cdot)$ defined on $\{(x, \pi, \eta) \,|\, (x, \pi) \in \mathcal{X}_a, \eta \in \mathcal{A}_a(x, \pi)\}$ yields the existence of a measurable selector $\eta_a(k, \cdot, \cdot)$ on $\mathcal{X}_a$ such that

$$\eta_a(k, x, \pi) \in \mathcal{A}_a(x, \pi), \quad V_k^a(x, \pi) = V_k(x, \pi, a, \eta_a(k, x, \pi)).$$

Therefore V_k^a is finite valued. [2, Proposition 7.32] provides the upper semi continuity of V_k^a. Now, V_k^* is simply given as

$$V_k^*(x,\pi) = \max\{V_k^a(x,\pi) \mid a \in \mathcal{A}_\mathcal{T},\, \mathcal{A}_a(x,\pi) \neq \emptyset\},$$

hence it is also upper semi continuous and finite valued. Note that the set always contains $V_k^0(x,\pi)$, hence the maximum exists.

To specify an optimal control we introduce

$$\mathcal{T}_a'(k) = \{(x,\pi) \in \mathcal{X}_a \mid V_k^a(x,\pi) \geq V_k^b(x,\pi) \text{ for all } b \in \mathcal{A}_\mathcal{T}, b \neq a\}, \quad a \in \mathcal{A}_\mathcal{T}.$$

Note that $\bigcup_{a\in\mathcal{A}} \mathcal{T}_a'(k) = \mathcal{X}$ since $\mathcal{X}_0 = \mathcal{X}$. To obtain disjoint sets, define

$$\mathcal{T}_a(k) = \mathcal{T}_a'(k) \setminus \bigcup_{b<a} \mathcal{T}_b(k) \quad \text{for } a = -n_\mathcal{T}, \ldots, -1,$$
$$\mathcal{T}_a(k) = \mathcal{T}_a'(k) \setminus \Big(\bigcup_{b<0} \mathcal{T}_b(k) \cup \bigcup_{b>a} \mathcal{T}_b(k) \Big) \quad \text{for } a = n_\mathcal{T}, \ldots, 1, 0.$$

Then a measurable selector can be defined by

$$\varphi_k(x,\pi) = \sum_{a\in\mathcal{A}_\mathcal{T}} (a, \eta_a(k,x,\pi))\, \mathbf{1}_{\mathcal{T}_a(k)}((x,\pi)), \quad (x,\pi) \in \mathcal{X}.$$

(ii) Let $K = (A_n, \eta_n)_{n=0,\ldots,N-1}$ be an admissible control strategy. Since the transition laws given by f_x and f_π are Markovian and Y_n is independent of $\mathcal{F}_{n-1}$, we have

$$\begin{aligned} J_N^K(x,\pi) &= \mathrm{E}_{x,\pi}[U(X_N^K)] \\ &= \mathrm{E}_{x,\pi}[\,\mathrm{E}_{x,\pi}[U(f_x(X_{N-1}^K, \pi_{N-1}^K, A_{N-1}, \eta_{N-1}, Y_N)) \mid \mathcal{F}_{N-1}]\,] \\ &= \mathrm{E}_{x,\pi}[V_1(X_{N-1}^K, \pi_{N-1}^K, A_{N-1}, \eta_{N-1})] \\ &\leq \mathrm{E}_{x,\pi}[V_1^*(X_{N-1}^K, \pi_{N-1}^K)], \end{aligned}$$

and further for $n = 1, \ldots, N$

$$\begin{aligned} &\mathrm{E}_{x,\pi}[V_{n-1}^*(X_{N-n+1}^K, \pi_{N-n+1}^K)] \\ &\quad = \mathrm{E}_{x,\pi}[\,\mathrm{E}_{x,\pi}[V_{n-1}^*(X_{N-n+1}^K, \pi_{N-n+1}^K) \mid \mathcal{F}_{N-n}]\,] \\ &\quad = \mathrm{E}_{x,\pi}[V_n(X_{N-n}^K, \pi_{N-n}^K, A_{N-n}, \eta_{N-n})] \\ &\quad \leq \mathrm{E}_{x,\pi}[V_n^*(X_{N-n}^K, \pi_{N-n}^K)], \end{aligned}$$

hence $J_N^K(x,\pi) \leq V_N^*(x,\pi)$ for all admissible control strategies K. By (i) the strategy K^* attains V_N^*, so $J_N^* = V_N^*$ and K^* is optimal. □

Definition 4.1 *The sets $\mathcal{T}_a(k)$, $a \in \mathcal{A}_\mathcal{T}$, $k = 1, \ldots, N$, defined in the proof of Theorem 4.1 are called trading regions (at time k to maturity).*

By Definition 2.1, $\mathcal{T}_a(k)$ corresponds for $a < 0$ to a region where selling is optimal, and for $a > 0$ where buying is optimal.

Since the structure for logarithmic utility $U = \log$ is much simpler leading to an additive DPE we shall now specialize the existence result. Due to the factorization of the wealth we can define a *reward function* $g : \mathcal{K} \times \{u, d\} \to \mathbb{R}$,

$$g(x, \pi, a, \eta, y) = \log\left(\frac{1 - \alpha_a(x) + \beta_a(x)\pi}{1 + \beta_a(x)\eta}\right) + \log(r + \eta(y - r)),$$

for which we get for a control strategy $K = (A_n, \eta_n)_{n=0,\dots,N-1}$

$$J_N^K(x, \pi) = \mathrm{E}_{x,\pi}[\log(X_N^K)] = \log(x) + \mathrm{E}_{x,\pi}[g(X_{n-1}^K, \pi_{n-1}^K, A_{n-1}, \eta_{n-1}, Y_n)].$$

Note that the first term of g corresponds to the transaction costs and the second term to the gain/loss due to the evolution of the stock.

We introduce value functions which are more appropriate for logarithmic utility. For $(x, \pi) \in \mathcal{X}$, $(a, \eta) \in \mathcal{A}(x, \pi)$ we define $v_0^*(x, \pi) = 0$ and for $k = 1, \dots, N$

$$\begin{aligned} v_k(x, \pi, a, \eta) &= \mathrm{E}\left[\, g(x, \pi, a, \eta, Y) + v_{k-1}^*(f_x(x, \pi, a, \eta, Y), f_\pi(\eta, y)) \,\right], \\ v_k^*(x, \pi) &= \sup_{(a,\eta)\in\mathcal{A}(x,\pi)} v_k(x, \pi, a, \eta), \qquad (4.4) \\ v_k^a(x, \pi) &= \sup_{\eta\in\mathcal{A}_a(x,\pi)} v_k(x, \pi, a, \eta), \quad a \in \mathcal{A}_\mathcal{T}, \end{aligned}$$

where (4.4) is the classic DPE when substituting the expression for v. As above v^* can be obtained as maximum of the v^a, $a \in \mathcal{A}_\mathcal{T}$.

Corollary 4.1 *Suppose $U = \log$. For the DPEs (4.4) selectors $\varphi_1, \dots, \varphi_N$ exist which attain the supremum, i.e. denoting $\varphi_k = (\varphi_k^1, \varphi_k^2)$*

$$v_k^*(x, \pi) = v_k(x, \pi, \varphi_k^1(x, \pi), \varphi_k^2(x, \pi)), \quad k = 1, \dots, N.$$

v_k^* is upper semi continuous and bounded from above, $k = 0, \dots, N$. The trading strategy K^* given by $(A_n^*, \eta_n^*) = \varphi_{N-n}(X_n^*, \pi_n^*)$, $n = 0, \dots, N-1$ is optimal, so

$$J_N^*(x, \pi) = J_N^{K^*}(x, \pi) = \log(x) + v_N(x, \pi), \quad (x, \pi) \in \mathcal{X}.$$

Proof We show that for $k = 0, \dots, N$,

$$v_k^*(x, \pi) = V_k^*(x, \pi) - \log(x), \quad (x, \pi) \in \mathcal{X},$$

V^* defined like in Section 4. That is by definition true for $k = 0$. Suppose it holds for some $k < N$. Then

$$\begin{aligned} &V_{k+1}^*(x, \pi) \\ &= \sup_{(a,\eta)\in\mathcal{A}(x,\pi)} \mathrm{E}[V_k^*(f_x(x, \pi, a, \eta, Y), f_\pi(\eta, Y))] \\ &= \sup_{(a,\eta)\in\mathcal{A}(x,\pi)} \mathrm{E}[\log(f_x(x, \pi, a, \eta, Y)) + v_k^*(f_x(x, \pi, a, \eta, Y), f_\pi(\eta, Y))] \\ &= \log(x) + \sup_{(a,\eta)\in\mathcal{A}(x,\pi)} \mathrm{E}[g(x, \pi, a, \eta, Y) + v_k^*(f_x(x, \pi, a, \eta, Y), f_\pi(\eta, Y))] \\ &= \log(x) + v_k^*(x, \pi). \end{aligned}$$

Hence the existence of an optimal strategy K^* and $J_N^*(x,\pi) = \log(x) + v_k^*(x,\pi)$ follows from Theorem 4.1. Further $(x,\pi,a,\eta) \in \mathcal{K}$ implies $1 - c(x,\pi,a,\eta)/x < 1$, hence

$$\begin{aligned} g(x,\pi,a,\eta,y) &= \log(1 - c(x,\pi,a,\eta)/x) + \log(r + \eta(y-r)) \\ &\leq 0 + \log(r + (u-r)) = \log(u). \end{aligned}$$

Thus the value functions v_k^* are bounded from above. □

Remark 4.1 (i) If we can determine the selectors η_a, $a \in \mathcal{A}_\mathcal{T}$, then the proof of Theorem 4.1 is constructive. The same applies to Corollary 4.1. We will exploit this fact in Section 6.

(ii) Note that for logarithmic utility we have all ingredients for the formulation as a decision model like e.g. specified in [15]: The state space $\mathcal{X}$, the action space $\mathcal{A}$, the feasible actions $\mathcal{A}(x,\pi)$ given $(x,\pi) \in \mathcal{X}$, the measurable set $\mathcal{K}$ of feasible state-action pairs, the transition law f, and a measurable reward function $g : \mathcal{K} \to I\!R$. Nevertheless, even for logarithmic utility the selection results cannot be used directly as pointed out above. It is easier to do it for each type of trading.

(iii) Under transaction costs there are different ways to evaluate the terminal wealth, compare the discussion in [14]. We take at the terminal time the utility of the current wealth thus assuming that the investor can liquidate his position in the stocks without paying transaction costs. If instead we require that the investor has to sell all his stocks we get further feasibility conditions on the control strategy corresponding to the requirement that the investor can afford the transaction fees at the terminal time. Using that criterion we had time dependent feasible state-action sets $\mathcal{A}(n,x,\pi)$.

Example 4.1 Suppose $U = \log$ and that no transaction fees have to be paid. In our model we have to use at least types of trading $\mathcal{A}_\mathcal{T} = \{-1, 0, +1\}$. But without costs

$$\mathrm{E}[g(x,\pi,a,\eta)] = p\, \log(r + \eta(u-r)) + (1-p)\, \log(r + \eta(d-r)),$$

hence a maximization in η yields the optimal risky fraction

$$\hat{\eta} = \frac{(E[Y]-r)r}{(u-r)(r-d)} \tag{4.5}$$

and the optimal type of trading $a = \mathrm{sign}(\pi - \hat{\eta})$. The optimal value $v_1^*(x,\pi) = \mathrm{E}[\log(r + \hat{\eta}(Y-r))]$ does not depend on x, π, a. In the next period by Corollary 4.1 we hence have to carry out the same maximization as above yielding again $\hat{\eta}$. By induction $v_k(x,\pi)$, $k \in I\!N$, does not depend on x, π and hence $\hat{\eta}$ is always the unique optimal choice.

5 Selling, buying, and holding in one period

To illustrate the structure of the trading regions depending on the transaction costs we concentrate from now on on logarithmic utility. In this section we discuss the case of one period in which some explicit results can be derived. To simplify the notation we denote e.g. $\eta = \eta_0$, $a = A_0$, $Y = Y_1$, $v = v_1$ etc. and we concentrate on the transaction costs of Example 2.2 which by (3.9) become

$$c(x, \pi, a, \eta) = \frac{a^2 \Gamma(x) + a\,\gamma(\eta - \pi)}{1 + a\,\gamma\,\eta}\, x. \tag{5.1}$$

So we now have simply $\mathcal{A}_{\mathcal{T}} = \{-1, 0, +1\}$, where $a = 1$ corresponds to buying stocks, $a = -1$ to selling, and $a = 0$ to holding the stocks (no trading). For $a \in \{-1, +1\}$ we define $r_a = (1 + a\,\gamma)r$.

Assumption 5.1 *We assume that*

(A1) $u > r_{+1}$ *and* $r_{-1} > d$,

(A2) $\displaystyle \frac{r_{+1} - d}{u - d} \le p \le \frac{(r_{-1} - d)u}{r_{-1}(u - d)}$,

The condition (A1) rules out arbitrage possibilities (for the bank). (A2) is not essential but it reduces the number of cases we have to consider. It means that we consider moderate stocks for which the optimal risky fractions after buying or selling defined in (5.2) lie in $[0, 1]$. That the interval is not empty can be seen as a condition on γ. E.g. $\gamma \le (u-r)(r-d)/(r(u+d))$ is sufficient for the existence of a p satisfying (A2).

In Lemma 5.2 and Theorem 5.1 we prove that the optimal risky fractions after selling and buying are given by the constants

$$\hat{\eta}_a = \frac{(E[Y] - r_a)r}{(u - r_a)(r_a - d) - a\,\gamma(E[Y] - r_a)r}, \quad a \in \{-1, +1\}. \tag{5.2}$$

Lemma 5.1 *(i)* $0 \le \mathrm{E}[Y] - r_a \le (u - r_a)(r_a - d)/r_a$, $a \in \{-1, +1\}$.
(ii) $(u - r_a)(r_a - d) - a\,\gamma(E[Y] - r_a)r > 0$, $a \in \{-1, +1\}$.
(ii) $\hat{\eta}_a \in [0, 1]$, $a \in \{-1, +1\}$, *and* $\hat{\eta}_{+1} \le \hat{\eta} \le \hat{\eta}_{-1}$, *where* $\hat{\eta}$ *is given in (4.5) and equality holds if and only if* $\gamma = 0$.

Proof (i) follows from Assumption 5.1. (ii) can be derived using (i) and for $a = +1$ also $(u - r_{+1})(r_{+1} - d) > (u - r)(r - d)$ and $1 - \gamma\, r/r_{+1} = 1/(1 + \gamma)$. (iii) follows directly from the definition of $\hat{\eta}_a$ in (5.2) using (ii). □

Note that the feasible action sets simplify to

$$\mathcal{A}_{-1}(x, \pi) = [0, \min\{1, \pi/(1 - \Gamma(x)\}] \quad \text{if} \quad 1 - \Gamma(x) - \gamma\,\pi > 0, \tag{5.3}$$
$$\mathcal{A}_{+1}(x, \pi) = [\pi/(1 - \Gamma(x)), 1] \quad \text{if} \quad 1 - \Gamma(x) > 0. \tag{5.4}$$

Lemma 5.2 *Suppose* $a \in \{-1, +1\}$, $(x, \pi) \in \mathcal{X}$ *satisfy* $\mathcal{A}_a(x, \pi) \neq \emptyset$. *Then*

(i) $\hat{\eta}_a$ *is the unique maximum of* $v(x,\pi,a,\cdot)$ *on* $[0,1]$.
(ii) $v^a(x,\pi)=v(x,\pi,\hat{\eta}_a)$ *if* $\hat{\eta}_a\in\mathcal{A}_a(x,\pi)$.
(iii) $\hat{\eta}_a\notin\mathcal{A}_a(x,\pi)$ *implies* $v^0(x,\pi)>v(x,\pi,a,\eta)$ *for all* $\eta\in\mathcal{A}_a(x,\pi)$.

Proof (i) For $a\in\mathcal{A}_{\mathcal{T}}$,

$$\begin{aligned}\partial_\eta v(x,\pi,a,\eta)&=-\frac{a\,\gamma}{1+a\,\gamma\,\eta}+\mathrm{E}\Big[\frac{Y-r}{r+\eta(Y-r)}\Big],\\&=\frac{(E[Y]-r_a)-\big((u-r_a)(r_a-d)-a\,\gamma(E[Y]-r_a)r\big)\eta}{(1+a\,\gamma\,\eta)(r+\eta(u-r))(r-\eta(r-d))}\,.\end{aligned}$$

Since $\eta\in[0,1]$ the denominator is strictly positive. Using Lemma 5.1 (ii) it hence follows that $\partial_\eta v(x,\pi,a,\eta)<,=,>0$ if $\eta>,=,<\hat{\eta}_a$. So $\hat{\eta}_a$ is the unique maximum. Due to Lemma 5.1 (iii) it lies in $[0,1]$. (ii) follows from (i).

(iii) Suppose $\hat{\eta}_{-1}\notin\mathcal{A}_{-1}(x,\pi)$ and $\eta\in\mathcal{A}_{-1}(x,\pi)$. Then $1-\Gamma(x)>0$ and $\hat{\eta}_{-1}\geq\pi/(1-\Gamma(x))>\eta$. Therefore

$$v(x,\pi,-1,\eta)<v(x,\pi,-1,\pi/(1-\Gamma(x)))\tag{5.5}$$

since $v(x,\pi,-1,\cdot)$ is strictly increasing on $[0,\eta_{-1}]$ as we showed in (i) above. Due to

$$\begin{aligned}v(x,\pi,-1,\pi/(1-\Gamma(x)))&\leq\log(1-\Gamma(x))+\mathrm{E}[\log(r+\pi/(1-\Gamma(x))(Y-r))]\\&=\mathrm{E}[\log((1-\Gamma(x))r+\pi(Y-r))]\\&\leq\mathrm{E}[\log(r+\pi(Y-r))]=v^0(x,\pi)\end{aligned}$$

the claim follows for $a=-1$ from (5.5); and for $a=+1$ by an analogous argument. □

Lemma 5.2 (iii) implies that trading can only be optimal, if $\hat{\eta}_a\in\mathcal{A}_a(x,\pi)$ for $a=-1$ or $a=+1$. In these cases (5.3), (5.4) imply $\pi\geq\hat{\eta}_{-1}(1-\Gamma(x))$ and $\pi\leq\hat{\eta}_{+1}(1-\Gamma(x))$, respectively. This simplifies the search for the boundaries of the trading regions for whose description we introduce (using $\inf\emptyset=\infty$, $\sup\emptyset=-\infty$)

$$\begin{aligned}\pi_{-1}(x)&=\inf\{\pi\in[\hat{\eta}_{-1}(1-\Gamma(x)),1]\,|\,v_{-1}(x,\pi)>v_0(x,\pi)\},\\\pi_{+1}(x)&=\sup\{\pi\in[0,\hat{\eta}_{+1}(1-\Gamma(x))]\,|\,v_{+1}(x,\pi)>v_0(x,\pi)\},\end{aligned}$$

and $\mathcal{T}_a(x)=\{\pi\in[0,1]\,|\,(x,\pi)\in\mathcal{T}_a\}$, where $\mathcal{T}_a$ was defined in the proof of Theorem 4.1. The thresholds for $\mathcal{T}_{-1}(x)$, $\mathcal{T}_{+1}(x)$ to be empty will be

$$\begin{aligned}C_{-1}&=(1-\gamma)\Bigg(1-\Big(\frac{(r_{-1}-d)u}{r_{-1}(u-d)p}\Big)^p\Big(\frac{d(u-r_{-1})}{r_{-1}(u-d)(1-p)}\Big)^{1-p}\Bigg),\\C_{+1}&=1-\Big(\frac{r_{-1}-d}{(u-d)p}\Big)^p\Big(\frac{u-r_{-1}}{(u-d)(1-p)}\Big)^{1-p}.\end{aligned}$$

Theorem 5.1 *For all* $x>0$

(i) For all $\pi \in [0,1]$ *an optimal action is given by* $(a^*, \eta^*) = \varphi(x,\pi)$*, where*

$$\varphi(x,\pi) = \begin{cases} (a, \hat\eta_a), & \text{if } \pi \in \mathcal{T}_a(x), \quad a \in \{-1,+1\}, \\ (0,\pi), & \text{if } \pi \in \mathcal{T}_0(x), \end{cases}$$

where

$$\mathcal{T}_{+1}(x) = [0, \pi_{+1}(x)), \quad \mathcal{T}_{-1}(x) = (\pi_{-1}(x), 1].$$

Further $\mathcal{T}_0(x) = [0,1] \setminus (\mathcal{T}_{+1}(x) \cup \mathcal{T}_{-1}(x)) \supseteq [(1-\Gamma(x))\hat\eta_{+1}, \hat\eta_{-1}]$.

(ii) $v^*(x,\cdot)$ *is continuous.*

(iii) $\mathcal{T}_a(x) = \emptyset$ *if and only if* $\Gamma(x) \geq C_a$, $a \in \{-1,+1\}$.

(iv) $\hat\eta_{+1} \leq \hat\eta \leq \hat\eta_{-1}$ *where equality holds if and only if* $\gamma = 0$.

(v) $\pi_{+1}(x) \leq \hat\eta_{+1}$ *and* $\hat\eta_{-1} \leq \pi_{-1}(x)$ *where equality holds if and only if* $\Gamma(x) = 0$.

Proof Since x is fixed we drop it as argument to simplify our notation.

(a) For $a \in \{-1,+1\}$ we define $h_a : \{\pi \in [0,1] \,|\, 1-\Gamma + a\gamma\pi > 0\} \to \mathbb{R}$ by

$$h_a(\pi) = v(x,\pi,a,\hat\eta_a) - v(x,\pi,0,\pi).$$

So $\mathcal{T}_a = \{\pi \in [0,1] \,|\, \mathcal{A}_a(x,\pi) \neq \emptyset,\ h_a(\pi) > 0\}$. Due to Lemma 5.2 (iii) $\mathcal{A}_a(x,\pi) \neq \emptyset$ and $h_a(\pi) > 0$ imply $\pi \geq (1-\Gamma)\hat\eta_{-1}$ for $a = -1$ and $\pi \leq (1-\Gamma)\hat\eta_{+1}$ for $a = +1$. Therefore $\mathcal{T}_{-1} \subseteq [\pi_{-1}, 1]$ and $\mathcal{T}_{+1} \subseteq [0, \pi_{+1}]$. We compute

$$\begin{aligned}\partial_\pi h_a(\pi) &= \frac{a\gamma}{1-\Gamma+a\gamma\pi} + \mathrm{E}\Big[\frac{Y-r}{r+\pi(Y-r)}\Big] \\ &= \frac{\psi_a\pi - \big(\mathrm{E}[Y]-r_a\big) - (\mathrm{E}[Y]-r)\big)r\,\Gamma}{(1-\Gamma+a\gamma\pi)(r+\pi(u-r))(r-\pi(r-d))},\end{aligned}$$

where

$$\psi_a = (u-r_a)(r_a-d) - a\gamma r(E[Y]-r_a) - (u-r)(r-d)\Gamma. \tag{5.6}$$

The denominator is strictly positive. If $\psi_a = 0$ the sign of $\partial_\pi h_a$ is constant. In the case $\psi_a > 0$

$$\partial_\pi h_a(\pi) < \ (=, >)\, 0 \quad \text{for} \quad \pi < \ (=, >)\, \pi_{0,a} \tag{5.7}$$

and if $\psi_a < 0$:

$$\partial_\pi h_a(\pi) > \ (=, <)\, 0 \quad \text{for} \quad \pi < (=, >)\, \pi_{0,a}, \tag{5.8}$$

where

$$\pi_{0,a} = \big((E[Y]-r_a) - (\mathrm{E}[Y]-r)\Gamma\big)r/\psi_a.$$

Note that $\pi_{0,a}$ might not lie in the domain of h_a.

(b) If $\pi_{-1} = \inf\{\pi \in [(1-\Gamma)\hat\eta_{-1}, 1] \,|\, h_{-1}(\pi) > 0\} = \infty$ then $\mathcal{T}_{-1} = \emptyset = (\pi_{-1}, 1]$. And $\mathcal{T}_{+1} = \emptyset$ if $\pi_{+1} = \sup\{\pi \in [0, (1-\Gamma)\hat\eta_{+1}] \,|\, h_{+1}(\pi) > 0\} = -\infty$. So for each $a \in \{-1,+1\}$ we have only to consider the case $\pi_a \in [0,1]$

(c) $\pi_a \in [0,1]$ implies $h_a(\pi_a) = 0$ since h_a is continuous and $h_a((1-\Gamma)\hat{\eta}_a) \le 0$.

(d) Suppose $\Gamma = 0$. Then $\pi_{0,a} = \hat{\eta}_a$, $\psi_a > 0$ by Lemma 5.1 (ii), and $h_a(\hat{\eta}_a) = 0$. From (5.7) hence follows $h_{-1}(\pi) > 0$ for all $\pi \in (\hat{\eta}_{-1}, 1]$. So $\pi_{-1} = \hat{\eta}_{-1} = \pi_{0,-1}$ and $\mathcal{T}_{-1} = (\pi_{-1}, 1]$. From (5.7) also follows $\mathcal{T}_{+1} = [0, \pi_{+1})$ analogously. So $\mathcal{T}_0 = [\pi_{+1}, \pi_{-1}] = [\hat{\eta}_{+1}, \hat{\eta}_{-1}]$.

(e) Suppose $\Gamma > 0$. By (a) $\pi_{-1} \ge (1-\Gamma)\hat{\eta}_{-1}$ and $\pi_{+1} \le (1-\Gamma)\hat{\eta}_{+1}$. As in the proof of Lemma 5.2 $h_a((1-\Gamma)\hat{\eta}_a) < 0$. Therefore the structure of $\partial_\pi h_a$ derived in (a) and (c) imply

$$\partial_\pi h_{+1}(\pi_{+1}) < 0 \quad \text{and} \quad \partial_\pi h_{-1}(\pi_{-1}) > 0. \tag{5.9}$$

Since $h_{-1}(\hat{\eta}_{-1}) < 0$ and $(1-\Gamma)\hat{\eta}_{-1} < \hat{\eta}_{-1}$, (a) and (c) even imply

$$\pi_{+1} < (1-\Gamma)\hat{\eta}_{+1} < \hat{\eta}_{-1} < \pi_{-1} \tag{5.10}$$

Furthermore, because $\pi_{0,a} = (\hat{\eta}_a^n - \Gamma\hat{\eta}^n)/(\hat{\eta}_a^d - \Gamma\,\hat{\eta}^d)$, where $\hat{\eta}_a^n$, $\hat{\eta}^n$, $\hat{\eta}_a^d$, $\hat{\eta}^d$ denote the numerators and denominators in the representations (5.2), (4.5), and because by Lemma 5.1 (iii) $\hat{\eta}_{+1} \le \hat{\eta} \le \hat{\eta}_{-1}$, it is easy to see that

$$\begin{aligned} \pi_{0,-1} &\le (\ge)\, \hat{\eta}_-, &\text{if} \quad \psi_{-1} &< (>)\, 0, \\ \pi_{0,+1} &\ge (\le)\, \hat{\eta}_+, &\text{if} \quad \psi_{+1} &< (>)\, 0. \end{aligned} \tag{5.11}$$

(e1) Suppose $\psi_a < 0$. Then for $a = -1$ we have $\pi_{0,-1} \le \hat{\eta}_{-1}$. Thus (5.8) implies that $\partial_\pi h_{-1}(\pi) < 0$ for all $\pi > \hat{\eta}_{-1}$ hence $\mathcal{T}_{-1} = \emptyset$. For $a = 1$ we have $\hat{\eta}_{+1} \le \pi_{0,+1}$ hence $\partial_\pi h_{+1}(\pi) > 0$ for all $\pi < \hat{\eta}_{+1}$ yielding $\mathcal{T}_{+1} = \emptyset$.
(e2) Suppose $\psi_a > 0$. For $a = +1$ we have $\pi_{0,+1} \le \hat{\eta}_{+1}$ by (5.11). If $\pi_{+1} > -\infty$ it follows by (5.9) and (5.7) that $\partial_\pi h_{+1}(\pi) < 0$ for all $\pi \le \pi_{+1}$ hence $\mathcal{T}_{+1} = (\pi_{+1}, 1]$. For $a = -1$ using $\pi_{0,-1} \ge \hat{\eta}_{-1}$ it follows similarly that $\mathcal{T}_{-1} = (\pi_{-1}, 1]$ if $\Gamma(x) < 1-\gamma$. But in the case $\Gamma \ge 1-\gamma$ the domain of h_{-1} is only $[0, (1-\Gamma)/\gamma]$ and outside of this domain selling cannot be optimal (ruined by the transaction costs). So if π_{-1} would exist in $[0,1]$ we can have a shorter interval for $\mathcal{T}_{-1}$ than claimed. But $h_{-1}(\pi) \to -\infty$ if $\pi \to (1-\Gamma)/\gamma$. Therefore $\partial_\pi h_a(\pi) < 0$ for all $(1-\Gamma)/\gamma - \varepsilon < \pi|(1-\Gamma)/\gamma$ for some $\varepsilon > 0$. So (5.7) implies $\pi_{0,-1} \ge (1-\Gamma)/\gamma$ and therefore $\partial_\pi h_{-1}(\pi) < 0$ for all π in the domain of h_{-1}. So $\mathcal{T}_{-1} = \emptyset$ by (5.9).

(e3) Suppose $\psi_a = 0$. Then $\partial_\pi h_a$ has always the same sign. Thus by (5.9) $\pi_{+1} > -\infty$ implies $\partial_\pi h_{+1} < 0$ and thereby $\mathcal{T}_{+1} = [0, \pi_{+1})$. Similarly $\pi_{-1} < \infty$ implies $\mathcal{T}_{-1} = (\pi_{-1}, 1]$ if $\Gamma < 1-\gamma$. With the same argument as in (e2) the derivative has to be negative in the case $\Gamma \ge 1-\gamma$, hence $\mathcal{T}_{-1} = \emptyset$.

(i) follows now from (d)… (e3) and Corollary 4.1. In (ii) $v^*(x,\cdot)$ is continuous since $h_a(\pi_a) = 0$ and v^a, v^0 are continuous. For (iii) a lengthy computation yields that the conditions are equivalent to $h_{-1}(1) > 0$ and $h_{+1}(0) > 0$, respectively. (iv) is Lemma 5.3 (iii), and (d) for $\Gamma = 0$ and (5.10) for $\Gamma > 0$ yield (v). □

Without proof we state some further properties of the boundaries of the trading regions.

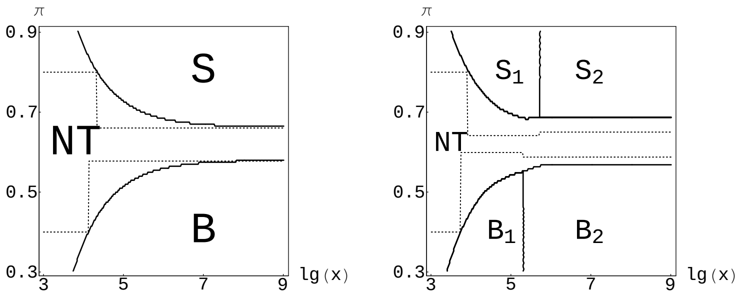

Fig. 1 Constant and proportional costs in (a) one period and (b) two periods

Proposition 5.1 *(i) If Γ is constant (fixed costs), so is π_a, $a \in \{-1, +1\}$.*
(ii) Suppose Γ is strictly decreasing. Then π_{-1} is strictly decreasing on $\{x > 0 \mid \mathcal{T}_{-1}(x) \neq \emptyset\}$ and π_{+1} is strictly increasing on $\{x > 0 \mid \mathcal{T}_{+1}(x) \neq \emptyset\}$.
(iii) Moreover $\lim_{x\to\infty} \Gamma(x) = 0$ implies $\lim_{x\to\infty} \pi_a(x) = \hat{\eta}_a$, $a \in \{-1, +1\}$.

6 Some examples

For all examples we use $U = \log$ and the parameters

$$u = 1.25, \quad d = 0.8, \quad r = 1.04, \quad p = 0.6.$$

Without transaction costs the optimal risky fraction, see Example 4.1, is $\hat{\eta} = 0.619048$.

Example 6.1 We consider a combination of constant and proportional costs given by (5.1) with $\Gamma(x) = 10/x$ and $\gamma = 0.002$, or in the notation of (2.8) by $\alpha_a(x) = a^2 10/x$, $\beta_a(x) = 0.002\,a$ for $a \in \{-1, 0, +1\}$. For one period the trading regions are plotted in Figure 1 (a). The no-trading region, the selling region and the buying region are indicated by NT, S, and B, respectively. For initial values $\pi = 0.8$ and $\pi = 0,4$ the optimal new risky fraction lies on the upper dotted line (selling) and on the lower dotted line (buying). Like stated in Theorem 5.1 the optimal new risky fraction depends neither on x nor on π if we trade at all.

The trading regions for two periods are given in Figure 1 (b). For their computation we use the DPE in Corollary 4.1. They look similar to the one-period case. But we observe that the trading regions for selling and buying split up in $\mathcal{T}_{-1} = S_1 \cup S_2$ and $\mathcal{T}_{+1} = B_1 \cup B_2$, respectively. Actually, S_1 corresponds to subsequent trading $(0, 0)$, S_2 to $(-1, 0)$, B_1 to $(0, 0)$, and B_2 to $(0, 1)$, where (a_u, a_d) means that we choose in the next period the type of trading a_u if the stock prices go up and a_d if they go down.

This is due to the fact that with increasing wealth the constant costs become relatively small, hence at some point it might be optimal to do another trade in the next period. At these points the boundaries between

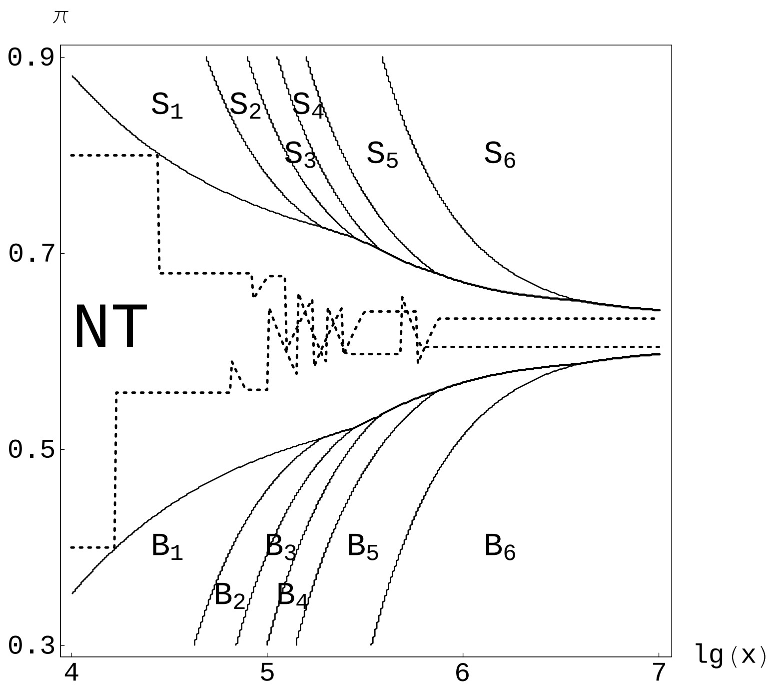

Fig. 2 Trading regions in one period for Example 6.2

the trading regions are not differentiable (as functions in x) and we can observe a jump in the optimal new risky fractions (dotted curves as above). Furthermore, what cannot be seen from the figure, if we do another trade in the next period, the optimal risky fraction is no longer constant in x and there is a small dependence on π, hence the boundary between S_1 and S_2 is not vertical.

Example 6.1 shows that many of the properties derived for one period do not carry over to the multi-period case. In the next example we consider a more complex cost structure for which even in one period the nice properties derived in the last section do not hold.

Example 6.2 We now consider the constant plus piecewise proportional costs of Example 2.1. So we have 13 types of trading $a \in \{-6, \dots, 6\}$ and use transaction costs which are in the notation of (2.8) given by

$$\alpha_a(x) = \operatorname{sign}(a)^2 10, \quad \beta_a(x) = \operatorname{sign}(a)\gamma_{|a|},$$

where $\gamma = 10^{-3}(0, 2.94, 2.8, 2.1, 1.4, 1.05, 0.7)^\top$. For one period the trading regions in Figure 2 are computed using the DPE in Corollary 4.1.

Now we observe that the buying and selling regions split in several subregions which correspond to a certain level of the proportional costs. Again we included the optimal risky fractions for initial $\pi = 0.8$ and initial $\pi = 0.4$. So when we increase x for fixed $\pi = 0.8$ it is for small x not optimal to sell, hence the optimal risky fraction (upper dotted curve) coincides with the initial risky fraction. Then we reach the selling region $S_1 = \mathcal{T}_{-1}$ and a certain constant new risky fraction is optimal. But with increasing x the corresponding transaction volume increases so we get closer to the interval where lower proportional costs have to be paid. If we are quite close it is optimal to sell more stocks such that the transaction is bigger and we only have to pay these lower costs, corresponding to reaching $S_2 = \mathcal{T}_{-2}$ and a jump in the optimal risky fraction. With increasing x we have to sell less stocks to get in favour of the lower costs, so the new risky fraction grows linearly until we reach the new constant optimal risky fraction corresponding to the lower costs. For the next two jumps in the costs we do not reach this constant fraction but for $a = -5, -6$ the risky fraction becomes constant for x high enough.

In Example 6.2 we could compute the trading regions for more periods using the DPE. We then observe that the regions split also depending on which subsequent trading is optimal, like we saw it in Example 6.1 for two periods. But due to the 13 types of trading and the maximization step in the recursion the algorithm needs some time. In the next example we consider fees for which the structure of the trading regions becomes much simpler and hence a fast algorithm can be implemented.

Example 6.3 We consider a combination of fixed costs $\delta \in (0,1)$ and proportional costs $\gamma \in [0,1)$, so

$$\alpha_a(x) = a^2\delta, \quad \beta_a(x) = a\,\gamma, \quad a \in \{-1,0,1\}.$$

Then the reward function g and hence the selectors η_a, the boundaries π_a, and value functions v do not depend on x. In fact we have $J_N^*(x,\pi) = \log(x) + v_N^*(\pi)$, where $v_N^*(\pi) = \sup_{(a,\eta)\in\mathcal{A}(x,\pi)} v_N(\pi,a,\eta)$,

$$\begin{aligned} v_N(\pi,a,\eta) = {}& \log(1 - a^2\delta + a\,\gamma\,\pi) \\ & - \log(1 + a\,\gamma\,\eta) + \mathrm{E}[\log((1-\eta)r^\tau + \eta Y_1 \dots Y_\tau)] \\ & + \log(1 - A_\tau^2\delta + A_\tau\gamma\pi_\tau) + w_{A_\tau}(N-\tau), \end{aligned} \tag{6.1}$$

where

$$\tau = \inf\{n > 0 \mid \pi_t \notin [\pi_{+1}(n), \pi_{+1}(n)]\} \wedge N,$$

$A_\tau = -1$ if $\pi_\tau > \pi_{-1}(\tau)$ and $A_\tau = +1$ if $\pi_\tau < \pi_{+1}(\tau)$, where

$$\pi_n = \frac{\eta Y_1 \dots Y_n}{(1-\eta)r^n + \eta Y_1 \dots Y_n},$$

on $\{\tau \geq n\}$. By an induction it can be seen that $w_a(0) = 0$ and

$$w_a(k) = v_k^a(\pi) - \log(1 - a^2\delta + a\,\gamma\,\pi), \quad k = 1, \dots, N-1,$$

do not depend on π, and hence the maximization over η yields an optimal $\eta_a(N)$ which is also independent of π.

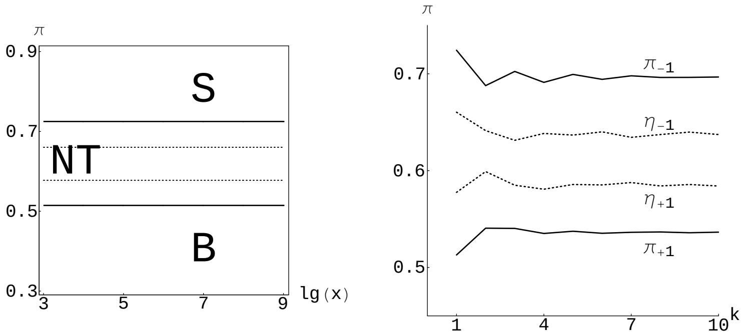

Fig. 3 Fixed and proportional costs: (a) Trading regions in one period and (b) depending on time

Table 2 Optimal values for fixed and proportional costs

N	$\eta_{-1}(N)$	$\eta_{+1}(N)$	$\pi_{-1}(N)$	$\pi_{+1}(N)$	$w_{-1}(N)$	$w_{+1}(N)$
1	0.660333	0.577557	0.724336	0.512908	0.049511	0.047034
2	0.641312	0.599012	0.687770	0.540564	0.097658	0.095177
3	0.631311	0.584919	0.702428	0.540340	0.145761	0.143315
4	0.638363	0.580806	0.691129	0.535111	0.193907	0.143315
5	0.636709	0.585672	0.699363	0.537388	0.242022	0.239569
10	0.637257	0.584085	0.696669	0.536461	0.482657	0.480199
20	0.639709	0.584091	0.696433	0.536069	0.963921	0.961461
30	0.639719	0.584088	0.696467	0.536082	1.44518	1.44272

So at time k to maturity there exist for each $a \in \{-1, 1\}$ constant optimal $\eta_a(k)$, boundaries of the trading regions $\pi_a(k)$, and optimal values $w_a(k)$. Going backwards and having computed these optimal constants for $k = 1, \ldots, N-1$, $a \in \{-1, +1\}$, the new constants at time N can be computed using (6.1). This backward algorithm is much faster than using the DP algorithm directly since we cannot compute the value functions v^* explicitly hence we had to use maximization steps in the recursion for solving the DPE.

For fixed and proportional costs $\delta = 0.0001$ and $\gamma = 0.002$ we observe that for one period the corresponding trading regions in Figure 3 (a) do not depend on x or π, a feature that also holds in the multi-period case as explained above. In Figure 3 (b) the corresponding optimal boundaries depending on the time to maturity are plotted for 10 periods and in Table 2 some of the corresponding values up to 30 periods are given.

For the algorithm to work as described we need to impose some conditions on the terms in (6.1) which depend on η, if we want to make sure that the optimal risky fractions always lie in $[0, 1]$, as we did in Assumption 5.1 (A2) for the one-period case.

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