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Lattice polygons and the number $2i + 7$

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LATTICE POLYGONS AND THE NUMBER $\mathbf{2i+7}$

CHRISTIAN HAASE AND JOSEF SCHICHO

Abstract. In this note we classify all triples (a, b, i) such that there is a convex lattice polygon P with area a, and p respectively i lattice points on the boundary respectively in the interior. The crucial lemma for the classification is the necessity of $b \leq 2\,i{+}7$. We sketch three proofs of this fact: the original one by Scott [Sco76], an elementary one, and one using algebraic geometry.

As a refinement, we introduce an onion skin parameter ℓ: how many nested polygons does P contain? Then we use the "12" of Poonen and Villegas [PV00] to give sharper bounds.

0. Introduction

0.1. **How it all began.** When the second author applied a result on algebraic surfaces to the special case of toric surfaces, he got an inequality for lattice polygons that looks pretty harmless. Nevertheless, he could not prove it in an elementary way. The first author – originally an anonymous referee – came up with an elementary proof. The result was then identified as due to Scott [Sco76] by yet another referee.

But once you get started to draw lattice polygons on graph paper and to discover relations between their numerical invariants, it is not so easy to stop. The gentle reader has been warned! So, it was just unavoidable that the authors came up with new inequalities: Scott's inequality can be sharpened if one takes another invariant into account, which is defined by peeling off the skins of the polygons like an onion (see section 4).

0.2. **Lattice polygons.** We want to study convex lattice polygons: convex polygons all whose vertices have integral coordinates. As it turns out, we need to consider non-convex polygons as well. Even non-simple polygons – polygons with self intersection – will prove useful later on.

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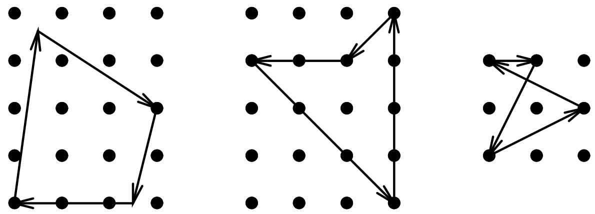

FIGURE 1. Polygons – convex, lattice, non-simple.

A closed polygon P is simple if it has no self-intersections. Simple polygons divide the plane into an interior and an exterior region. The quantities that we are interested in are the area $a = a(P)$ of the interior, and the numbers $b = b(P)$, $i = i(P)$ of lattice points on the boundary of respectively strictly inside the interior part. In what follows, we will say "polygon" for "simple polygon", and emphasize when we allow non-simple situations.

The classic result that relates the quantities a, b, and i is Pick's Formula. It describes the affine plane spanned by $(a(P), b(P), i(P))$ for lattice polygons P.

Theorem 1 (Pick's Formula [Pic99])**.**

$$a = i + \frac{b}{2} - 1$$

The crucial observation is that it is true for $b = 3$, $i = 0$: every triangle without lattice points but the vertices has area $a = 1/2$ $\big(= i + b/2 - 1 \big)$.[1] Then the general formula is obtained by double counting and Euler characteristic (compare the Book proof [AZ04, §11.3]).

0.3. **Lattice equivalence.** The geometry we are dealing with is not the plane geometry one is used to. For example, the number of lattice points in a polygon is *not* preserved by rigid motions of the plane. So we cannot regard two polygons as equivalent if they are related by a rigid motion. In this paragraph we explain the correct notion of equivalence for our purposes, and collect some properties for later use.

A lattice equivalence (of the plane) is a lattice preserving affine map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$. Two subsets of the plane are lattice equivalent if there is a lattice equivalence which maps one to the other.

Orientation preserving lattice equivalences form a group, the semi direct product $\mathrm{SL}_2\mathbb{Z} \ltimes \mathbb{Z}^2$.

So Φ has the form $\Phi(\mathbf{x}) = A\mathbf{x} + \mathbf{y}$ for a matrix A, and a vector $\mathbf{y}$. By lattice preserving we mean $\Phi(\mathbb{Z}^2) = \mathbb{Z}^2$ which implies that both A and $\mathbf{y}$

[1] In fact, all triangles with $b = 3$, $i = 0$ are lattice equivalent (see § 0.3).

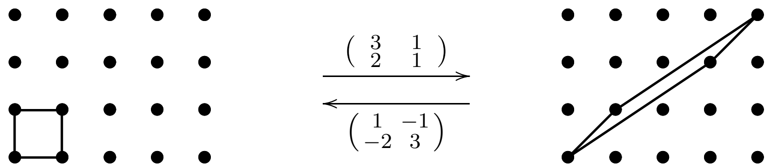

FIGURE 2. Two lattice equivalent quadrangles.

have integral entries, and the same is true for the inverse transformation $\Phi^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} - A^{-1}\mathbf{y}$. Hence $\det A = \pm 1$. All measures that we are interested in remain invariant under lattice equivalence.

Lemma 2. *Two lattice equivalent polygons have the same area as well as the same number of lattice points in the interior and on the boundary.*

On the other hand, angles and Euclidian lengths are not preserved. For example, the quadrangle in Figure 2 on the right looks to us like a perfect square. So it is not natural to work with an inner product in this context.[2] The correct notion of length of a lattice segment is the number of lattice points it contains minus one. We call a lattice segment of length one primitive (i.e., its end points are the only lattice points it contains). In this sense, b is the length of P. The following exercise will help to get a feeling for what lattice equivalence means.

Exercise 3. *Given a vertex* $\mathbf{x}$ *of a polygon* P*, show that there is a unique orientation preserving lattice equivalence* Φ *so that* $\Phi(\mathbf{x}) = (0,0)^t$*, and* $\Phi(P)$ *contains the segments* $[(1,0)^t, (0,0)^t]$ *and* $[(0,0)^t, (-p,q)^t]$ *with coprime* $0 < p \leq q$*.*

0.4. **Why algebraic geometry?** Toric geometry is a powerful link connecting discrete and algebraic geometry. At the heart of this link is the simple correspondence

$$\begin{array}{ccc} \text{lattice point} & & \text{Laurent monomial} \\ \mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{Z}^m & \longleftrightarrow & \mathbf{x}^{\mathbf{p}} = x_1^{n_1} \cdot \ldots \cdot x_m^{n_m} \in \Bbbk[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \end{array}$$

It was invented by Demazure [Dem70] for a totally different purpose (to study algebraic subgroups of the Cremona group) in algebraic geometry. Stanley used it to classify the possible face numbers of simplicial convex polytopes [**?**]. Today, we have many interactions in algebraic and discrete geometry via toric geometry (e.g. [Kho77, Stu91, Roj94]). A quite surprising new application in computer-aided design has been found in [Kra01].

[2]...and consequently, we will distinguish $\mathbb{R}^2$ and its dual space $(\mathbb{R}^2)^*$ of linear functionals.

For any finite set $S = \{\mathbf{p}_0, \dots, \mathbf{p}_n\} \subset \mathbb{Z}^m$, the associated projective toric variety $X(S)$ is the projective closure of the image $V(S)$ of the rational map $\mathbb{C}^m \to \mathbb{P}^n$ defined by $\mathbf{x} \to (\mathbf{x}^{\mathbf{p}_0}:\dots:\mathbf{x}^{\mathbf{p}_n})$. Its projective normalization in the field $\mathbb{C}(\mathbf{x})$ is also a toric variety, associated to the convex hull of S. Algebraic geometers have a preference for normal varieties, which is why toric geometry deals a lot with convex polytopes. Lattice equivalent polytopes define the same projective toric variety.

As to be expected, there is a dictionary translating toric geometry to lattice geometry (see [Ful93]). For instance, the dimension of $V(S)$ is equal to the dimension of the affine hull of S. If the dimension is m, then the degree of $V(S)$ is equal to $m!$ times the volume of the convex hull P of S. The number of interior points of P is equal to the *sectional genus*, which is defined as the geometric genus of a generic hyperplane section of $X(S)$. The number of vertices of P is equal to the Euler characteristic of $X(P)$. Pick's formula appears as a consequence of the Theorem of Riemann-Roch.

0.5. **Inequalities.** Pick's theorem is not the only relation between these three invariants of convex lattice polygon (invariant under unimodular lattice transformations). There is the rather obvious constraint $b \geq 3$. Moreover, [Sco76] showed that if $i \geq 2$, then $b \leq 2\ i + 6$, and if $i = 1$, then $b \leq 2\ i + 7$, For $i = 0$, b can be an arbitrary integer greater than or equal to 3.

Are the above the only restrictions for a, i, b? The answer is yes, and we will give a graphical proof (see Figures 3-6).

Scott's proof is elementary and short enough to be included in this paper. We give two other proofs for the same result. One of them uses toric geometry. Actually, it is nothing but the observation that the inequality is a consequence of a well-known inequality [Sch99] in algebraic geometry applied to toric varieties. The third proof is elementary, but the main technique of vertex clipping also has a toric interpretation: it corresponds to the blowing up construction for toric varieties.

For the inequality $b \leq 2\ i + 6$, we have arbitrary large examples where equality holds (see Figure 5); but for all these examples, all interior points are collinear. Under the additional assumption that the interior points are not collinear, the inequality can be strengthened to $b \leq i + 9$ (see the remark after Lemma 12). The coefficient in front of the i can be improved further by introducing the *level* of a convex lattice polygon: roughly speaking, this is the number of times one can pass to the convex hull of the interior lattice points.

1. Examples

Let us approach the question which parameters are possible for convex lattice polygons by looking at some examples. Can we bound i or a in terms of b ? From Pick's formula we obtain immediately $a \geq i + \frac{1}{2}$ and $a \geq \frac{b}{2} - 1$. What about bounds in the opposite direction? Figure 3 shows examples with $b = 3$ and arbitrarily high a and i. (No isoperimetric inequality.) Can we bound b in terms of i? Well, there is

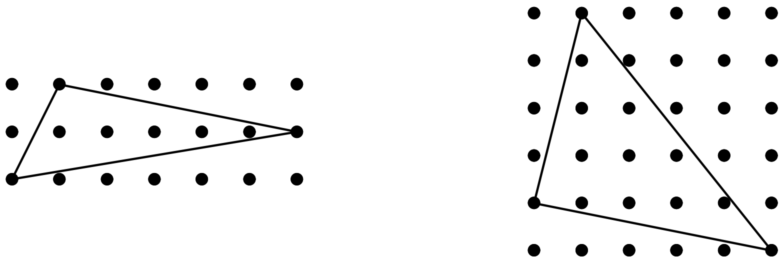

Figure 3. $b = 3$ and $a \gg 0$

the family of Figure 4 with $i = 0$ and arbitrary b. But as we will see,

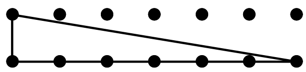

Figure 4. $i = 0$ and $b \gg 0$.

as soon as $i > 0$, no such family can exist. We can achieve $b = 2i+7$ if $i = 1$, and $b = 2i + 6$ if $i \geq 2$ (see Figure 5). Figure 6 shows that any

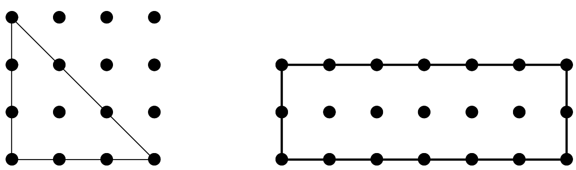

Figure 5. $b = 2i + 7$ respectively $b = 2i + 6$

pair (b, i) with $4 \leq b \leq 2i + 5$ can be realized.

Before we really get going, here is a little caveat. Most of our considerations break down in dimension 3. Pick's formula has no analogue. Already tetrahedra without lattice points but the vertices can have arbitrary volume, as was first pointed out by John Reeve [Ree57] (compare Figure 7).

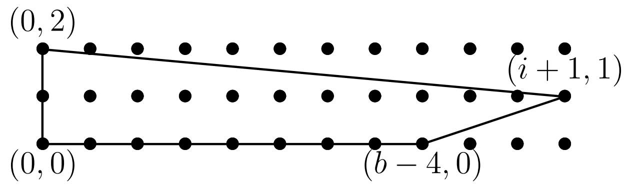

FIGURE 6. $4 \leq b \leq 2i + 5$

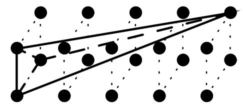

FIGURE 7. Reeve's simplices

2. "**12**"

In this section we develop the necessary notions to state the result of Poonen and Villegas that the lengths of a polygon and its dual add up to 12. While we try to approximate a self-contained exposition, this section can be thought of as an advertisement for their excellent paper [PV00].

Consider a primitive oriented segment $s = [\mathbf{x}, \mathbf{y}]$, i.e., $\mathbf{x}$ and $\mathbf{y}$ are the only lattice points s contains. Call s admissible if the triangle $\mathrm{conv}(\mathbf{0}, \mathbf{x}, \mathbf{y})$ contains no other lattice points. Equivalently, s is admissible if the determinant $\mathrm{sign}(s) = \left| \begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix} \right|$ is equal to ± 1. The length of a sequence $(s^{(1)}, \ldots, s^{(n)})$ of admissible segments is $\sum \mathrm{sign}(s^{(k)})$. The analogous definitions will be used for segments $[\mathbf{a}, \mathbf{b}]$ in dual space.

The dual of an admissible segment is the unique integral normal vector $\mathbf{a} = \mathbf{a}(s)$ such that $\langle \mathbf{a}, \mathbf{x} \rangle = \langle \mathbf{a}, \mathbf{y} \rangle = 1$. For a closed polygon with segments $(s^{(1)}, \ldots, s^{(n)})$, the dual polygon walks through the normal vectors $\mathbf{a}(s^{(k)})$.

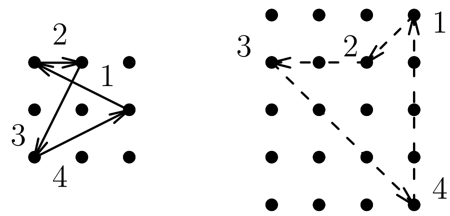

FIGURE 8. A polygon and its dual.
Their lengths are $1 - 1 + 1 + 1$ respectively $1 + 2 + 3 + 4$.

Theorem 4 ([PV00]). *The sum of the lengths of an admissible polygon and its dual is* 12 *times the winding number.*

Heuristically, the winding number counts how many times a polygon winds around the origin. Dual polygons will have equal winding number. In this article, we will only be concerned with polygons of winding number one.

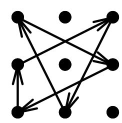

FIGURE 9. A polygon of winding number -2.

3. THREE PROOFS OF $\mathbf{b \leq 2i+7}$

Let P be a convex lattice polygon with interior lattice points. Denote a its surface, i the number of interior lattice points, and b the number of lattice points on P's boundary. In view of Pick's theorem $a = i+b/2-1$, the following three inequalities are equivalent.

Proposition 5. *If* $i > 0$*, then*

$$b \leq 2i + 7 \tag{1}$$

$$a \leq 2i + 5/2 \tag{2}$$

$$b \leq a + 9/2 \tag{3}$$

with equality only for the triangle of Figure 5 on the left.

3.1. **Scott's proof.** Translate P so that P fits tightly into a box $[0, p'] \times [0, p]$. Without loss of generality, we can assume $2 \leq p \leq p'$ (remember, $i > 0$). If P intersects the top and the bottom edge of the box in segments of length $q \geq 0$ and $q' \geq 0$ respectively, then $b \leq q + q' + 2p$, and $a \geq p\,(q + q')/2$. (Compare Figure 10.)

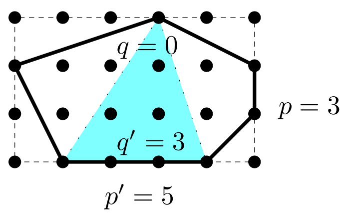

FIGURE 10. P in a box.

We distinguish three cases

- $p = 2$, or $q + q' \geq 4$, or $p = q + q' = 3$
- $p = 3$, and $q + q' \leq 2$

- $p \geq 4$, and $q + q' \leq 3$.

The above inequalities are already sufficient to deal with the first two cases.

- We have
$$\begin{aligned} 2b - 2a &\leq 2(q + q' + 2p) - p\,(q + q') \\ &= (q + q' - 4)(2 - p) + 8 \quad \leq 9, \end{aligned}$$
which shows (3) in Proposition 5. (With equality if and only if $p = q + q' = 3$, $a = 9/2$, $b = 9$.)[3]
- The estimate $b \leq q + q' + 2p \leq 8$ together with $i \geq 1$ show that inequality (1) in Proposition 5 is strictly satisfied.

The only case where we have to work a little is case three. Choose points $\mathbf{x} = (x_1, p)$, $\mathbf{x}' = (x_1', 0)$, $\mathbf{y} = (0, y_2)$, and $\mathbf{y}' = (p', y_2')$ in P so that $\delta = |x_1 - x_1'|$ is as small as possible. Then $a \geq p(p' - \delta)/2$ (compare Figure 11).

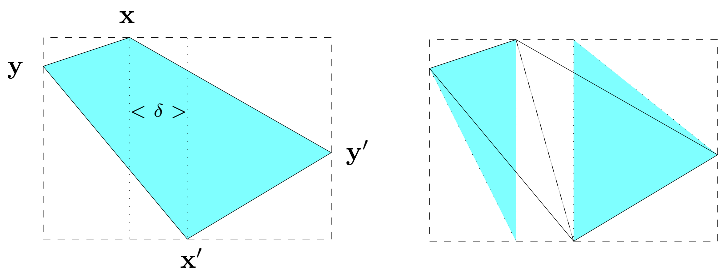

FIGURE 11. Case three. Two triangles of total area $p(p' - \delta)/2$.

Now the task is to apply lattice equivalences so that δ becomes small.

Exercise 6. *It is possible to choose* $\delta \leq (p - q - q')/2$.

This lattice equivalence will leave q, q', p unchanged, because it fixes the x_1-axis. We can still assume $p \leq p'$ by starting out with a minimal possible p. Thus, we obtain $a \geq p(p + q + q')/4$, and

$$\begin{aligned} 4(b - a) &\leq 8p + 4q + 4q' \; - \; p(p + q + q') \\ &= p(8 - p) - (p - 4)(q + q') \leq p(8 - p) \leq 16 \end{aligned}$$

because $p \geq 4$ in case three. This proves that inequality (3) in Proposition 5 is strictly satisfied. □

[3]It is an exercise to show that the only P with these parameters is the triangle in Figure 5 on the left.

3.2. **Clipping off vertices.** This proof proceeds by induction on i. If $i = 1$, we can check the inequalities on all 16 lattice–equivalence classes of such P. We could also use "12", and observe that the dual polygon must have length at least 3.

For the induction step we want to 'chop off a vertex'. If $i \geq 2$, and $b \leq 10$, nothing is to show. So, let (up to lattice–equivalence) $\mathbf{0}$ and $(1,0)^t$ lie in the interior of P, and let $b \geq 11$. Then we can further assume that there are ≥ 5 boundary lattice points with positive second coordinate.

If v is a vertex among them which is not unimodular (i.e., the triangle formed by v together with its two neighbors on the boundary has area $> 1/2$), denote $P' = \operatorname{conv}(P \cap \mathbb{Z}^2 \setminus v)$. This omission affects our parameters as follows: $a' = a - k/2$, $b' = b + k - 2$, and $i' = i - k + 1$, where k is the lattice length of the boundary of P' that is visible from v. Because v was not unimodular, we have $k \geq 2$. Because there are other lattice points with positive second coordinate, at least one of $\mathbf{0}$ or $(1,0)^t$ remains in the interior of P', and we can use induction.

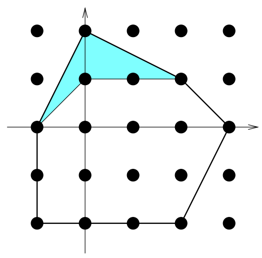

FIGURE 12. Clipping off a non–unimodular vertex.

If all vertices with positive second coordinate are unimodular, simiiarly omit one vertex together with its two boundary neighbors. The parameters change as follows: $a' = a - k/2 + 1$, $b' = b + k - 4$, and $i' = i - k + 1$, where $k \geq 2$ is the lattice length of the boundary of P' that is visible from the removed points. As observed above, there remain lattice points with positive second coordinate in P' so that at least one of $\mathbf{0}$ or $(1,0)^t$ stays in the interior of P'.

3.3. **Algebraic geometry.** We use the letters d and p to denote the degree and the sectional genus of an algebraic surface. The inequality $p \leq (d-1)(d-2)/2$ holds for arbitrary algebraic surfaces. If the surface is rational, i.e. if it has a parametrization by rational functions, then there are more inequalities.

Theorem 7.

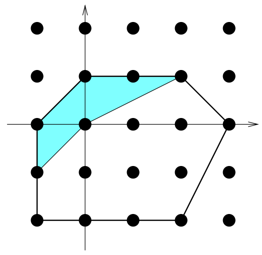

FIGURE 13. Clipping off a unimodular vertex (and its neighbors).

- *If* $p = 1$*, then* $d \leq 9$*.*
- *If* $p \geq 2$*, then* $d \leq 4p + 4$*.*

Rational surfaces with $p = 1$ are called Del Pezzo surfaces. The degree bound 9 is due to [dP87]. The bound $d \leq 4p + 4$ has already been known in [Jun90]. A modern proof can be found in [Sch99].

Toric surfaces are rational, and Scott's inequality is equivalent to Theorem 7 for toric surfaces.

4. Onion skins

For any convex lattice polygon P define a sequence of convex lattice polygons $P = P^{(0)}, P^{(1)}, \ldots$ by $P^{(n+1)} = \operatorname{conv}(\operatorname{int}(P^{(n)}) \cap \mathbb{Z}^2)$. There is a unique notion of level $\ell = \ell(P)$ so that $\ell(P^{(1)}) = \ell(P) - 1$, and $\ell(kP) = k\ell(P)$ for all k and P:

- $\ell(P) = n$ if $P^{(n)}$ is a point or a segment,
- $\ell(P) = n + 1/3$ if $P^{(n)} = \Delta$,
- $\ell(P) = n + 2/3$ if $P^{(n)} = 2\Delta$, and
- $\ell(P) = n + 1/2$ if $P^{(n)}$ is any other polygon without interior lattice points.

Here Δ stands for (a polygon lattice equivalent to) the standard triangle $\operatorname{conv}[(0,0)^t, (1,0)^t, (0,1)^t]$. This weird definition will unify the formulas in Theorem 8 below.

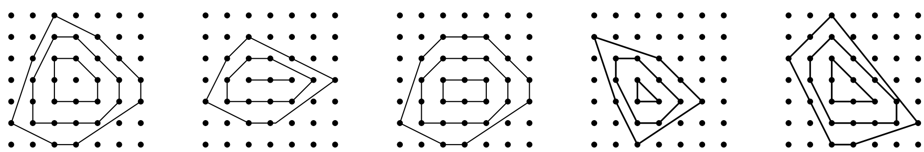

FIGURE 14. Polygons of levels
$\ell = 3$, $\ell = 2$, $\ell = 5/2$, $\ell = 7/3$, and $\ell = 8/3$.

Theorem 8. *Let P be a convex lattice polygon of area a and level $\ell \geq 1$ with b and i lattice points on the boundary and in the interior, respectively. Then $(2\ell-1)b \leq 2i+9\ell^2-2$, or equivalently $2\ell b \leq 2a+9\ell^2$, or equivalently $(4l-2)a \leq 9\ell^2+4l(i-1)$, with equality if and only if P is a multiple of Δ.*

It is easy to see that for $\ell > 1$, these inequality really strengthen the old $b \leq 2i+6$. We give two elementary proofs. One is similar to Scott's proof. The other is a bit longer, but it gives more insight into the process of peeling onion skins. For instance, it reveals that the set of all polygons P such that $P^{(1)} = Q$ for some fixed Q is either empty or has a largest element.

4.1. **Moving out edges.** Using this technique, it is actually possible to sharpen the bound in the various (sub)cases. E.g.,

- if $P^{(\ell)}$ is a point, but $P^{(\ell-1)} \neq 3\Delta$, then $(2\ell-1)b \leq 2i+8\ell^2-2$;
- if $P^{(\ell)}$ is a segment, then $(2\ell-1)b \leq 2i+8\ell^2-2$ with equality if and only if P is lattice–equivalent to a polygon with vertices $\mathbf{0}, (r,0)^t, (2pq+r,2p)^t, (0,2p)^t$ for integers $p \geq 1$, $q, r \geq 0$ such that $pq+r \geq 3$;
- if $P^{(\ell)}$ has no interior lattice points but is not a multiple of Δ, then $(2\ell-1)b \leq 2i+8\ell^2-2$.

We reduce the proof to the case that P is obtained from $P^{(1)}$ by 'moving out the facets by one'. This is done in the following three lemmas. Finally, Lemma 12 yields the induction step in the proof of Theorem 8.

Lemma 9. *Suppose that the inequality $a_1x_1+a_2x_2 \leq b$ defines a facet of $P^{(1)}$. (I.e., it is satisfied by all points in $P^{(1)}$, and there are two distinct points in $P^{(1)}$ satisfying equality.)*

Then $a_1x_1+a_2x_2 \leq b+1$ is valid for P.

That means, if we move all the facets of $Q = P^{(1)}$ out by one, we obtain a superset $Q^{(-1)}$ of P.

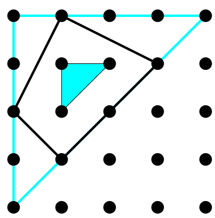

FIGURE 15. If $Q = P^{(1)}$ then $P \subseteq Q^{(-1)}$.

Proof. Suppose to the contrary, that $\mathbf{0}$ and $(1,0)^t$ lie in the facet $x_2 \le 0$ of $P^{(1)}$, and P has a vertex $\mathbf{v}$ with $v_2 > 1$. Then the triangle formed by the three points has area $v_2/2 \ge 1$. It must therefore contain another lattice point which lies in the interior of P, and has positive second coordinate. □

For arbitrary Q, $Q^{(-1)}$ does not necessarily have integral vertices. But not every lattice polygon arises as $P^{(1)}$ for some P. A necessary condition is that the polygon has good angles.

Lemma 10. *If $P^{(1)}$ is 2–dimensional, then for all vertices $\mathbf{v}$ of $P^{(1)}$, the cones generated by $P^{(1)} - v$ are lattice–equivalent to a cone generated by $(1,0)^t$ and $(-1,k)^t$, for some integer $k \ge 1$.*

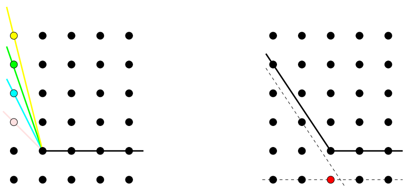

FIGURE 16. Good angles and a bad angle.

Proof. Assume (up to lattice–equivalence) that $\mathbf{v} = \mathbf{0}$, and the rays of the cone in question are generated by $(1,0)^t$ and $(-p,q)^t$, with coprime $0 < p \le q$ (compare Exercise 3). By Lemma 9, all points of P satisfy $x_2 \ge -1$ and $qx_1 + px_2 \ge -1$. But this implies $x_1 + x_2 \ge -1 + \frac{p-1}{q}$. So, if $p > 1$, because P has integral vertices, we have $x_1 + x_2 \ge 0$ for all points of P. This contradicts the fact that $\mathbf{0} \in P^{(1)}$. □

For a vertex $\mathbf{v}$ of a convex lattice polygon define the shifted vertex $\mathbf{v}^{(-1)}$ as follows. Let $\langle \mathbf{a}, \mathbf{x} \rangle \le b$ and $\langle \mathbf{a}', \mathbf{x} \rangle \le b'$ be the two facets that intersect in $\mathbf{v}$. (With $\mathbf{a}$ and $\mathbf{a}'$ primitive.) The unique solution to $\langle \mathbf{a}, \mathbf{x} \rangle = b + 1$ and $\langle \mathbf{a}', \mathbf{x} \rangle = b' + 1$ is denoted $\mathbf{v}^{(-1)}$. According to Lemma 10, when we deal with $P^{(1)}$ then $\mathbf{v}^{(-1)}$ is a lattice point. (In the situation of the lemma, it is $(0,-1)^t$.)

We obtain a characterization of when $Q = P^{(1)}$ for some P.

Lemma 11. *For a convex lattice polygon Q, the following are equivalent:*

- *$Q = P^{(1)}$ for some convex lattice polygon P.*
- *$Q^{(-1)}$ has integral vertices.*

Thus given Q, $P = Q^{(-1)}$ actually is *the* maximal polygon with $P^{(1)} = Q$. We will (and can) restrict to this situation when we prove the induction step $\ell \rightsquigarrow \ell+1$ for Theorem 8.

Proof. If $Q^{(-1)}$ has integral vertices, then its interior lattice points span Q. For the converse direction, if $Q = P^{(1)}$ then we claim that

$$Q^{(-1)} = \operatorname{conv}\{\mathbf{v}^{(-1)} \ : \ \mathbf{v} \text{ vertex of } Q\} \ .$$

(Observe that the $\mathbf{v}^{(-1)}$ are integral by Lemma 10.)
"$\subseteq$": In fact, this inclusion holds for arbitrary Q:

denote $\mathbf{a_1}, \ldots, \mathbf{a_n}$ and $b_1, \ldots, b_n$ the primitive normal vectors respectively right hand sides of the facets of Q in cyclic order. Also denote $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the vertices of Q so that facet number k is the segment $[\mathbf{v}_k, \mathbf{v}_{k+1}]$ ($k \mod n$).

For a point $\mathbf{y} \in Q^{(-1)}$, let $\langle \mathbf{a}_k, \mathbf{x}\rangle \leq b_k$ be a facet of Q that maximizes $\langle \mathbf{a}, \mathbf{y}\rangle - b$ over all facets. So if $\langle \mathbf{a}_k, \mathbf{y}\rangle - b_k \leq 0$ then $\mathbf{y} \in Q$. Otherwise

- $b_k \leq \langle \mathbf{a}_k, \mathbf{y}\rangle \leq b_k + 1$,
- $\langle \mathbf{a}_k, \mathbf{y}\rangle - b_k \geq \langle \mathbf{a}_{k-1}, \mathbf{y}\rangle - b_{k-1}$, and
- $\langle \mathbf{a}_k, \mathbf{y}\rangle - b_k \geq \langle \mathbf{a}_{k+1}, \mathbf{y}\rangle - b_{k+1}$,

which describes (a subset of) the convex hull of $\mathbf{v}_k, \mathbf{v}_k^{(-1)}, \mathbf{v}_{k+1}, \mathbf{v}_{k+1}^{(-1)}$.
"$\supseteq$": For this inclusion we actually use $Q = P^{(1)}$. Then for each k, P (and therefore $Q^{(-1)}$ by Lemma 9) contains a point $\mathbf{w}_k$ with $\langle \mathbf{a}_k, \mathbf{w}_k\rangle = b_k + 1$. In order to show that $\mathbf{v}_k^{(-1)} \in Q^{(-1)}$, observe that none of the other facet normals belongs to the cone generated by $\mathbf{a}_{k-1}$ and $\mathbf{a}_k$. So for $m \neq k, k-1$,

$$\begin{array}{lcccccc} \text{either} & \langle \mathbf{a}_m, \mathbf{v}_k\rangle & < & \langle \mathbf{a}_m, \mathbf{w}_{k-1}\rangle & \leq & b_m + 1, \\ \text{or} & \langle \mathbf{a}_m, \mathbf{v}_k\rangle & < & \langle \mathbf{a}_m, \mathbf{w}_k\rangle & \leq & b_m + 1. \end{array}$$

□

Finally, we can show the key lemma for our induction step.

Lemma 12. *Let $b^{(1)}$ denote the number of lattice points on the boundary of $P^{(1)}$. Then $b \leq b^{(1)} + 9$, with equality if and only if P is a multiple of Δ.*

This immediately shows that $b \leq b^{(1)} + 9 \leq i + 9$ if $P^{(1)}$ is 2–dimensional.

Proof. We can assume that P arises from $P^{(1)}$ by relaxing all facet defining inequalities by one. Then for each of the vertices $v_1^{(1)}, \ldots, v_n^{(1)}$ of $P^{(1)}$ there is a corresponding vertex $v_1, \ldots, v_n$ of P. Consider the (possibly non convex, non simple) admissible polygon $v_1 - v_1^{(1)}, \ldots, v_n - v_n^{(1)}$. It is admissible because there are no lattice points between P and

$P^{(1)}$. One can think of it as what remains of P when $P^{(1)}$ shrinks to a point. Each segment measures the difference (with the correct sign) between the corresponding facets of P and $P^{(1)}$. I.e., the length of that polygon is precisely $b - b^{(1)}$.

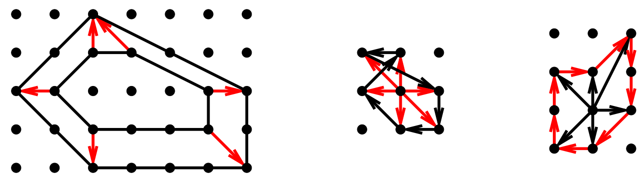

FIGURE 17. An admissible polygon from $(P, P^{(1)})$, and its dual.

Now the dual polygon will walk through the normal vectors of $P^{(1)}$. Therefore all segments will count with positive length, and there cannot be less than 3. Also, there is a unique one with 3 segments, which is the dual to 3Δ. Thus $b - b^{(1)} \leq 12 - 3$ with equality only for multiples of Δ. □

Proof of Theorem 8. Induction on ℓ. For $\dim P^{(\ell)} = 0$.

- For $\ell = 1$, the inequality $b \leq 2i + 7$ was proved earlier.
- For $\ell = 4/3$, we have $i = 3$, and $P \subseteq 4\Delta$. So $b \leq 12$.
- For $\ell = 5/3$, we have $i = 6$, and $P \subseteq 5\Delta$. So $b \leq 15$.
- For $\ell = 3/2$, Lemma 12 reads $b \leq i + 8$ which is stronger than what we need.

If $\ell \geq 2$, we have

$$\begin{aligned}(2\ell - 1)b &\leq (2\ell - 1)b^{(1)} + 9(2\ell - 1)\\ &= 2b^{(1)} + (2(\ell - 1) - 1)b^{(1)} + 9(2\ell - 1)\\ &\leq 2b^{(1)} + 2i^{(1)} + 9(\ell - 1)^2 - 2 + 9(2\ell - 1)\\ &= 2i + 9\ell^2 - 2\end{aligned}$$

□

4.2. **Generalizing Scott's proof.** As in subsection 3.1, we tightly fit P into a box $[0, p'] \times [0, p]$, with $p \leq p'$. Let q and q' be the length of the top and bottom edge (see Figure 10). We again apply lattice equivalence transformations such that p is as small as possible, and that P has point on the top and bottom edge with horizontal distance smaller than or equal to $(p - q - q')/2$. Again, we obtain the following inequalities:

$$b \geq q + q' + 2p \tag{4}$$

$$a \geq p(q + q')/2 \tag{5}$$

$$a \geq p(p + q + q')/4 \tag{6}$$

Set $x := p/\ell$ and $y := (q + q')/\ell$. Then $x \geq 2$, because passing to $P^{(1)}$ reduces the height at least by 2. From (4) and (5), we get

$$\frac{2\ell b - 2a - 9\ell^2}{\ell^2} \leq 2(q + q' + 2p)/\ell - p(q + q')/\ell^2 - 9$$
$$= -xy + 4x + 2y - 9,$$

and from 4 and 6, we get

$$\frac{4\ell b - 4a - 18\ell^2}{\ell^2} \leq 4(q + q' + 2p)/\ell - p(p + q + q')/\ell^2 - 18$$
$$= -x^2 - xy + 8x + 4y - 18.$$

For $x \geq 2$ and $y \geq 0$, at least one of the two polynomials $-xy + 4x +$

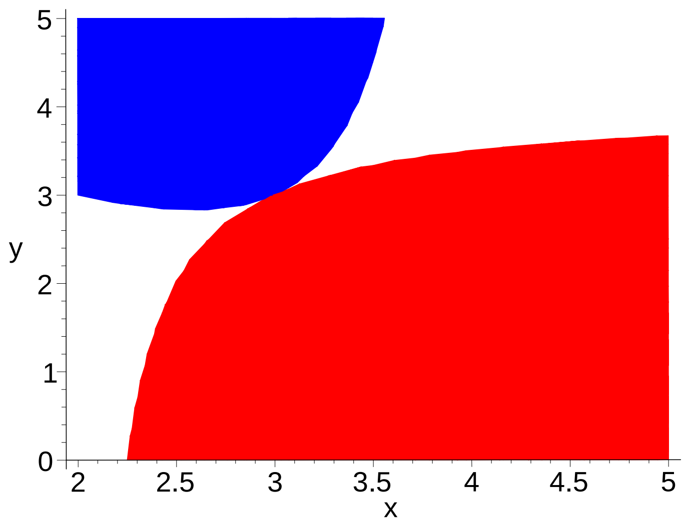

FIGURE 18. At least one of the polynomials is ≤ 0.

$2y - 9$ and $-x^2 - xy + 8x + 4y - 18$ is zero or negative, as it can be seen in Figure 18. There is only one point where both upper bounds reach zero, namely $(x, y) = (3, 3)$, and this is the only case where equality can hold in Theorem 8. It is an exercise to show that equality actually holds only for multiples of Δ.

5. Conclusion

An algebraic geometry proof for Theorem 8 is under construction. A translation of the "passing to the convex hull of interior points" is available: this is the adjoint surface. As in the toric case, it is true that the set of all surfaces with a fixed common adjoint surface is either empty or has a largest element in the sense of minimal model theory (see [Sch03]).

The level of an algebraic surface can be defined as the supremum of all rational numbers l such that $H+lK$ is effective, where H is a divisor of hyperplane sections and K is a canonical divisor. Equivalently, this is the unique number $\ell = p/q$ such that the p-th adjoint of the q-uple embedding of the given surface degenerates.

The algebraic geometry version of Theorem 8 then reads $(2l-1)d \leq 9l^2+4l(p-1)$. This should be true for arbitrary rational surfaces. If we can construct the proof, then this would be another example were an elementary theorem in lattice geometry leads to a theorem in algebraic geometry.

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