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β-expansions in algebraic function fields over finite fields

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1 Introduction

In the present paper, we define a new kind of digit system in algebraic function fields over finite fields. There are striking analogies of these digit systems to the well known β-expansions defined in $\mathbb{R}_+$. Results corresponding to classical theorems as well as open problems will be proved. In order to pursue this analogy we will recall the definition of real β-expansions and state the classical theorems corresponding to our results.

β-expansions of real numbers were introduced by Rényi [15]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors (cf. [1, 2, 7, 8, 9, 14, 17]). Given $\beta > 1$, the *β-transformation* $T = T_\beta$ is defined for $x \in [0, 1)$ by $T(x) = \beta x - \lfloor \beta x \rfloor$.

If $\beta = b$ is an integer and if

$$x = \frac{x_1}{b} + \frac{x_2}{b^2} + \frac{x_3}{b^3} + \ldots = .x_1 x_2 x_3 \ldots .$$

is the b-ary expansion of x, then $Tx = .x_2 x_3 \ldots$. Thus, T acts as a one-sided shift on the sequence of b-ary digits of x.

If β is not an integer, we can use T to obtain an expansion for which the above description is still valid. By iterating this map and considering

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its trajectory

$$x \xrightarrow{x_1} T(x) \xrightarrow{x_2} T^2(x) \xrightarrow{x_3} \dots$$

with $x_j = \lfloor \beta T^{j-1} x \rfloor$, we obtain

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots.$$

We will call the sequence

$$d_\beta(x) = .x_1 x_2 \dots$$

the β-expansion of x. It is easy to prove that an infinite sequence of non-negative integers $(x_i)_{i \geq 1}$ is the β-expansion of $x \in [0,1)$ if and only if

$$x_i \beta^{-i} + x_{i+1} \beta^{-i-1} + \dots < \beta^{-i+1} \tag{1.1}$$

for every $i \geq 1$. We say that $d_\beta(x)$ is finite when $x_i = 0$ for all sufficiently large i. This is the case when there in an integer $i \geq 0$ such that $T^i x = 0$.

Now let $x \geq 1$. Then there is an integer n such that $\beta^n \leq x < \beta^{n+1}$. We define in a similar manner

$$d_\beta(x) = x_{-n} \dots x_{-1} x_0 . x_1 x_2 \dots.$$

Let

$$\begin{aligned} \mathbf{Per}(\beta) &= \{x \in \mathbb{R}_+ : d_\beta(x) \text{ is eventually periodic}\} \quad \text{and} \\ \mathbf{Fin}(\beta) &= \{x \in \mathbb{R}_+ : d_\beta(x) \text{ is finite}\}. \end{aligned}$$

Recall that a *Pisot number* is an algebraic integer $\beta > 1$ for which all algebraic conjugates γ with $\gamma \neq \beta$ satisfy $|\gamma| < 1$. A *Salem number* is an algebraic integer $\beta > 1$ for which all algebraic conjugates γ with $\gamma \neq \beta$ satisfy $|\gamma| \leq 1$ with at least one conjugate having $|\gamma| = 1$.

Theorem 1.1 (Bertrand and Schmidt [7, 17]) *If β is a Pisot number, then* $\mathbf{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+$.

If $\beta = b$ is an integer, then $\mathbb{Q}(b) = \mathbb{Q}$. Since every rational integer $b > 1$ is a Pisot number, Theorem 1.1 is a natural generalization of the well known fact that $x \in \mathbb{Q}$ if and only if $d_b(x)$ is eventually periodic.

There exists a partial converse of Theorem 1.1.

Theorem 1.2 (Schmidt [17]) *If* $\mathbb{Q} \cap \mathbb{R}_+ \subset \mathbf{Per}(\beta)$, *then* β *is a Pisot or Salem number.*

Remark 1.3 Schmidt stated his results for numbers in the unit interval. It is trivial that $\mathbf{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+ \iff \mathbf{Per}(\beta) \cap [0,1) = \mathbb{Q}(\beta) \cap [0,1)$ and $\mathbb{Q} \cap \mathbb{R}_+ \subset \mathbf{Per}(\beta) \iff \mathbb{Q} \cap [0,1) \subset \mathbf{Per}(\beta) \cap [0,1)$. We have changed his notation in order to emphasize the analogies to our results.

Schmidt conjectured, that the converse of Theorem 1.2 is also true, i.e. if β is a Salem number, then $\mathbb{Q} \cap \mathbb{R}_+ \subset \mathbf{Per}(\beta)$ (The Pisot case is trivially included in Theorem 1.1). In the setting of algebraic function fields, we can prove even more, since we can prove an analogue of Theorem 1.1 also for Salem elements.

A similar situation occurs in the case of finite expansions. We say that a number $\beta > 1$ has the *finiteness property* or property **(F)**, if

$$\mathbf{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+. \qquad \textbf{(F)}$$

This property was introduced in [9].

Theorem 1.4 (Frougny and Solomyak [9])

(i) If $\mathbb{Z}_+ \subset \mathbf{Fin}(\beta)$, *then* β *is a Pisot or Salem number.*

(ii) Condition **(F)** *implies that* β *is a Pisot number.*

Several classes of Pisot numbers are known, such that **(F)** holds. On the other hand, there exist examples of Pisot numbers, such that **(F)** is not fulfilled (cf. [4, 9, 10]).

In the case of algebraic function fields, we are able to prove that no such exceptional cases exist. Furthermore, we can prove that the analogues of conditions in (i) and (ii) are equivalent.

This paper is organized as follows. In section two, we will define $\mathbb{F}((x^{-1}))$, the field of pole like formal Laurent series about ∞ as well as the analogues to Pisot and Salem numbers in $\mathbb{F}((x^{-1}))$. Furthermore, we will provide a simple algorithm to compute the coefficients of Pisot and Salem elements in $\mathbb{F}((x^{-1}))$.

In section three, we will define the expansion algorithm for $\mathbb{F}((x^{-1}))$ and prove that there are no dependencies between consecutive digits.

Section four is devoted to periodic expansions. We will prove an extended analogue of Theorem 1.1 as well as an analogue of Theorem 1.2.

In section five, we will give a complete characterization of all bases, which give raise to finite expansions. In contrast, such a simple characterization is hardly expected in the real case.

2 Pisot and Salem elements in a field of formal Laurent series over a finite field

Let $\mathbb{F}$ be a finite field, $\mathbb{F}[x]$ the ring of polynomials, $\mathbb{F}(x)$ the field of rational functions. Let $\mathbb{F}((x^{-1}))$ be the field of formal Laurent series of the form

$$z = \sum_{k=-\infty}^{\ell} z_k x^k, \quad z_k \in \mathbb{F} \tag{2.1}$$

where

$$\ell = \deg z := \begin{cases} \max\{k : z_k \neq 0\} & \text{for} \quad z \neq 0 \\ -\infty & \text{for} \quad z = 0. \end{cases}$$

Define the absolute value

$$|z| = \begin{cases} c^{\deg z} & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0. \end{cases}$$

where $c > 1$ is an arbitrary but fixed real number. Note that the set of possible values of $|\cdot|$ is a discrete set. Then $\mathbb{F}((x^{-1}))$ is the closure of $\mathbb{F}(x)$ with respect to $|\cdot|$.

Thus $\mathbb{F}[x]$ is the analogue to $\mathbb{Z}_+$, $\mathbb{F}(x)$ is the analogue to $\mathbb{Q}$ and $\mathbb{F}((x^{-1}))$ is the analogue to $\mathbb{R}$. If $\beta \in \mathbb{F}((x^{-1}))$, then $\mathbb{F}(x, \beta)$ is the analogue to $\mathbb{Q}(\beta)$, $\mathbb{F}[x, \beta]$ is the analogue to $\mathbb{Z}[\beta]$ and $\mathbb{F}[x, \beta^{-1}]$ is the analogue to $\mathbb{Z}[\beta^{-1}]$.

Since $|\cdot|$ is not archimedean, $|\cdot|$ fullfills the strict triangle inequality $|x + y| \leq \max(|x|, |y|)$ and $|x + y| = \max(|x|, |y|)$ if $|x| \neq |y|$. $\mathbb{F}((x^{-1}))$ together with $|\cdot|$ becomes a Banach space.

For $a \in \mathbb{F}((x^{-1}))$ and $r \in \mathbb{R}_+$, set

$$D(a, r) = \{z \in \mathbb{F}((x^{-1})) : |z - a| < r\}.$$

Let z be as in (2.1). Define the integer (polynomial) part

$$\lfloor z \rfloor = \sum_{k=0}^{\ell} z_k x^k$$

where the empty sum, as usual, is defined to be zero. Therefore $\lfloor z \rfloor \in \mathbb{F}[x]$ and $z - \lfloor z \rfloor \in D(0,1)$ for all $z \in \mathbb{F}((x^{-1}))$. Note that $\lfloor z + w \rfloor = \lfloor z \rfloor + \lfloor w \rfloor$, $\lfloor -z \rfloor = -\lfloor z \rfloor$ and $|\lfloor z \rfloor| \leq |z|$. The following two definitions are due to [6].

Definition 2.1 *An element $\beta = \beta_1 \in \mathbb{F}((x^{-1}))$ is called Pisot element if it is an algebraic integer over $\mathbb{F}[x]$, $|\beta| > 1$ and $|\beta_j| < 1$ for all Galois conjugates β_j.*

Definition 2.2 *An element $\beta = \beta_1 \in \mathbb{F}((x^{-1}))$ is called Salem element if it is an algebraic integer over $\mathbb{F}[x]$, $|\beta| > 1$, $|\beta_j| \leq 1$ for all Galois conjugates β_j, and there exists at least one Galois conjugate β_k, such that $|\beta_k| = 1$.*

In general, β and its Galois conjugates are hard to compute. Therefore, the conditions in the Definitions 2.1 and 2.2 are difficult to verify. By considering the Newton polygon (cf. [12, 13]) of the minimal polynomial, the following, more useful equivalences [6, Theorem 12.1.1] can be derived.

Theorem 2.3 *Let $\beta \in \mathbb{F}((x^{-1}))$ be algebraic integer over $\mathbb{F}[x]$ and*

$$p(y) = y^n - a_1 y^{n-1} - \ldots - a_n, \quad a_i \in \mathbb{F}[x] \tag{2.2}$$

be its minimal polynomial. Then

(i) β is a Pisot element if and only if $\deg a_1 > 0$ *and* $\deg a_1 > \max_{j=2}^{n} \deg a_j$.

(ii) β is a Salem element if and only if $\deg a_1 > 0$ *and* $\deg a_1 = \max_{j=2}^{n} \deg a_j$.

Remark 2.4 In [5, Theorem 1.1.] it is proved, that $\mathbb{F}((x^{-1}))$ contains Pisot elements of all degrees for arbitrary fields $\mathbb{F}$.

In Theorem 2.6, a method to compute the coefficients of Pisot or Salem elements is given. For the proof, we will need the following auxiliary result.

Lemma 2.5 *If $z, w \in \mathbb{F}((x^{-1}))$ with $|z| = |w|$, then $|z^n - w^n| \leq |z-w||z|^{n-1}$ for all $n \in \mathbb{Z}$.*

Proof. The statement is trivial for $n = 0$. For $n > 0$, we have

$$\begin{aligned} |z^n - w^n| &= |z-w||z^{n-1} + z^{n-2}w + \ldots + w^{n-1}| \\ &\leq |z-w| \max_{j=0}^{n-1} |z^{n-1-j} w^j| \\ &= |z-w||z|^{n-1} \end{aligned}$$

and

$$\begin{aligned} |z^{-n} - w^{-n}| &= |w^n - z^n||z^{-n}w^{-n}| \\ &\leq |w - z||w|^{n-1}|z^{-n}w^{-n}| \\ &= |z - w||z|^{-n-1}. \end{aligned}$$

□

Theorem 2.6 *Let β be a Pisot or Salem element and* (2.2) *be its minimal polynomial. Then the recurrence*

$$\begin{aligned} y_1 &= a_1 \\ y_{k+1} &= a_1 + \frac{a_2}{y_k} + \ldots + \frac{a_n}{y_k^{n-1}} \quad \textit{for } k \geq 1 \end{aligned}$$

fulfills

$$\lim_{k\to\infty} y_k = \beta.$$

Proof. First we prove by induction that $|y_k| = |a_1|$ for all $k \geq 1$. For $k = 1$ this assertion is trivial.

Let $|y_k| = |a_1|$ or equivalently, $\deg y_k = \deg a_1$. For $j = 2, \ldots, n$, it follows from $\deg a_1 > 0$ and $\deg a_1 \geq \deg a_j$ that

$$\begin{aligned} \deg a_j/y_k^{j-1} &= \deg a_j - (j-1)\deg y_k \\ &\leq \deg a_1 - 1 \deg a_1 \\ &= 0 < \deg a_1. \end{aligned}$$

Thus $\deg y_{k+1} = \deg a_1$ or equivalently $|y_{k+1}| = |a_1|$. From Lemma 2.5, we get

$$\begin{aligned} |y_{k+1} - y_k| &= \left| a_2 \left(\frac{1}{y_k} - \frac{1}{y_{k-1}} \right) + \ldots + a_n \left(\frac{1}{y_k^{n-1}} - \frac{1}{y_{k-1}^{n-1}} \right) \right| \\ &\leq \max \left(\frac{|a_2|}{|a_1|^2}, \ldots, \frac{|a_n|}{|a_1|^n} \right) |y_k - y_{k-1}|. \end{aligned}$$

Since $|a_1| > 1$ and $|a_1| \geq |a_j|$, the left factor is constant and less than 1. Thus, the sequence converges. □

Example 2.7 Let $p(y) = y^2 + xy + x$ over $\mathbb{Z}_2$. Since $\deg a_1 = \deg a_2$, its zero must be a Salem element. Then the above sequence converges to

$$\beta = \lim_{k\to\infty} y_k = x + \sum_{k=0}^{\infty} \frac{1}{x^{2^k-1}}.$$

An elementary computation verifies that $p(\beta) = 0$. Since $p(y)$ is quadratic, Vieta's formula implies

$$\beta_2 = a_1 - \beta = \sum_{k=0}^{\infty} \frac{1}{x^{2^k-1}}.$$

3 β-expansions in $\mathbb{F}((x^{-1}))$

Let $\beta, z \in \mathbb{F}((x^{-1}))$ with $|\beta| > 1$, $|z| < 1$. A *representation in base β (or β-representation)* of z is an infinite sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{F}[x]$ such that

$$z = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}. \tag{3.1}$$

A particular β-representation – called the *β-expansion* – can be computed by a greedy algorithm.

This algorithm works as follows. Set $r^{(0)} = z$ and let $d_j = \lfloor \beta r^{(j-1)} \rfloor$, $r^{(j)} = \beta r^{(j-1)} - d_j$ for $j \geq 1$. This procedure yields a representation of z of the form (3.1). Note that $|d_j| < |\beta|$ and $|r^{(j)}| < 1$ for all j. The β-expansion of z will be denoted by

$$d_\beta(z) = .d_1 d_2 \ldots .$$

Note that

$$r^{(k)} = \beta^k \left(z - \sum_{l=1}^{k} d_l \beta^{-l} \right).$$

An equivalent definition of the β-expansion is obtained by using the β-transformation on $D(0,1)$ which is given by the mapping

$$\begin{aligned} T_\beta : D(0,1) &\mapsto D(0,1) \\ z &\mapsto \beta z - \lfloor \beta z \rfloor. \end{aligned}$$

Then $d_\beta(z) = (d_i)_{i=1}^{\infty}$ if and only if $d_i = \lfloor \beta T_\beta^{i-1}(z) \rfloor$. Note that $d_\beta(z)$ is finite if and only if there is a $k \geq 0$ such that $T_\beta^k(z) = 0$.

Now let $z \in \mathbb{F}((x^{-1}))$ be an element with $|z| \geq 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq |z| < |\beta|^{k+1}$. Hence $|z/\beta^{k+1}| < 1$ and we can represent z by shifting $d_\beta(z/\beta^{k+1})$ by k digits to the left. Therefore, if $d_\beta(z) = .d_1 d_2 d_3 \ldots$, then $d_\beta(\beta z) := d_1.d_2 d_3 \ldots$.

In the sequel, we will use the following notations:

$$\begin{aligned} \mathbf{Per}(\beta) &= \{z \in \mathbb{F}((x^{-1})) \ : \ d_\beta(z) \text{ is eventually periodic}\} \quad \text{and} \\ \mathbf{Fin}(\beta) &= \{z \in \mathbb{F}((x^{-1})) \ : \ d_\beta(z) \text{ is finite}\}. \end{aligned}$$

It is clear that $\mathbf{Fin}(\beta) \subset \mathbf{Per}(\beta)$ but $\mathbf{Fin}(\beta) \neq \mathbf{Per}(\beta)$.

The following theorem provides an analogue to the condition mentioned in (1.1).

Theorem 3.1 *An infinite sequence of polynomials $(d_j)_{j\geq 1}$ is the β-expansion of $z \in D(0,1)$ if and only if $|d_j| < |\beta|$ for $j \geq 1$. Therefore, consecutive digits of z are independent.*

Proof. If $(d_i)_{i\geq 1}$ is the β-expansion of $z \in D(0,1)$, then the expansion algorithm implies that $|d_i| < |\beta|$ for $i \geq 1$.

Now let $z \in D(0,1)$ and $z = d_1/\beta + d_2/\beta^2 + \ldots$ be a β-representation with $|d_j| < |\beta|$ for $j \geq 1$. Let $z = d'_1/\beta + d'_2/\beta^2 + \ldots$ be the β-expansion of z. Then

$$0 = \frac{d_1 - d'_1}{\beta} + \frac{d_2 - d'_2}{\beta^2} + \ldots$$

with $|d_i - d'_i| < |\beta|$. Suppose that not all summands are zero. Let $i = \min\{j : d_j - d'_j \neq 0\}$. Since

$$\left| \frac{d_i - d'_i}{\beta^i} \right| \geq \frac{1}{|\beta|^i}$$

and

$$\left| \frac{d_{i+1} - d'_{i+1}}{\beta^{i+1}} + \ldots \right| \leq \max\left(\left| \frac{d_{i+1} - d'_{i+1}}{\beta^{i+1}} \right|, \ldots \right) < \frac{1}{|\beta|^i},$$

we run into a contradiction. □

Remark 3.2 In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if $z, w \in \mathbb{F}((x^{-1}))$, we have $d_\beta(z + w) = d_\beta(z) + d_\beta(w)$ digitwise.

Example 3.3 Take $p(y)$ from Example 2.7. Since $\beta^2 + x\beta + x = 0$ and $1 = -1$ in $\mathbb{Z}_2$, we obtain

$$x = \frac{\beta^2}{\beta+1} = \beta + 1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \dots.$$

Thus $d_\beta(x) = 11.11\dots$. Therefore, $x \in \mathbf{Per}(\beta)$ but $x \notin \mathbf{Fin}(\beta)$.

4 Periodic expansions

The aim of the current section is to prove Theorems 4.1 and 4.3. In the case of Pisot elements, Theorem 4.1 provides an analogue to the classical result of Bertrand and Schmidt [7, 17] mentioned in the introduction. The corresponding problem on real β-expansions by Salem-numbers is still open. Theorem 4.3 is an improved analogue of Schmidt's partial converse of the classical result.

Theorem 4.1 *Let β be a Pisot or Salem element. Then*

$$\mathbf{Per}(\beta) = \mathbb{F}(x, \beta).$$

Proof. The proof for $\mathbf{Per}(\beta) \subset \mathbb{F}(x,\beta)$ is trivial. To prove $\mathbb{F}(x,\beta) \subset \mathbf{Per}(\beta)$, we mainly follow [11, Proposition 7.2.19].

It is sufficient to prove the result for $\mathbb{F}(x,\beta) \cap D(0,1)$. Let $z \in \mathbb{F}(x,\beta) \cap D(0,1)$. Then

$$z = q^{-1} \sum_{i=0}^{n-1} p_i \beta^i$$

with $q, p_i \in \mathbb{F}[x]$ and $\deg q$ as small as possible. Let $(d_k)_{k\geq 1}$ be the β-expansion of z and

$$r_j^{(k)} = \beta_j^k \left(q^{-1} \sum_{i=0}^{n-1} p_i \beta_j^i - \sum_{\ell=1}^{k} d_\ell \beta_j^{-\ell} \right) \tag{4.1}$$

for $j = 1, \dots, n$. Therefore, $r_1^{(k)} = r^{(k)}$ and $r_j^{(k)}$, $j = 2, \dots, n$ are the Galois conjugates of $r^{(k)}$.

We have $|r_1^{(k)}| = |r^{(k)}| < 1$ for all k. For $j = 2, \ldots, n$, from $|\beta_j| \leq 1$ and $|d_\ell| < |\beta|$ follows

$$\begin{aligned} |r_j^{(k)}| &\leq \max\left(|\beta_j|^k |r_j^{(0)}|, \max_{\ell=1}^{k}(|d_\ell \beta_j^{k-\ell}|)\right) \\ &\leq \max\left(|r_j^{(0)}|, |\beta|\right) < \infty. \end{aligned}$$

Here the strict triangle inequality has been applied (This is the crucial step which doesn't work in the setting of real β-expansions by Salem-numbers). Therefore, $|r_j^{(k)}|$ is bounded for all k and j. We need a technical result.

Lemma 4.2 *Let $R^{(k)} = (r_1^{(k)}, \ldots, r_n^{(k)})$ and $B = (\beta_j^{-i})_{1 \leq i,j \leq n}$. Then for every $k \geq 0$, there exists a unique n-tuple $W^{(k)} = (w_1^{(k)}, \ldots, w_n^{(k)}) \in \mathbb{F}[x]^n$ such that $R^{(k)} = q^{-1} W^{(k)} B$.*

Proof. We use induction by k. Note that $\beta^n = a_1 \beta^{n-1} + \ldots + a_n$. Thus

$$\begin{aligned} r^{(0)} = z &= q^{-1} \sum_{i=0}^{n-1} p_i \beta^i \\ &= q^{-1} \left(\frac{w_1^{(0)}}{\beta} + \ldots + \frac{w_n^{(0)}}{\beta^n} \right). \end{aligned}$$

Now $r^{(k+1)} = \beta r^{(k)} - d_{k+1}$, and hence

$$\begin{aligned} r^{(k+1)} &= q^{-1} \left(w_1^{(k)} + \frac{w_2^{(k)}}{\beta} + \ldots + \frac{w_n^{(k)}}{\beta^{n-1}} - q d_{k+1} \right) \\ &= q^{-1} \left(\frac{w_1^{(k+1)}}{\beta} + \ldots + \frac{w_n^{(k+1)}}{\beta^n} \right). \end{aligned}$$

Since $q, w_i^{(k)} \in \mathbb{F}[x]$, the induction also works for the Galois conjugates $r_j^{(k)}$ of $r^{(k)}$.

□

Now we proceed with the proof of Theorem 4.1. Let $H^{(k)} = q R^{(k)}$. Since $|r_j^{(k)}|$ is bounded for every j, the sequence $\|H^{(k)}\|$ is bounded. As the matrix B is invertible, for every $k \geq 1$,

$$\|H^{(k)} B^{-1}\| = \|W^{(k)}\| = \|(w_1^{(k)}, \ldots, w_n^{(k)})\| = \max_{1 \leq j \leq n} |w_j^{(k)}| < \infty.$$

Thus there exist p and m such that $W^{(m+p)} = W^{(m)}$, and therefore, $r^{(m+p)} = r^{(m)}$ which implies that the β-expansion of z is eventually periodic.

□

The following Theorem is the converse of Theorem 4.1.

Theorem 4.3 *Let* $\mathbb{F}[x] \subset \mathbf{Per}(\beta)$. *Then* β *is a Pisot or Salem element.*

Remark 4.4 In Theorem 1.2, $\mathbb{Q} \cap \mathbb{R}_+ \subset \mathbf{Per}(\beta)$ is needed. The analogue to this condition would be $\mathbb{F}(x) \subset \mathbf{Per}(\beta)$ which is, of course, stronger than $\mathbb{F}[x] \subset \mathbf{Per}(\beta)$. Thus, Theorem 4.3 is an improved analogue of Theorem 1.2.

Proof. From
$$\frac{\lfloor\beta\rfloor}{\beta} = 1 - \frac{\beta - \lfloor\beta\rfloor}{\beta}$$
and $|\beta - \lfloor\beta\rfloor| < 1$ we obtain
$$\lfloor\beta\rfloor = \beta + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \dots \quad \text{where } |d_i| < |\beta|.$$
Since $\lfloor\beta\rfloor \in \mathbb{F}[x]$, the expansion of $\lfloor\beta\rfloor$ must be eventually periodic. Therefore,
$$\lfloor\beta\rfloor = \beta + \frac{d_1}{\beta} + \dots + \frac{d_k}{\beta^k} + \frac{d_{k+1}}{\beta^{k+1}} + \dots + \frac{d_{k+p}}{\beta^{k+p}} + \frac{d_{k+1}}{\beta^{k+p+1}} + \dots + \frac{d_{k+p}}{\beta^{k+2p}} + \dots.$$
Thus
$$\beta^k\left(\lfloor\beta\rfloor - \beta - \frac{d_1}{\beta} - \dots - \frac{d_k}{\beta^k}\right) = \beta^{k+p}\left(\lfloor\beta\rfloor - \beta - \frac{d_1}{\beta} - \dots - \frac{d_{k+p}}{\beta^{k+p}}\right).$$
If the β-expansion of $\lfloor\beta\rfloor$ is finite, the right hand side of this equation is zero. In both cases β must be an algebraic integer.

Suppose that β has a Galois conjugate $\beta_j \neq \beta$ with $|\beta_j| > 1$. Choose m with
$$|\beta^m - \beta_j^m| > \max\left(1, \left|\frac{\beta}{\beta_j}\right|\right).$$
Since
$$\frac{\lfloor\beta^m\rfloor}{\beta^m} = 1 - \frac{\beta^m - \lfloor\beta^m\rfloor}{\beta^m}$$

and $|\beta^m - \lfloor \beta^m \rfloor| < 1$, we have

$$\lfloor \beta^m \rfloor = \beta^m + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \dots \quad \text{where } |d_i| < 1.$$

Since $\lfloor \beta^m \rfloor \in \mathbb{F}[x]$, the expansion must be eventually periodic. Consider the β-expansion of $z = \lfloor \beta^m \rfloor - \beta^m \in D(0,1)$. Let $r^{(k)}$, $r_j^{(k)}$ be as in the proof of Theorem 4.1. Equation (4.1) yields

$$\lfloor \beta^m \rfloor = \beta^m + \sum_{\ell=1}^{k} \frac{d_\ell}{\beta^\ell} + \frac{r^{(k)}}{\beta^k} \quad \text{for } k \geq 0.$$

Since $\lfloor \beta^m \rfloor \in \mathbb{F}[x]$ and β_j is an Galois conjugate of β, it must fulfill the equation

$$\lfloor \beta^m \rfloor = \beta_j^m + \sum_{\ell=1}^{k} \frac{d_\ell}{\beta_j^\ell} + \frac{r_j^{(k)}}{\beta_j^k} \quad \text{for } k \geq 0.$$

Since the expansion of z is eventually periodic, the $r^{(k)}$ take only finitely many values. Thus, the same is true for $r_j^{(k)}$. Hence, $\lim_{k\to\infty} r^{(k)}/\beta^k = 0$. If $|\beta_j| > 1$, then $\lim_{k\to\infty} r_j^{(k)}/\beta_j^k = 0$. Therefore,

$$\beta^m - \beta_j^m + \sum_{\ell=1}^{\infty} d_\ell \left(\frac{1}{\beta^\ell} - \frac{1}{\beta_j^\ell} \right) = 0.$$

From

$$\begin{aligned} \left| \sum_{\ell=1}^{\infty} d_\ell \left(\frac{1}{\beta^\ell} - \frac{1}{\beta_j^\ell} \right) \right| &\leq \max \left(\max_{\ell=1}^{\infty} \left| \frac{d_\ell}{\beta^\ell} \right|, \max_{\ell=1}^{\infty} \left| \frac{d_\ell}{\beta_j^\ell} \right| \right) \\ &< \max \left(1, \left| \frac{\beta}{\beta_j} \right| \right) \end{aligned}$$

we get a contradiction.

□

We can combine Theorem 4.1 and Theorem 4.3 to obtain

Corollary 4.5 *An element $\beta \in \mathbb{F}((x^{-1}))$ is a Pisot or Salem element if and only if*

$$\mathbb{F}[x] \subset \mathbf{Per}(\beta).$$

5 Finite expansions

The present section is devoted to the characterization of $\mathbf{Fin}(\beta)$, the set of finite expansions. Contrary to the case of real β-expansions, we can prove a complete characterization result in our setting. We will need the following

Lemma 5.1 *Let β be an arbitrary element of $\mathbb{F}((x^{-1}))$ with $\deg\beta > 0$, and let $z \in \mathbb{F}[x, \beta^{-1}]$ have a purely periodic β-expansion. Then $z \in \mathbb{F}[x, \beta]$.*

Proof. Assume $z \in \mathbb{F}[x, \beta^{-1}]$ is purely periodic with period n. Let $d_\beta(z) = .d_1d_2\ldots$. Since $z \in \mathbb{F}[x, \beta^{-1}]$, there is a m such that $\beta^{mn}z \in \mathbb{F}[x, \beta]$. Therefore

$$z = \beta^{mn}z - d_1\beta^{mn-1} - \ldots - d_{mn} \in \mathbb{F}[x, \beta].$$

□

Theorem 5.2 *Let $\beta \in \mathbb{F}((x^{-1}))$ be a Pisot element. Then*

$$\mathbf{Fin}(\beta) = \mathbb{F}[x, \beta^{-1}]. \qquad \textbf{(F)}$$

Remark 5.3 Note that **(F)** is true if and only if for every $z \in \mathbb{F}[x, \beta^{-1}]$, there is a $k \geq 0$ such that $T^k(z) = 0$.

Proof. Since it is trivial that $\mathbf{Fin}(\beta) \subset \mathbb{F}[x, \beta^{-1}]$, we need to prove only the other direction. Let

$$\beta^n - a_1\beta^{n-1} - \ldots - a_n = 0 \qquad (5.1)$$

with $\deg a_1 > \deg a_j$ for $j > 1$.

From Theorem 4.1 it follows that $\mathbb{F}[x, \beta^{-1}] \subset \mathbb{F}(x, \beta) \subset \mathbf{Per}(\beta)$. Thus $z \in \mathbb{F}[x, \beta^{-1}]$ has an eventually periodic expansion. Therefore, it can be decomposed into $z = z_f + z_p$, where $d_\beta(z_f)$ is finite and $d_\beta(z_p)$ is purely periodic. Hence, by Lemma 5.1, $z_p \in \mathbb{F}[x, \beta]$. Since

$$r^{(k)} = \beta^k(z_f + z_p) - \sum_{l=1}^{k} d_l\beta_j^{k-l},$$

there is an integer k such that $r^{(k)} \in \mathbb{F}[x, \beta]$, and we can restrict our attention to $\mathbb{F}[x, \beta]$. For $i = 1, \ldots, n$, let

$$v_i := \beta^{i-1} - a_1\beta^{i-2} - \ldots - a_{i-1} \qquad (5.2)$$

$$= \frac{a_i}{\beta} + \ldots + \frac{a_n}{\beta^{n-i+1}} \qquad (5.3)$$

and $V = \{v_1, \ldots, v_n\}$. Note that $v_1 = 1$.

Then V is a base of $\mathbb{F}[x, \beta]$ over $\mathbb{F}[x]$. Hence, for every $z \in \mathbb{F}[x, \beta]$, there are $z_1, \ldots, z_n \in \mathbb{F}[x]$ such that

$$z = z_1 v_1 + \ldots + z_n v_n.$$

For

$$z = \tilde{z}_0 + \ldots + \tilde{z}_{n-1}\beta^{n-1} \in \mathbb{F}[x, \beta],$$

let

$$\tilde{e}(z) = [\tilde{z}_0, \ldots, \tilde{z}_{n-1}]_\beta^T$$

the vector of coordinates with respect to the standard base $\{1, \beta, \ldots, \beta^{n-1}\}$. Analogously, we will denote the vector of coordinates with respect to V by

$$e(z) = [z_1, \ldots, z_n]_V^T.$$

Using (5.2), the coordinates with respect to V can be computed from the standard coordinates from the system

$$\begin{bmatrix} \tilde{z}_0 \\ \vdots \\ \vdots \\ \tilde{z}_{n-1} \end{bmatrix}_\beta = \begin{bmatrix} 1 & -a_1 & \cdots & -a_{n-1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix}_V.$$

If we define $a_k = 0$ for $k > n$ and using (5.3), we can write

$$z = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_j z_{j+i-1} \right) \beta^{-i}. \qquad (5.4)$$

In base V, multiplication by β is represented by the matrix

$$M = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Define the vectors $\mathbf{v} = [v_1, \ldots, v_n]^T$ and $\mathbf{e} = [1, 0, \ldots,]^T$. We consider the greedy algorithm for $z \in \mathbb{F}[x, \beta]$ with respect to V. Write $\mathbf{z} = e(z)$. Since

$$T_\beta(z) = \beta z - \lfloor \beta z \rfloor,$$

we can write the β-transformation in base V as

$$\begin{array}{rcl} T : \mathbb{F}[x,\beta] & \mapsto & \mathbb{F}[x,\beta] \\ \mathbf{z} & \mapsto & M\mathbf{z} - (\lfloor M\mathbf{z}\cdot\mathbf{v}\rfloor)\mathbf{e}. \end{array}$$

Furthermore we have $e(T_\beta(z))) = T(e(z))$, which shows that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{F}[x,\beta] & \xrightarrow{T_\beta} & \mathbb{F}[x,\beta] \\ \downarrow e & & \downarrow e \\ \mathbb{F}[x]^n & \xrightarrow[T]{} & \mathbb{F}[x]^n. \end{array}$$

With respect to V, we can express T as follows:

$$T : \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_V \to \begin{bmatrix} -\lfloor z_1 v_2 + \ldots + z_{n-1} v_n \rfloor \\ z_1 \\ \vdots \\ z_{n-1} \end{bmatrix}_V .$$

Thus $\mathbb{F}[x,\beta]$ together with T provides an analogon to the so called *shift radix system* defined in [3]. However, due to our notation, the indices here are in the reverse direction as in [3].

For $j = 2, \ldots, n$ it follows that

$$\begin{array}{rcl} \deg \dfrac{a_j}{\beta} & = & \deg a_j - \deg \beta \\ & < & \deg a_1 - \deg a_1 = 0 \end{array}$$

and therefore

$$\deg v_j < 0.$$

Now

$$\begin{array}{rcl} \deg\left(-\lfloor z_1 v_2 + \ldots + z_{n-1} v_n \rfloor\right) & \leq & \max_{i=1}^{n-1} \deg\left(z_i v_{i+1}\right) \\ & < & \max_{i=1}^{n-1} \deg z_i. \end{array} \tag{5.5}$$

For $k \geq 0$, let

$$z^{(k)} = (z_1^{(k)}, \ldots, z_n^{(k)})_V^T = e(T^k(z)).$$

Thus, **(F)** is true if and only if there is a $k \geq 0$, such that $(z_1^{(k)}, \dots, z_n^{(k)})^T = (0, \dots, 0)^T$ or equivalently, since $z^{(k)} \in \mathbb{F}[x]^n$,

$$\max_{i=1}^{n} \deg z_i^{(k)} = 0.$$

It follows from (5.5) that

$$\deg z_1^{(1)} \leq \max_{i=1}^{n-1} \left(\deg z_i^{(0)} - 1\right).$$

From

$$(z_1^{(1)}, \dots, z_n^{(1)})^T = (z_1^{(1)}, z_1^{(0)}, \dots, z_{n-1}^{(0)})^T,$$

we obtain

$$\begin{aligned}
\deg z_1^{(2)} &\leq \max_{i=1}^{n-1} \left(\deg z_i^{(1)} - 1\right) \\
&= \max\left(\max_{i=1}^{n-1} \deg z_i^{(0)} - 2, \max_{i=1}^{n-2} \deg z_i^{(0)} - 1\right) \\
&= \max\left(\deg z_1^{(0)} - 1, \dots, \deg z_{n-2}^{(0)} - 1, \deg z_{n-1}^{(0)} - 2\right).
\end{aligned}$$

Analogously

$$\begin{aligned}
\deg z_1^{(3)} &\leq \max_{i=1}^{n-1} \left(\deg z_i^{(2)} - 1\right) \\
&= \max\left(\deg z_1^{(0)} - 1, \dots, \deg z_{n-3}^{(0)} - 1, \deg z_{n-2}^{(0)} - 2, \deg z_{n-1}^{(0)} - 2\right).
\end{aligned}$$

After $n-2$ such steps we obtain

$$\deg z_1^{(n-1)} \leq \max\left(\deg z_1^{(0)} - 1, \deg z_2^{(0)} - 2, \dots, \deg z_{n-1}^{(0)} - 2\right).$$

Thus

$$\begin{aligned}
\max_{i=1}^{n} \deg z_i^{(n)} &= \max_{i=1}^{n} \deg z_1^{(n+1-i)} \\
&\leq \max_{i=1}^{n} \left(\deg z_i^{(0)} - 1\right).
\end{aligned}$$

Going on this way, we will find a number h such that $\max_{i=1}^{n} \deg z_i^{(h)} = 0$ and we are done.

□

The following theorem forms the converse of Theorem 5.2.

Theorem 5.4 *If* $\mathbb{F}[x,\beta^{-1}] \subset \mathbf{Fin}(\beta)$*, then* β *is a Pisot element.*

Proof. Applying arguments from complex analysis, the proof of the corresponding result for real β-expansions [9, Lemma 1 (b)] is pretty short. Unfortunately, this technique doesn't work in our context. Therefore, we will adapt the idea of [16, Lemma 2.4].

Suppose that

$$\max_{j=2}^{n} \deg a_j \geq \deg a_1.$$

We will construct an element $z \in \mathbb{F}[x,\beta]$ which does not have a finite representation. Let v_i be as in (5.2). Define

$$i_0 = i_0(a_1,\ldots,a_n) := \max\{\ i\ :\ \deg a_i = \max_{j=1}^{n} \deg a_j\}$$

and

$$j_0 = j_0((z_1,\ldots,z_n)_V^T) := \begin{cases} \min\{\ i\ :\ \deg z_i = \max_{j=1}^{n} \deg z_j\} & \text{if } \max_{j=1}^{n} \deg z_j > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Select

$$z = z^{(0)} := (z_1^{(0)},\ldots,z_n^{(0)})_V^T \quad \text{with} \quad j_0(z^{(0)}) + 1 \leq i_0 \tag{5.6}$$

(take for instance $z^{(0)} := (x,0,\ldots,0)_V^T$). Thus

$$1 < i_0 \leq n \quad \text{and} \quad 1 \leq j_0 < n.$$

Furthermore,

$$\begin{aligned} &\deg v_{i_0} = \deg a_{i_0} - \deg\beta \geq 0, && \deg v_i < \deg v_{i_0} \text{ for } i > i_0, \\ &\deg z_{j_0}^{(0)} = \max_{j=1}^{n-1} \deg z_j^{(0)} > 0, && \deg z_j^{(0)} < \deg z_{j_0}^{(0)} \text{ for } j < j_0. \end{aligned}$$

We will show that $z^{(0)}$ has an infinite representation by proving that

$$j_0(z^{(k)}) + 1 \leq i_0 \quad \text{for all } k \geq 0. \tag{5.7}$$

By the definition of j_0, this implies that $z^{(k)} \neq 0$ for all $k \geq 0$ and we are done. Let

$$z^{(k)} = (z_1^{(k)},\ldots,z_n^{(k)})_V^T.$$

We will prove (5.7) by induction.

Since (5.7) holds for $k = 0$ by (5.6), we can proceed to the induction step.

Suppose that (5.7) holds for a certain k and note that

$$z^{(k+1)} = (z_1^{(k+1)}, \ldots, z_n^{(k+1)})_V^T = (z_1^{(k+1)}, z_1^{(k)}, \ldots, z_{n-1}^{(k)})_V^T. \tag{5.8}$$

We distinguish two cases.

Case 1. $j_0 := j_0(z^{(k)}) < i_0 - 1$. By (5.8) and because $j_0 < n$, we have

$$\begin{aligned}\max_{j=1}^{n} \deg z_j^{(k+1)} &= \max(\deg z_{j_0}^{(k)}, \deg z_1^{(k+1)}) \\ &= \max(\deg z_{j_0+1}^{(k+1)}, \deg z_1^{(k+1)}) > 0.\end{aligned}$$

Thus $j_0(z^{(k+1)}) = 1$ or $j_0(z^{(k+1)}) = j_0 + 1$. Both of these inequalities imply that

$$j_0(z^{(k+1)}) \leq i_0 - 1$$

and we are done.

Case 2. $j_0 := j_0(z^{(k)}) = i_0 - 1$. The definitions of i_0 and j_0 imply that

$$\begin{aligned}&\deg v_{i_0} \geq 0, && \deg v_i < \deg v_{i_0} \text{ for } i > i_0, \\ &\deg z_{i_0-1}^{(k)} > 0, && \deg z_j^{(k)} < \deg z_{i_0-1}^{(k)} \text{ for } j < i_0 - 1.\end{aligned}$$

Thus

$$\deg (z_{i_0-1}^{(k)} v_{i_0}) > \deg (z_{i-1}^{(k)} v_i) \quad \text{for } i \neq i_0.$$

This implies that no cancellations occur in the highest power of x in the sum

$$z_1^{(k)} v_2 + \ldots + z_{n-1}^{(k)} v_n.$$

Hence,

$$\begin{aligned}\deg (z_1^{(k)} v_2 + \ldots + z_{n-1}^{(k)} v_n) &= \deg z_{i_0-1}^{(k)} + \deg v_{i_0} \\ &> 0,\end{aligned}$$

and therefore

$$\begin{aligned}\deg (-\lfloor z_1^{(k)} v_2 + \ldots + z_{n-1}^{(k)} v_n \rfloor) &= \deg (z_1^{(k)} v_2 + \ldots + z_{n-1}^{(k)} v_n) \\ &= \deg (z_{i_0-1}^{(k)} v_{i_0}) \\ &\geq \deg z_{i_0-1}^{(k)} \\ &= \max_{j=1}^{n-1} \deg z_j^{(k)}.\end{aligned}$$

This implies that

$$\deg z_1^{(k+1)} \geq \max_{j=2}^{n} \deg z_j^{(k+1)}.$$

Thus,

$$j_0(z^{(k+1)}) = 1 \leq i_0 - 1$$

and we are done also in this case. □

It turns out that condition **(F)** is equivalent to a seemingly weaker condition.

Theorem 5.5 *$\mathbb{F}[x, \beta^{-1}] \subset \mathbf{Fin}(\beta)$ if and only if $\mathbb{F}[x] \subset \mathbf{Fin}(\beta)$.*

Proof. Of course, if $\mathbb{F}[x, \beta^{-1}] \subset \mathbf{Fin}(\beta)$, then $\mathbb{F}[x] \subset \mathbf{Fin}(\beta)$. To prove the converse, consider an element

$$z = z_0 + \frac{z_1}{\beta} + \ldots + \frac{z_\ell}{\beta^\ell} \in \mathbb{F}[x, \beta^{-1}] \quad \text{where } z_i \in \mathbb{F}[x].$$

There exist finite expansions $z_i = \sum_j d_{ij}/\beta^j$. Therefore $z_i/\beta^i = \sum_j d_{ij}/\beta^{i+j}$. Adding up the corresponding digits, we obtain the β-expansion of z, which is again finite.

□

We can combine Theorem 5.2 together with Theorem 5.4 and Theorem 5.5 to obtain

Corollary 5.6 *An element $\beta \in \mathbb{F}((x^{-1}))$ is a Pisot element if and only if*

$$\mathbb{F}[x] \subset \mathbf{Fin}(\beta).$$

Remark 5.7 The real analogue of condition $\mathbb{F}[x] \subset \mathbf{Fin}(\beta)$ is $\mathbb{Z}_+ \subset \mathbf{Fin}(\beta)$. In the case of real expansions, one can only prove that β is a Pisot or Salem number. Thus Corollary 5.6 is an improved analogue of Theorem 1.4 (i).

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