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Abstract

In this paper, a uniformly elliptic second order boundary value problem in 2D is discretized by the p-version of the finite element method. An inexact Dirichlet-Dirichlet domain decomposition pre-conditioner for the system of linear algebraic equations is investigated. The ingredients of such a pre-conditioner are an pre-conditioner for the Schur complement, an pre-conditioner for the sub-domains and an extension operator operating from the edges of the elements into their interior. Using methods of multi-resolution analysis, we propose a new method in order to compute the extension efficiently. We prove that this type of extension is optimal, i.e. the $H^1(\Omega)$-norm of the extended function is bounded by the $H^{0.5}(\partial\Omega)$-norm of the given function. Numerical experiments show the optimal performance of the described extension.

1 Introduction

We investigate the following boundary value problem. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a domain which can be decomposed into elements R_s. Find $u \in \hat{H}_0^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cup \Gamma_2 \subset \partial\Omega$ such that

$$a_\triangle(u, v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_2} f_1 v := \langle f, v\rangle + \langle f_1, v\rangle_{\Gamma_2} \tag{1.1}$$

holds for all $v \in \hat{H}_0^1(\Omega)$. Problem (1.1) will be discretized by means of the p-version of the finite element method using quadrilateral ($d = 2$) or hexahedral ($d = 3$) elements. Let $\mathcal{R}_d = (-1, 1)^d$ be the reference element and $\Phi_s : \mathcal{R}_d \to R_s$ be the mapping to the element R_s. We define the finite element space $\mathbb{M} := \{u \in \hat{H}_0^1(\Omega), u\,|_{R_s} = \tilde{u}(\Phi_s^{-1}(x, y)), \tilde{u} \in \mathbb{Q}_p\}$, where $\mathbb{Q}_p$ is the space of all polynomials of maximal degree p in each variable. Let $(\zeta_1, \ldots, \zeta_N)$ be a basis for $\mathbb{M}$. The Galerkin projection of (1.1) onto $\mathbb{M}$ leads to the linear system of algebraic finite element equations

$$\mathcal{A}\underline{u} = \underline{f}, \quad \text{where} \quad \mathcal{A} = [a_\Delta(\zeta_j, \zeta_i)]_{i,j=1}^N, \quad \underline{f}_p = [\langle f, \zeta_i\rangle + \langle f_1, \zeta_i\rangle_{\Gamma_2}]_{i=1}^N. \tag{1.2}$$

Using the vector $\underline{u}$, an approximation u_h of the exact solution u of (1.1) can be built by the usual finite element isomorphism. The error $u_h - u$ tends to zero in a suitably chosen norm, if the mesh-size parameter h of the element R_s tends to zero. Therefore for the practical implementation of such algorithms, it is important to choose a discretization parameter h as small as possible in order to obtain a sufficiently accurate approximation u_h to the exact solution u. Then, the dimension n of the vector $\underline{u} \in \mathbb{R}^N$ will be nearly proportional to h^{-d}, where d is the dimension of the domain in which the partial differential equation is solved. Using finite elements of low order (h-version of the FEM), the corresponding system matrix $\mathcal{A}$ is a sparse matrix and is positive definite for elliptic problems. More precisely, the number of nonzero elements is of order N. For $d > 1$, the matrix $\mathcal{A}$ has a banded-like structure. For todays computers, it is no problem to store such a sparse matrix of dimensions up to some millions.

Instead of the h-version of the finite element method, collocation methods, [26], and finite elements of high order (p-version), see e.g. [36] and the references therein, have become more popular for twenty years. For the h-version of the FEM, the polynomial degree p of the shape functions on the elements is kept constant and the mesh-size h is decreased. This is in contrast to the the p-version of the FEM in which the polynomial degree p is increased and the mesh-size h is kept constant. The advantage of the p-version in comparison to the h-version is that the approximate solution u_p converges faster to the exact solution u, if u is sufficiently smooth. For example, for the potential equation $-\triangle u = f$ with u analytic, the error in the H^1-Sobolev norm fulfills $\| u - u_p \|_1 \leq Ce^{-rp}$ (with some constant $r > 0$ independent of p) in contrast to the algebraic convergence order of the h-version with $\| u - u_h \|_1 \leq Ch$. Thus, the dimension of the FEM ansatz space can be reduced while obtaining an approximate solution with the same accuracy as in the h-version of the FEM. Both ideas, mesh refinement and increasing the polynomial degree, can be combined. This is called the hp-version of the FEM.

In both cases, h- and p-version of the FEM, the stiffness matrix is usually ill-conditioned. Therefore, efficient pre-conditioning techniques are required in order to solve (1.2).

Pre-conditioners for (1.2) can be built by sub-structuring techniques (DD-methods). We refer to [31], [10], [11], [29], [12], [32], [13], [30], in the case of the h-version of the FEM and to [17], [3], [2], [25], [23], [24], [27] for the p-version of the FEM. From the algebraic point of view, the symmetric positive definite (spd) stiffness matrix $\mathcal{A}$ is splitted into

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} I & \mathcal{A}_{12}\mathcal{A}_{22}^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{S} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathcal{A}_{22}^{-1}\mathcal{A}_{21} & I \end{bmatrix} \tag{1.3}$$

with the Schur-complement $\mathcal{S} = \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$. Then, the pre-conditioner

$$\mathcal{C} = \begin{bmatrix} I & -\mathcal{E}^T \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_{22} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -\mathcal{E} & I \end{bmatrix} \tag{1.4}$$

is introduced. The following result has been proved in [29], [30], [19], [34] and is the key in order to analyze the preconditioner $\mathcal{C}$.

Lemma 1.1 *Let $\mathcal{C}_S$ and $\mathcal{C}_{22}$ be spd pre-conditioners for $\mathcal{S}$ and $\mathcal{A}_{22}$, i.e.*

$$\begin{aligned} (\mathcal{C}_S\underline{v}, \underline{v}) \leq (\mathcal{S}\underline{v}, \underline{v}) \leq C_S\,(\mathcal{C}_S\underline{v}, \underline{v}) \quad & \forall \underline{v}, \\ c_I\,(\mathcal{C}_{22}\underline{v}, \underline{v}) \leq (\mathcal{A}_{22}\underline{v}, \underline{v}) \leq C_I\,(\mathcal{C}_{22}\underline{v}, \underline{v}) \quad & \forall \underline{v}. \end{aligned}$$

Moreover, let

$$\left(\mathcal{A} \begin{bmatrix} I \\ \mathcal{E} \end{bmatrix} \underline{g}, \begin{bmatrix} I \\ \mathcal{E} \end{bmatrix} \underline{g}\right) \leq c_E^2\,(\mathcal{C}_S\underline{g}, \underline{g}) \quad \forall \underline{g}.$$

Then, the inequalities

$$c\,(\mathcal{C}\underline{v}, \underline{v}) \leq (\mathcal{A}\underline{v}, \underline{v}) \leq C\,(\mathcal{C}\underline{v}, \underline{v}) \quad \forall \underline{v}$$

hold with $c = \frac{1}{2(C_S(1+c_E^2)-1)} \min\{1, c_I\}$ and $C = 2\max\{c_E^2, C_I\}$.

In the p-version of FEM, the splitting of the matrix $\mathcal{A}$ is naturally given by a splitting of the ansatz functions. Let n_v, n_e, n_f, and n_i be the number of vertices not belonging to $\overline{\Gamma}_1$, number of edges not belonging to $\overline{\Gamma}_1$, number of faces not belonging to $\overline{\Gamma}_1$ (for $d=3$ only) and number of elements, respectively. To each vertex corresponds 1, to each edge correspond $p-1$, to each face correspond $(p-1)^2$ and to each element correspond $(p-1)^d$, $d=2,3$, basis functions. Thus, the dimension of the ansatz space is $N=n_v+(p-1)n_e+(p-1)^2n_i$ for $d=2$ and $N=n_v+(p-1)n_e+(p-1)^2n_f+(p-1)^3n_i$ for $d=3$.
For $d=2$, we define the functions $\zeta_1,\dots,\zeta_{n_v}$ as the usual piecewise bilinear hat functions. The functions $\zeta_{n_v+(j-1)(p-1)+1},\dots,\zeta_{n_v+j(p-1)}$ correspond to the edge e_j of the mesh, and vanish on all other edges, i.e. satisfy the condition $\zeta_{n_v+(j-1)(p-1)+k-1}\,|_{e_l}=\delta_{j,l}p_k$, where p_k is a polynomial of degree p, $k=2,\dots,p$. The support of an edge function is formed by those two elements, which have this edge e_j in common. The remaining basis functions are interior bubble functions consisting of a support containing one element only. For $d=3$, the functions are analogously defined, in addition to the vertex, edge, and interior functions, we have so called face bubble functions associated to a face of the mesh. Now, the basis functions ζ_i are divided into three (four for $d=3$) groups,

- the vertex functions,
- the edge bubble functions,
- face bubble functions, (for $d=3$ only),
- the interior bubbles.

Corresponding to the division of the shape functions, the matrix $\mathcal{A}_d$ is splitted into the blocks

$$\mathcal{A}_2=\begin{bmatrix}\mathcal{A}_v & \mathcal{A}_{v,e} & \mathcal{A}_{v,i}\\ \mathcal{A}_{e,v} & \mathcal{A}_e & \mathcal{A}_{e,i}\\ \mathcal{A}_{i,v} & \mathcal{A}_{i,e} & \mathcal{A}_i\end{bmatrix},\quad\text{or}\quad \mathcal{A}_3=\begin{bmatrix}\mathcal{A}_v & \mathcal{A}_{v,e} & \mathcal{A}_{v,f} & \mathcal{A}_{v,i}\\ \mathcal{A}_{e,v} & \mathcal{A}_e & \mathcal{A}_{e,f} & \mathcal{A}_{e,i}\\ \mathcal{A}_{f,v} & \mathcal{A}_{f,e} & \mathcal{A}_f & \mathcal{A}_{f,i}\\ \mathcal{A}_{i,v} & \mathcal{A}_{i,e} & \mathcal{A}_{i,f} & \mathcal{A}_i\end{bmatrix},$$

where the indices v, e, f and i denote the blocks corresponding to the vertex, edge bubble, face bubble and interior bubble functions for $d=2$ and $d=3$, respectively.
For $d=2$, the simpler matrix

$$\mathcal{C}=\begin{bmatrix}\mathcal{A}_v & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathcal{A}_e & \mathcal{A}_{e,i}\\ \mathbf{0} & \mathcal{A}_{i,e} & \mathcal{A}_i\end{bmatrix}, \tag{1.5}$$

is investigated. It has been proved in [3] that the condition number $\kappa(\mathcal{C}^{-1}\mathcal{A}_2)$ grows as $1+\log p$. Therefore, the vertex unknowns can be determined separately. Pre-conditioners for $\mathcal{A}_v$ are multi-grid methods, [21], or BPX-pre-conditioners, [37], [14]. Computing the other unknowns, we factorize the remaining 2 by 2 block

$$\begin{bmatrix}\mathcal{A}_e & \mathcal{A}_{e,i}\\ \mathcal{A}_{i,e} & \mathcal{A}_i\end{bmatrix}=\begin{bmatrix}I & \mathcal{A}_{e,i}\mathcal{A}_i^{-1}\\ \mathbf{0} & I\end{bmatrix}\begin{bmatrix}\mathcal{S} & \mathbf{0}\\ \mathbf{0} & \mathcal{A}_i\end{bmatrix}\begin{bmatrix}I & \mathbf{0}\\ \mathcal{A}_i^{-1}\mathcal{A}_{i,e} & I\end{bmatrix} \tag{1.6}$$

with the Schur-complement $\mathcal{S}=\mathcal{A}_e-\mathcal{A}_{e,i}\mathcal{A}_i^{-1}\mathcal{A}_{i,e}$.
For $d=3$, we follow an approach of [27]. An elimination of the matrix $\mathcal{A}_i$ gives

$$\begin{bmatrix}\mathcal{A}_f & \mathcal{A}_{f,i}\\ \mathcal{A}_{i,f} & \mathcal{A}_i\end{bmatrix}=\begin{bmatrix}I & \mathcal{A}_{f,i}\mathcal{A}_i^{-1}\\ \mathbf{0} & I\end{bmatrix}\begin{bmatrix}\mathcal{S} & \mathbf{0}\\ \mathbf{0} & \mathcal{A}_i\end{bmatrix}\begin{bmatrix}I & \mathbf{0}\\ \mathcal{A}_i^{-1}\mathcal{A}_{i,f} & I\end{bmatrix} \tag{1.7}$$

with the Schur-complement $\mathcal{S}=\mathcal{A}_f-\mathcal{A}_{f,i}\mathcal{A}_i^{-1}\mathcal{A}_{i,f}$ for the last two blocks of $\mathcal{A}_3$. The coupling of the face and interior unknowns to the block of the vertex and edge unknowns is more difficult. In [27], the coupling is removed by a second extension operator.

For $d = 2, 3$, the matrix $\mathcal{A}_i$ is a block diagonal matrix, each block $\mathcal{A}_{R_s}$ corresponds to one element R_s, i.e.

$$\mathcal{A}_i = \text{blockdiag}[\mathcal{A}_{R_s}]_{s=1}^{n_i}. \tag{1.8}$$

Therefore, in order to compute the interior unknowns, we have to solve a Dirichlet problem on each quadrilateral or hexehadron. The edge unknowns are computed via the Schur-complement $\mathcal{S}$. Due to Lemma 1.1, an inexact DD-pre-conditioner for (1) includes a pre-conditioner for $\mathcal{A}_i$, a pre-conditioner for the Schur-complement $\mathcal{S}$ and an extension operator $\mathfrak{E}$ operating from the edges of the quadrilateral/faces of the hexahedron into its interior in order to replace the matrix $\mathcal{A}_i^{-1}\mathcal{A}_{i,e}$ or $\mathcal{A}_i^{-1}\mathcal{A}_{i,f}$ by a matrix $-\mathcal{E}$.
In [25], Jensen/Korneev have proved the following result which allows us to restrict ourselves to the case of the reference element in order to derive pre-conditioners for $\mathcal{A}_i$, $\mathcal{S}$ and to find the extension operator $\mathcal{E}$.

Lemma 1.2 *Let $\partial R_s \in C^{(t)}$, $t \geq 2$, $s = 1, \ldots, n_i$, where $C^{(t)}$ denotes the class of all boundaries which consist of a finite number of t times continuously differentiable curves and the angles of these curves at their intersection points on ∂R_s are distinct from 0 and 2π. Let $\tilde{A}$ be the result of assembling the element stiffness matrices on the reference element $\mathcal{R}_2 = [-1, 1]^2$ instead of the element stiffness matrix corresponding to the element R_s. Then, $\kappa(\tilde{A}^{-1}A) = \mathcal{O}(1)$.*

In [23], [24] and [25], two pre-conditioners for the Schur-complement are proposed. The pre-conditioners use basis transformations from the integrated Legendre polynomials to the Chebyshev polynomials or a Lagrange basis. Another pre-conditioning techniques can be found in [1], [18]. Moreover, an approach using multi-resolution bases for the Schur-complement in 3D, [27], can be applied to the 2D case as well.
For the interior solver, several pre-conditioners $\mathcal{C}_i$ are developed in [25], [6], [9], [28], [7] for $d = 2$ and in [9] for $d = 3$. The condition number of $\mathcal{C}_i^{-1}\mathcal{A}_i$ is bounded independent of the polynomial degree, [8]. The operation $\mathcal{C}_i^{-1}\underline{w}$ requires $\mathcal{O}(p^d)$ operations. All pre-conditioners use interpretations of the matrix $\mathcal{A}_{\mathcal{R}_d}$ as an h-version FEM discretization matrix of a degenerated elliptic problem, published in [6].
In order to replace $\mathcal{A}_i^{-1}\mathcal{A}_{i,e}$ in (1.6) or $\mathcal{A}_i^{-1}\mathcal{A}_{i,f}$ in (1.7), an efficient extension operator $\mathfrak{E} \leftrightarrow \mathcal{E}$ is required, cf. Lemma 1.1. Given a polynomial $g \in \mathbb{P}_p$ on one edge(face) of $\mathcal{R}_d$, find a polynomial $u \in (\mathbb{Q}_p)$ with

$$\| u \|_{H^1(\mathcal{R}_d)} \leq c_E \| g \|_{H^{0.5}(\partial\mathcal{R}_d)} \quad \forall g \in \mathbb{P}_p \tag{1.9}$$

under the constraint $u = g$ on $\partial\mathcal{R}_d$, where the constant c_E is independent of the polynomial degree p. Usually, the system

$$\mathcal{A}_i\underline{u} = -\mathcal{A}_{i,e}\underline{g} \quad \text{or} \quad \mathcal{A}_i\underline{u} = -\mathcal{A}_{i,f}\underline{g} \tag{1.10}$$

can be treated by a preconditioned Chebyshev iteration with the pre-conditioner $\mathcal{C}_i$ for $\mathcal{A}_i$, see e.g. [27]. Since this operation is required in each pre-conditioning step, cf. (1.3), the system solve (1.10) is too expensive. In the case of the h-version of the FEM, i.e. the function g is a piecewise linear function, Nepomnyaschikh, [20], [34], derived several techniques in order to develop such an extension operator without the solution of (1.10). For the p-version of the FEM, a pioneering work is done by [3]. The practical and efficient implementation of this extension operator in certain polynomial bases, i.e. integrated Legendre polynomials, has been an open question.
This paper is dedicated to the development of fast solvers for the system (1.2) arising from the discretization by the p-version of the FEM. Efficient tools are Domain Decomposition methods. We mainly focus on one ingredient of the inexact Domain Decomposition pre-conditioner, the extension operator. On the one hand, we will prove the estimate (1.9). On the other hand, we will show that this extension can be computed in $\mathcal{O}(p^d)$ operations. Moreover, we will show that this technique can be extended to tensor product discretizations of the h-version of the finite element

method as well. Our technique uses a basis for the given function g in (1.9) which is stable in L_2 and H^1. With methods of multi-resolution analysis, cf. [16], [15], [35] for the h-version of the FEM, and [9] for the p-version of the FEM, such bases can be determined.
The paper is organized as follows. Section 2 deals with purely algebraic investigations. The assumptions are required in order to replace $-\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$ in (1.3) by a proper matrix $\mathcal{E}$ are analyzed. In section 3, the model problem in order to derive the extension operator is introduced. Then, this algebraic approach is used for the general definition of extension operators. In section 4, some examples of extension operators for the h- and the p-version of the FEM in $2D$ and $3D$ are given. In section 5, we show the numerical performance of the proposed extension operators.
Throughout this paper, the notation $\underline{g}$ is used for vectors of the Euklidian space, where the corresponding function $g(x) = \sum_{j=1}^n g_j\phi_j(x)$ is denoted by g. We write $g \leftrightarrow \underline{g}$. Moreover, let $I = (-1, 1)$. The parameter p denotes the polynomial degree and $\mathbb{P}_p$ and $\mathbb{Q}_p$ denote the space of all polynomials of degree p in one and two variables. $\mathbb{P}_{p,00}$ denotes the subspace of $\mathbb{P}_p$ satisfying $q(\pm 1) = 0$.

2 Premilinaries from Linear Algebra

In this section, we prove some auxiliary results from Linear Algebra.

Lemma 2.1 *Let*

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

be a symmetric positive definite matrix. Moreover let $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}$. *The matrix* $S = K_{11} - K_{12}K_{22}^{-1}K_{21}$ *denotes the Schur-complement. Then, the solution* $\underline{x^*} = \begin{bmatrix} \underline{g} \\ \underline{u^*} \end{bmatrix}$ *of* $K_{22}\underline{u} = -K_{21}\underline{g}$ *is the optimal solution of*

$$(K\underline{x}, \underline{x}) \to \min_{\underline{u}}, \quad where \quad \underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}. \tag{2.1}$$

This solution satisfies

$$\left(S\underline{g}, \underline{g}\right) = (K\underline{x^*}, \underline{x^*}). \tag{2.2}$$

Proof: Linear Algebra.□
Let us assume that

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}$$

is a spd pre-conditioner for the matrix K, i.e.

$$c_1 (C\underline{x}, \underline{x}) \le (K\underline{x}, \underline{x}) \le c_2 (C\underline{x}, \underline{x}) \quad \forall \underline{x}. \tag{2.3}$$

We investigate the optimization problem

$$(C\underline{x}, \underline{x}) \to \min_{\underline{u}}, \quad \underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}. \tag{2.4}$$

The next lemma gives a relation between the solutions of the optimization problems (2.1) and (2.4).

Lemma 2.2 *Let $\underline{x_0} = \begin{bmatrix} \underline{g} \\ \underline{u_0} \end{bmatrix}$ be the optimal solution of* (2.4) *and let $\underline{x^*}$ be the optimal solution of* (2.1)*. Then, we have*

$$\left(K\underline{x_0}, \underline{x_0}\right) \leq \frac{c_2}{c_1} \left(K\underline{x^*}, \underline{x^*}\right).$$

The constants c_1 and c_2 are the constants of the spectral equivalence relations (2.3)*.*

Proof: Using (2.3), one easily concludes

$$\left(K\underline{x_0}, \underline{x_0}\right) \leq c_2 \left(C\underline{x_0}, \underline{x_0}\right). \tag{2.5}$$

Since $\underline{x_0}$ is the optimal solution of (2.4), we have

$$\left(C\underline{x_0}, \underline{x_0}\right) \leq \left(C\underline{x^*}, \underline{x^*}\right). \tag{2.6}$$

Using (2.3) again, one derives

$$\left(C\underline{x^*}, \underline{x^*}\right) \leq c_1^{-1} \left(K\underline{x^*}, \underline{x^*}\right). \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), the assertion follows. □

Corollary 2.3 *Let $\underline{x_0}$ be the optimal solution of* (2.4)*. Then, we have*

$$\frac{c_2}{c_1} \left(S\underline{g}, \underline{g}\right) \geq \left(K\underline{x_0}, \underline{x_0}\right) \tag{2.8}$$

for all $\underline{g} \in \mathbb{R}^n$.

In the following, let us assume that the matrix K has tensor product structure: Let

$$A_i = \begin{bmatrix} \alpha_i & \underline{a}_i^T \\ \underline{a}_i & A_{i,0} \end{bmatrix} \in \mathbb{R}^{m\times m}, \quad i = 1, 2,\ A_{i,0} \in \mathbb{R}^{m-1\times m-1}$$

and $B_i \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrices. Moreover, let

$$K = A_1 \otimes B_1 + A_2 \otimes B_2 = \begin{bmatrix} \alpha_1 B_1 + \alpha_2 B_2 & \underline{a}_1^T \otimes B_1 + \underline{a}_2^T \otimes B_2 \\ \underline{a}_1 \otimes B_1 + \underline{a}_2 \otimes B_2 & A_{1,0} \otimes B_1 + A_{2,0} \otimes B_2 \end{bmatrix} \in \mathbb{R}^{mn\times mn}. \tag{2.9}$$

Assume further that D_1 and D_2 are pre-conditioners for B_1 and B_2 satisfying the spectral equivalence relations

$$c_{1,i} \left(D_i \underline{v}, \underline{v}\right) \leq \left(B_i \underline{v}, \underline{v}\right) \leq c_{2,i} \left(D_i \underline{v}, \underline{v}\right) \quad \forall \underline{v}, i = 1, 2. \tag{2.10}$$

For the matrix K, we define the pre-conditioner

$$C = A_1 \otimes D_1 + A_2 \otimes D_2 = \begin{bmatrix} \alpha_1 D_1 + \alpha_2 D_2 & \underline{a}_1^T \otimes D_1 + \underline{a}_2^T \otimes D_2 \\ \underline{a}_1 \otimes D_1 + \underline{a}_2 \otimes D_2 & A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2 \end{bmatrix} \in \mathbb{R}^{mn\times mn}. \tag{2.11}$$

Proposition 2.4 *Let C be defined via* (2.11)*. Then,*

$$\min\{c_{1,1}, c_{1,2}\} \left(C\underline{v}, \underline{v}\right) \leq \left(K\underline{v}, \underline{v}\right) \leq \max\{c_{2,1}, c_{2,2}\} \left(C\underline{v}, \underline{v}\right) \quad \forall \underline{v}.$$

Proof: The proof is trivial. □

Theorem 2.5 *Let K and C be defined via* (2.9) *and* (2.11)*. Moreover, let us assume that the matrices D_i satisfy relations* (2.10)*. Let*

$$\underline{\hat{u}} = -(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2)^{-1}(\underline{a}_1 \otimes D_1 + \underline{a}_2 \otimes D_2)\underline{g}. \tag{2.12}$$

Then,

$$\left(K \begin{bmatrix} \underline{g} \\ \underline{\hat{u}} \end{bmatrix}, \begin{bmatrix} \underline{g} \\ \underline{\hat{u}} \end{bmatrix}\right) \leq \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}} \left(S\underline{g}, \underline{g}\right), \quad \forall \underline{g} \in \mathbb{R}^m.$$

Proof: Use Corollary 2.3. By Proposition 2.4, the assertion follows immediately. □
In the next lemma, we consider the special case that D_1 and D_2 are diagonal matrices, i.e.

$$D_1 = \mathrm{diag}\left[d_j^{(1)}\right]_{j=1}^n \quad \text{and} \quad D_2 = \mathrm{diag}\left[d_j^{(2)}\right]_{j=1}^n . \tag{2.13}$$

Then, the solution of (2.12) can easily be computed by solving n $(m-1)\times(m-1)$ linear systems with linear combinations of the matrices $A_{1,0}$ and $A_{2,0}$.

Lemma 2.6 *Let $\underline{\hat{u}}$ be defined via (2.12), where D_1 and D_2 are defined via (2.13). Then, $\underline{\hat{u}} = \left[\hat{u}_1^{(1)}, \hat{u}_1^{(2)}, \ldots, \hat{u}_1^{(n)}, \hat{u}_2^{(1)}, \ldots, \hat{u}_{m-1}^{(n)}\right]^T$, where*

$$\underline{\hat{u}}^{(j)} = -g_j(d_j^{(1)} A_{1,0} + d_j^{(2)} A_{2,0})^{-1}(d_j^{(1)}\underline{a}_1 + d_j^{(2)}\underline{a}_2) \in \mathbb{R}^{m-1}, \quad j = 1, \ldots, n.$$

Proof: We investigate the auxiliary problem

$$\underline{\tilde{u}} = -(D_1 \otimes A_{1,0} + D_2 \otimes A_{2,0})^{-1}(D_1 \otimes \underline{a}_1 + D_2 \otimes \underline{a}_2)\underline{g}.$$

The matrix $\tilde{C}_{22} = D_1 \otimes A_{1,0} + D_2 \otimes A_{2,0}$ is a block diagonal matrix, i.e.

$$\tilde{C}_{22} = \mathrm{blockdiag}\left[d_i^{(1)} A_{1,0} + d_i^{(2)} A_{2,0}\right]_{i=1}^n .$$

The inverse is given by

$$\tilde{C}_{22}^{-1} = \mathrm{blockdiag}\left[(d_i^{(1)} A_{1,0} + d_i^{(2)} A_{2,0})^{-1}\right]_{i=1}^n .$$

Moreover,

$$D_1 \otimes \underline{a}_1 + D_2 \otimes \underline{a}_2 = \mathrm{blockdiag}\left[d_i^{(1)}\underline{a}_1 + d_i^{(2)}\underline{a}_2\right]_{i=1}^n .$$

So,

$$\underline{\tilde{u}} = -\mathrm{blockdiag}\left[(d_i^{(1)} A_{1,0} + d_i^{(2)} A_{i,0})^{-1}(d_i^{(1)}\underline{a}_1 + d_i^{(2)}\underline{a}_2)g_i\right]_{i=1}^n .$$

Reordering the unknowns proves the assertion. □

3 Application to extension operators

3.1 Extensions from one face or edge

We consider the following problem:

$$-\triangle u = 0 \quad \text{in} \quad \Omega = I \times \omega, \tag{3.1}$$
$$u(1, y) = 0, \qquad u(-1, y) = g(y) \quad \text{on} \quad \omega, \tag{3.2}$$
$$u(x, y) = 0 \quad \text{on} \quad I \times \partial\omega. \tag{3.3}$$

Problem (3.1)-(3.3) is the typical model problem for the harmonic extension of a function g on ω into the interior of the domain Ω in the tensor product case. The weak formulation of (3.1)-(3.3) is: Find $u \in H_g(\Omega) = \{u = w + g, w \in H_0^1(\Omega)\}$ such that

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v = 0, \quad \forall v \in H_0^1(\Omega) \tag{3.4}$$

holds. Problem (3.1)-(3.3) is solved approximately by a tensor product version of the Finite Element Method (FEM). Let $\Xi = \{\xi_j(x)\}_{j=1}^m$ be a basis for the discretization of the interval $(-1,1)$ with $\xi_j(\pm 1) = 0$ for $j = 2, \dots, m$, $\xi_1(1) = 0$ and $\xi_1(-1) = 1$. Moreover, let $\mathbf{X} = \{\chi_j(y)\}_{j=1}^n$ be a basis for the discretization of ω satisfying $\chi_j(\partial\omega) = 0$. On Ω, we use the approximation space

$$\mathbb{V} = \text{span}\,\{\phi_{ij}(x,y)\}_{i,j=1}^{m,n}, \quad \text{where} \quad \phi_{ij}(x,y) = \xi_i(x)\chi_j(y).$$

In order to satisfy the boundary condition (3.2), the functions ϕ_{1j}, $1 \le j \le n$ are defined, i.e. we assume that $g(y) = \sum_{j=1}^n g_j \phi_1(-1,y)$ with the given coefficients g_j.
The Galerkin projection of problem (3.4) onto the space $\mathbb{V}$ leads to the following problem: Find $u = \sum_{j=1}^n g_j \phi_{1j} + \sum_{i=2}^m \sum_{j=1}^n u_{ij}\phi_{ij}$ such that

$$a(u, \phi_{ij}) = 0, \quad \forall i = 2, \dots, m, j = 1, \dots, n. \tag{3.5}$$

Introducing the matrices

$$\begin{aligned} K_{22} &= \left[a(\phi_{ij}, \phi_{lk})_{i=2,j=1;l=2,k=1}^{m,n;m,n}\right], \\ K_{21} = K_{12}^T &= \left[a(\phi_{ij}, \phi_{1k})_{i=2,j=1;k=1}^{m,n;n}\right], \\ K_{11} &= \left[a(\phi_{1j}, \phi_{1k})_{k;j=1}^{n}\right], \end{aligned}$$

problem (3.5) is equivalent to solve the system of linear algebraic equations

$$K_{22}\underline{u} = -K_{21}\underline{g} \tag{3.6}$$

with $\underline{u} = [u_{21}, \dots, u_{mn}]^T$ and $\underline{g} = [g_1, \dots, g_n]^T$.
We are not interested in solving the problem (3.5), (3.6) exactly, we only try to find a vector $\underline{u}$ satisfying the inequality

$$(K\underline{x}, \underline{x}) \le c_E^2 \left(S\underline{g}, \underline{g}\right) \quad \forall \underline{g} \tag{3.7}$$

under the constraint $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}$. The constant c_E should not depend on the discretization parameter. In the case of the h-Version of the FEM, several techniques in order to construct a vector $\underline{u} \leftrightarrow u$ satisfying (3.7) are given in [34], where the system (3.6) has not to be solved.
By the usual FEM-isomorphism, one has

$$(K\underline{x}, \underline{x}) = \mid u \mid_{H^1(\Omega)}^2 \quad \text{with} \quad \underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix} \leftrightarrow u \tag{3.8}$$

between the vector $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}$ and the corresponding function u. Due to (3.2) and the Poincare-Friedrichs inequality, the H^1-seminorm is equivalent to the H^1-norm. Thus, one obtains $c_\Omega^{-2} \parallel u \parallel_{H^1(\Omega)}^2 \le (K\underline{x}, \underline{x}) \le \parallel u \parallel_{H^1(\Omega)}^2$ for all u, where the constant c_Ω depends on the domain Ω only. Concerning a relation between the function g and the vector $\underline{g}$, the following assumption is required.

Assumption 3.1 *Let us assume that for every $g \in \text{span}\,\mathbf{X}$ there exists a function u such that*

$$\parallel u \parallel_{H^1(\Omega)} \le c_H \parallel g \parallel_{H^{0.5}(\partial\Omega)}, \tag{3.9}$$

under the constraints $u(-1,y) = g(y)$, $u(1,y) = 0$, $u(x, \partial\omega) = 0$, where the constant c does not depend on the discretization parameter.

Then, the following result can be proved.

Lemma 3.2 *Let us assume that Assumption 3.1 is satisfied and that $\partial\Omega$ is Lipschitz continuous. Let $g = \sum_{j=1}^{n} g_j \phi_j$ and $\underline{g} = [g_j]_{j=1}^{n}$. Then, we have the norm equivalence relation*

$$c_T^2 \parallel g \parallel_{H^{0.5}(\partial\Omega)}^2 \leq \left(S\underline{g}, \underline{g}\right) \leq c_H^2 \parallel g \parallel_{H^{0.5}(\partial\Omega)}^2, \tag{3.10}$$

where the constant c_H is the constant in (3.9) and c_T is the constant from the trace theorem.

Proof: The proof is similar to the arguments in order to derive formula (3.12.) in [25]. By Lemma 2.1, we have

$$\left(S\underline{g}, \underline{g}\right) = \min_{\underline{u}} \left(K\underline{u}, \underline{u}\right), \quad \text{where} \quad \underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix}.$$

Let $u^* \leftrightarrow \underline{x^*} = \begin{bmatrix} \underline{g} \\ \underline{u^*} \end{bmatrix}$ be the optimal solution of this minimization problem. Thus, by the trace theorem,

$$c_T^2 \parallel g \parallel_{H^{0.5}(\partial\Omega)}^2 \leq \parallel u^* \parallel_{H^1(\Omega)}^2 = \left(K\underline{x^*}, \underline{x^*}\right) = \left(S\underline{g}, \underline{g}\right).$$

Moreover, let $u_0 \leftrightarrow \underline{x_0} = \begin{bmatrix} \underline{g} \\ \underline{u_0} \end{bmatrix}$ be the extension of Assumption 3.1. Then,

$$\left(S\underline{g}, \underline{g}\right) \leq \left(K\underline{x_0}, \underline{x_0}\right) = \parallel u_0 \parallel_{H^1(\Omega)}^2 \leq c_H^2 \parallel g \parallel_{H^{0.5}(\partial\Omega)}^2,$$

which proves the lemma. □
This lemma is important in order to derive a pre-conditioner for the Schur-complement embedded in the Domain Decomposition pre-conditioner (1.4). Using this norm equivalence, most pre-conditioners for S can be constructed by finding a stable basis in $H^{0.5}(\partial\Omega)$. Moreover, we are able to define $\sqrt{\left(S\underline{g}, \underline{g}\right)}$ as an equivalent norm on $H^{0.5}(\partial\Omega)$.
Now, we want to apply the theory of the preceeding section in order to derive an extension operator satisfying relation (3.7). The main point are the relations (3.10) and (3.8) between the shape functions $u \in \mathbb{V}$ and vectors of the Euklidian space $\underline{x} \in \mathbb{R}^N$.

Lemma 3.3 *Let C be a pre-conditioner for K satisfying the spectral equivalence relations (2.3). Let $\underline{u} = -C_{22}^{-1} C_{12}^T \underline{g}$. Then, the extension $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix} \leftrightarrow u$ satisfies*

$$\parallel u \parallel_{H^1(\Omega)}^2 \leq \frac{c_2}{c_1} c_\Omega^2 c_H^2 \parallel g \parallel_{H^{0.5}(\partial\Omega)}^2 . \tag{3.11}$$

Proof: Use relations (3.10), (3.8), Theorem 2.5 and Poincare-Friedrichs inequality. □
The case of the extension operator $-C_{22}^{-1} K_{21}$, where C_{22} is a pre-conditioner for K_{22}, is investigated in [19]. Then, it is not possible to prove $\left(K\underline{u}, \underline{u}\right) \leq c \left(S\underline{g}, \underline{g}\right)$, where the constant c depends only on the condition number of $C_{22}^{-1} K_{22}$. In [19], an additional assumption concerning the preconditioned Schur-complement is required. The reason is that a pre-conditioner is considered for K_{22}, i.e. the corresponding Dirichlet problem only. Here, we consider a pre-conditioner for K, i.e. the corresponding Neumann problem.
In order to define an extension operator, we use a technique which is similar to the multi-level decomposition technique derived in [20]. In the space $\mathbb{Y} = \operatorname{span} \mathbf{X}$, a stable basis is used.

Assumption 3.4 *Let us assume that there exists a basis* $\Psi = \{\psi_j\}_{j=1}^n$ *in the space* $\mathbb{Y}$ *which is stable in* $L_2(\omega)$ *and* $H^1(\omega)$, *i.e. for any function* $g = \sum_{j=1}^n \tilde{g}_j \psi_j(y)$ *the norm equivalence relations*

$$c_{1,1} \sum_{j=1}^n d_j^{(1)} \tilde{g}_j^2 \le \quad \| g \|^2_{L_2(\omega)} \le \quad c_{2,1} \sum_{j=1}^n d_j^{(1)} \tilde{g}_j^2, \tag{3.12}$$

$$c_{1,2} \sum_{j=1}^n d_j^{(2)} \tilde{g}_j^2 \le \quad \| g \|^2_{H^1(\omega)} \le \quad c_{2,2} \sum_{j=1}^n d_j^{(2)} \tilde{g}_j^2 \tag{3.13}$$

hold with some numbers $d_j^{(i)} > 0$, $j = 1, \dots, n$, $i = 1, 2$. *The constants* $c_{i,j}$, $i, j = 1, 2$ *do not depend on the discretization parameter* n.

The computation of the nearly discrete harmonic extension of the function $g(y) = \sum_{j=1}^n g_j \chi_j(y)$ consists of three steps:

Algorithm 3.5 1. *Transform the function* g *into the basis* $\Psi = \{\psi_j\}_{j=1}^n$, *i.e.*

$$g(y) = \sum_{j=1}^n \tilde{g}_j \psi_j(y).$$

2. *For* $j = 1, \dots, n$, *let* u_j^* *be the solution of the problem*

$$\begin{aligned} \| u_j \|_{H_1(\Omega)} \quad &\to \quad \min_{f_i}, \quad \textit{where} \\ u_j(x,y) \quad &= \quad \psi_j(y) \left(\xi_1(x) + \sum_{i=2}^m \xi_i(x) f_i \right). \end{aligned}$$

Put

$$u(x,y) = \sum_{j=1}^n \tilde{g}_j u_j^*(x,y). \tag{3.14}$$

3. *Transform* $u(x,y) = \sum_{j=1}^n \tilde{g}_j u_j(x,y)$ *into the basis* $\{\phi_{ij}(x,y)\}_{i,j}$.

We introduce the following real numbers, vectors and matrices: Let

$$\begin{aligned} \alpha_1 = \langle \xi_1, \xi_1 \rangle_I, \qquad & \underline{a}_1 = [\langle \xi_1, \xi_j \rangle_I]_{j=2}^m, \qquad A_{1,0} = [\langle \xi_i, \xi_j \rangle_I]_{i,j=2}^m, \\ \alpha_2 = \langle D\xi_1, D\xi_1 \rangle_I, \qquad & \underline{a}_2 = [\langle D\xi_1, D\xi_j \rangle_I]_{j=2}^m, \qquad A_{2,0} = [\langle D\xi_i, D\xi_j \rangle_I]_{i,j=2}^m, \\ B_2 = [\langle \chi_i, \chi_j \rangle_\omega]_{i,j=1}^n, \quad & B_1 = [\langle D\chi_i, D\chi_j \rangle_\omega]_{i,j=1}^n, \\ B_2^\Psi = [\langle \psi_i, \psi_j \rangle_\omega]_{i,j=1}^n, \quad & B_1^\Psi = [\langle D\psi_i, D\psi_j \rangle_\omega]_{i,j=1}^n, \\ A_1 = \begin{bmatrix} \alpha_1 & \underline{a}_1^T \\ \underline{a}_1 & A_{1,0} \end{bmatrix} \quad & \text{and} \qquad A_2 = \begin{bmatrix} \alpha_2 & \underline{a}_2^T \\ \underline{a}_1 & A_{2,0} \end{bmatrix}, \end{aligned} \tag{3.15}$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the $L_2(\Omega)$-scalar product and Du is the gradient of u. Then, in the basis $\{\xi_i(x)\psi_j(y)\}_{i,j}$,

$$K^\Psi = \begin{bmatrix} K_{11}^\Psi & K_{12}^\Psi \\ K_{21}^\Psi & K_{22}^\Psi \end{bmatrix} = \begin{bmatrix} \alpha_1 B_1^\Psi + \alpha_2 B_2^\Psi & \underline{a}_1^T \otimes B_1^\Psi + \underline{a}_2^T \otimes B_2^\Psi \\ \underline{a}_1 \otimes B_1^\Psi + \underline{a}_2 \otimes B_2^\Psi & A_{1,0} \otimes B_1^\Psi + A_{2,0} \otimes B_2^\Psi \end{bmatrix}. \tag{3.16}$$

The subscript Ψ denotes that the basis Ψ instead of $\mathbf{X}$ is used in y-direction. Now, we are able to prove the following theorem.

Theorem 3.6 *Let us assume that Assumptions 3.1 and 3.4 are satisfied. Let u be the extension described in Algorithm 3.5, i.e.* $u = \mathfrak{E} g$. *Then,*

$$\| u \|_{H^1(\Omega)}^2 \leq c_H^2 c_\Omega^2 \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}} \| g \|_{H^{0.5}(\partial\Omega)}^2$$

with the constants in (3.12), (3.13) *and* (3.9).

Proof: Let

$$D_1 = \mathrm{diag}\left[d_j^{(1)}\right]_{j=1}^n \quad \text{and} \quad D_2 = \mathrm{diag}\left[d_j^{(2)}\right]_{j=1}^n$$

with the constants $d_j^{(i)}$ defined in (3.12) and (3.13). Then, for all $\underline{v} \leftrightarrow v(y) = \sum v_j \psi_j(y)$,

$$c_{1,i}\left(D_i \underline{v}, \underline{v}\right) \leq \left(B_i^\Psi \underline{v}, \underline{v}\right) \leq c_{2,i}\left(D_i \underline{v}, \underline{v}\right) \quad \forall \underline{v},\ i = 1, 2 \tag{3.17}$$

follows from (3.12) and (3.13). Hence, the relations (2.10) are satisfied. By Theorem 2.5, one obtains

$$\left(K^\Psi \begin{bmatrix} \underline{g} \\ \underline{\hat{u}} \end{bmatrix}, \begin{bmatrix} \underline{g} \\ \underline{\hat{u}} \end{bmatrix}\right) \leq \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}} \left(S^\Psi \underline{g}, \underline{g}\right). \tag{3.18}$$

Then, using the arguments of Lemma 3.3 (Assumption 3.1), i.e. $\begin{bmatrix} \underline{g} \\ \underline{\hat{u}} \end{bmatrix} \leftrightarrow u$, $\underline{g} \leftrightarrow g$, the assertion follows. □

Hence, an optimal extension can be computed by the technique described in Algorithm 3.5. In order to compute the extension efficiently, the following remarks are useful:

Remark 3.7

- *In 1., one basis change from the basis* $\mathbf{X}$ *to the basis* Ψ *has to be done, in 3., n basis changes from the* Ψ *to the basis* $\mathbf{X}$ *have to be done.*
- *2. means from the algebraic point of view the system solve* $\underline{\hat{u}} = -C_{22} C_{21} \underline{g}$. *This can be done by solving systems with linear combinations of the matrices* $A_{1,0}$ *and* $A_{2,0}$, *cf. Lemma 2.6. The matrices* $A_{1,0}$ *and* $A_{2,0}$ *are the 1D mass and 1D stiffness matrix with respect to the basis* Ψ.

Usually, the extension operator $E \leftrightarrow \mathfrak{E}$ is embedded in a domain decomposition pre-conditioner $\hat{C}$ for K, cf. (1.3),

$$\hat{C}^{-1} = \begin{bmatrix} I & \mathbf{0} \\ E & I \end{bmatrix} \begin{bmatrix} \hat{S}^{-1} & \mathbf{0} \\ \mathbf{0} & C_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & E^T \\ \mathbf{0} & I \end{bmatrix}, \tag{3.19}$$

where $\hat{S}$ is a pre-conditioner for the Schur-complement and C_{22} is a pre-conditioner for K_{22}. The basis Ψ is stable in $H^1(\omega)$ and $L^2(\omega)$. Thus, for all $g(y) = \sum_{j=1}^n \tilde{g}_j \psi_j(y)$, we have

$$\begin{aligned} c_{1,1} \sum_{j=1}^n d_j^{(1)} \tilde{g}_j^2 \leq \quad & \| g \|_{L_2(\omega)}^2 \leq \quad c_{2,1} \sum_{j=1}^n d_j^{(1)} \tilde{g}_j^2, \\ c_{1,2} \sum_{j=1}^n d_j^{(2)} \tilde{g}_j^2 \leq \quad & \| g \|_{H^1(\omega)}^2 \leq \quad c_{2,2} \sum_{j=1}^n d_j^{(2)} \tilde{g}_j^2. \end{aligned}$$

By the K-method of interpolation, cf. [27], it follows that this basis is stable in $H^{0.5}(\omega)$, too:

$$c_S \sqrt{c_{1,1} c_{1,2}} \sum_{j=1}^n \sqrt{d_j^{(1)} d_j^{(2)}} \tilde{g}_j^2 \leq \| g \|_{H^{0.5}(\omega)}^2 \leq C_S \sqrt{c_{2,2} c_{2,1}} \sum_{j=1}^n \sqrt{d_j^{(1)} d_j^{(2)}} \tilde{g}_j^2, \tag{3.20}$$

where c_S and C_S are some constants independent of the parameter n. Hence, in the basis Ψ, we can apply the diagonal matrix $(D_1 D_2)^{0.5}$ as pre-conditioner for the Schur-complement S^Ψ. Let W be the basis transformation matrix between the bases Ψ and $\mathbf{X}$, i.e. $\Psi = \mathbf{X}W$. Then, we introduce $\hat{S}^{-1} = W(D_1 D_2)^{-0.5} W^T$ as pre-conditioner for S.
For C_{22}^{-1}, we use the pre-conditioner

$$C_{22}^{-1} = (I \otimes W)(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2)^{-1}(I^T \otimes W^T) \tag{3.21}$$

and the extension operator is of the form

$$E = -C_{22}^{-1} C_{21} = -(I \otimes W)(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2)^{-1}(a_1 \otimes D_1 + a_2 \otimes D_2)W^{-1}. \tag{3.22}$$

Inserting this into (3.19), a simple computation gives the final pre-conditioner

$$\hat{C}^{-1} = \begin{bmatrix} W & \mathbf{0} \\ \mathbf{0} & (I \otimes W) \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ T & I \end{bmatrix} \begin{bmatrix} (D_1 D_2)^{-0.5} & \mathbf{0} \\ \mathbf{0} & D \end{bmatrix} \begin{bmatrix} I & T^T \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} W^T & \mathbf{0} \\ \mathbf{0} & (I \otimes W^T) \end{bmatrix}$$

with $D = (A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2)^{-1}$ and $T = -(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2)^{-1}(a_1 \otimes D_1 + a_2 \otimes D_2)$. Thus, if this pre-conditioner is used, the basis change $\mathbf{X} \to \Psi$ is not necessary.

Lemma 3.8 *The condition number estimate*

$$c_1 \left(\hat{C}\underline{v}, \underline{v}\right) \leq (K\underline{v}, \underline{v}) \leq c_2 \left(\hat{C}\underline{v}, \underline{v}\right) \quad \forall \underline{v}$$

is valid with $c_1 = \frac{1}{2} \frac{\min\{c_{1,1}, c_{1,2}\}}{\max\{c_{2,1}, c_{2,2}\}} \cdot \min\left\{c_{1,1}, c_{1,2}, c_T^2 c_S \sqrt{c_{1,1}, c_{1,2}}\right\}$ *and*
$c_2 = 2 \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}} \cdot \max\left\{c_{2,1}, c_{2,2}, c_H^2 C_S \sqrt{c_{2,1}, c_{2,2}}\right\}$.

Proof: We apply Lemma 1.1. By Theorem 3.6, the extension operator satisfies the estimate

$$\left(K \begin{bmatrix} I \\ E \end{bmatrix} \underline{g}, \begin{bmatrix} I \\ E \end{bmatrix} \underline{g}\right) \leq c_E^2 \left(S\underline{g}, \underline{g}\right) \quad \forall \underline{g} \tag{3.23}$$

with $c_E^2 = 2 \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}}$. Due to (3.17), cf. Proposition 2.4, the matrix $A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2$ is a pre-conditioner for $A_{1,0} \otimes B_1^\Psi + A_{2,0} \otimes B_2^\Psi$ with

$$\begin{aligned} \min\{c_{1,1}, c_{1,2}\} \left(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2 \underline{v}, \underline{v}\right) &\leq \left(A_{1,0} \otimes B_1^\Psi + A_{2,0} \otimes B_2^\Psi \underline{v}, \underline{v}\right) \\ \left(A_{1,0} \otimes B_1^\Psi + A_{2,0} \otimes B_2^\Psi \underline{v}, \underline{v}\right) &\leq \max\{c_{2,1}, c_{2,2}\} \left(A_{1,0} \otimes D_1 + A_{2,0} \otimes D_2 \underline{v}, \underline{v}\right) \quad \forall \underline{v}. \end{aligned}$$

By $B_i^\Psi = W^{-T} B_i W^{-1}$, $i = 1, 2$, and (3.21), one concludes

$$\min\{c_{1,1}, c_{1,2}\} \left(C_{22}\underline{v}, \underline{v}\right) \leq \left(K_{22}\underline{v}, \underline{v}\right) \leq \max\{c_{2,1}, c_{2,2}\} \left(C_{22}\underline{v}, \underline{v}\right) \quad \forall \underline{v}. \tag{3.24}$$

By (3.20), and Lemma 3.2, for $g(y) = \sum_{j=1}^n \tilde{g}_j \psi_j(y)$ we have

$$\left(S^\Psi \underline{g}, \underline{g}\right) \leq c_H^2 \parallel g \parallel_{H^{0.5}(\omega)}^2 \leq c_H^2 C_S \sqrt{c_{2,1} c_{2,2}} \sum_{j=1}^n \sqrt{d_j^{(1)} d_j^{(2)}} \tilde{g}_j^2 = c_H^2 C_S \sqrt{c_{2,1} c_{2,2}} \left((D_1 D_2)^{0.5} \underline{g}, \underline{g}\right).$$

The estimate $c_T^2 c_S \sqrt{c_{1,1} c_{1,2}} \left((D_1 D_2)^{0.5} \underline{g}, \underline{g}\right) \leq \left(S^\Psi \underline{g}, \underline{g}\right)$ can be proved in the same way. Transforming back into the basis $\mathbf{X}$, one obtains

$$c_T^2 c_S \sqrt{c_{1,1} c_{1,2}} \left(\hat{S}\underline{v}, \underline{v}\right) \leq \left(S\underline{v}, \underline{v}\right) \leq c_H^2 C_S \sqrt{c_{2,1} c_{2,2}} \left(\hat{S}\underline{v}, \underline{v}\right) \quad \forall \underline{v}. \tag{3.25}$$

Combining (3.23), (3.24), and (3.25) proves the assertion. □

3.2 The general case

In the last subsection, only an extension of a function living on one edge (in 2D) or face (in 3D) of the domain Ω has been investigated. This subsection deals with the general case of an extension from $\partial\Omega$ into Ω. Usually, the boundary of Ω is splitted into several faces (edges in 2D) ω_i, $i = 1, \dots, r$, $\overline{\cup_{i=1}^r \omega_i} = \partial\Omega$, $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$. On each ω_i, a function g_i with $g_i(\partial\omega_i) = 0$ is given. Then, we consider the problem: Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} \| u \|^2_{H^1(\Omega)} &\leq c \| g \|^2_{H^{0.5}(\partial\Omega)}, \\ u |_{\omega_i} &= g_i = g |_{\omega_i}, \quad 1, \dots, r. \end{aligned} \tag{3.26}$$

This problem can be solved by the method described in the previous subsection (Algorithm 3.5). Let u_i be the extension of

$$u_i |_{\omega_j} = g_i \delta_{ij} \quad \text{on } \omega_j \quad \text{and} \quad \| u_i \|^2_{H^1(\Omega)} \leq c_i \| g_i \|^2_{H^{0.5}(\omega_i)} . \tag{3.27}$$

Then, we define $u = \sum_{i=1}^r u_i$ as the extension of the function g into Ω. Obviously, $u |_{\omega_j} = \sum_{i=1}^r u_i |_{\omega_j} = \sum_{i=1}^r g_i \delta_{ij} = g_j$, i.e. condition (3.26) is fulfilled. Moreover, by the Cauchy-Schwarz inequality and (3.27), we obtain

$$\begin{aligned} \| u \|^2_{H^1(\Omega)} = \| \sum_{j=1}^r u_j \|^2_{H^1(\Omega)} &\leq r \sum_{j=1}^r \| u_j \|^2_{H^1(\Omega)} \\ &\leq r \sum_{j=1}^r c_j \| g_j \|^2_{H^{0.5}(\omega_j)} \leq r C(n) \max_{j=1,\dots,r} \{c_j\} \sum_{j=1}^r \| g_j \|^2_{H^{0.5}_{00}(\omega_j)} \\ &\leq C(n) r \max_{j=1,\dots,r} \{c_j\} \| g \|^2_{H^{0.5}(\partial\Omega)} . \end{aligned}$$

In the last estimate, a relation between the norms $\| \cdot \|^2_{H^{0.5}(\omega_j)}$ and $\| \cdot \|^2_{H^{0.5}_{00}(\omega_j)}$ is required. In the case of the p-version and h-version of the FEM in two dimensions , we have $C(n) \sim (1 + \log n)$, i.e. there is a logarithmic dependence on the discretization parameter. In the second norm, the coupling between two neighbouring faces or edges ω_i and ω_j has no influence in the norm, whereas an additional connectivity term is involved in the definition of the first norm.
So, with an optimal extension for problem (3.27), cf. Algorithm 3.5, an quasioptimal extension for (3.26) can be computed.

4 Examples

In this section, several examples for extension operators of the type proposed in Algorithm 3.5 are given. So, Assumption 3.4 has to be verified, i.e. a polynomial basis Ψ has to be derived which is stable in $H^1(I)$ and $L_2(I)$. Using methods of multi-resolution analysis, [16], [15], [9], [35], such a basis Ψ can be constructed in the case of h- and p-version of the FEM. We start with the p-version of the FEM.

4.1 Extension operators for the *p*-version of the FEM

In this subsection, we describe a method deriving an extension operator $\mathfrak{E}$, i.e. $\underline{u} = E\underline{g}$ in matrix representation, in the case of the p-version of the FEM. Usually, such an extension operator is embedded in a Domain Decomposition pre-conditioner of Dirichlet-Dirichlet-type, cf. section 1, or

[6]. The operator $\underline{u} = E\underline{g}$ will require $\mathcal{O}(p^2)$ arithmetical operations in 2D and $\mathcal{O}(p^3)$ arithmetical operations in 3D, i.e. it is arithmetically optimal. Moreover the extension satisfies the estimate

$$(K\underline{x}, \underline{x}) = \left(K \begin{bmatrix} I \\ E \end{bmatrix} \underline{g}, \begin{bmatrix} I \\ E \end{bmatrix} \underline{g} \right) \le c \left(S\underline{g}, \underline{g} \right) \quad \forall \underline{g} \in \mathbb{R}^m,$$

where c is independent of the polynomial degree p. A system solve $K_{22}\underline{u} = -K_{21}\underline{g}$ is not required. In sub-subsection 4.1.1 a method in order to construct stable polynomial bases in the spaces $H^1(I)$ and $L_2(I)$ is shown. In sub-subsections 4.1.2 and 4.1.3, we derive the extension operators for the 2D and 3D case.

4.1.1 A stable polynomial basis of $\mathbb{P}_{p,00}$ in $H^1(I)$ and $L_2(I)$

In this sub-subsection, we propose a basis in $\mathbb{P}_{p,00}$, which is stable in the spaces $L_2(I)$ and $H^1(I)$. For $i \in \mathbb{N}$, let $\ell_i(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i (x^2-1)^i$ be the i-th Legendre polynomial and let

$$\hat{\ell}_i(x) = \frac{\sqrt{(2i-3)(2i-1)(2i+1)}}{2} \int_{-1}^{x} \ell_{i-1}(s)\,\mathrm{d}s \tag{4.1}$$

be the i-th integrated Legendre polynomial ($i \ge 2$). Note that $\hat{\ell}_i(\pm 1) = 0$. Thus, $\{\hat{\ell}_i\}_{i=2}^p$ is a basis in the space $\mathbb{P}_{p,00}$. Moreover, we introduce the 1D mass and 1D stiffness matrix with respect to the basis $\{\hat{\ell}_i\}_{i=2}^p$, i.e.

$$L_{2,I} = \left[\langle \hat{\ell}_j, \hat{\ell}_i \rangle_I \right]_{i,j=2}^p \quad \text{and} \quad H_{1,I} = \left[\langle D\hat{\ell}_j, D\hat{\ell}_i \rangle_I \right]_{i,j=2}^p . \tag{4.2}$$

In [8], we have proved that

$$\begin{aligned} L_{2,I} &= \begin{bmatrix} 1 & 0 & -\gamma_2 & 0 & & \\ 0 & 1 & 0 & -\gamma_3 & & \\ -\gamma_2 & 0 & 1 & 0 & -\gamma_4 & \\ \vdots & & & \ddots & & \\ 0 & \dots & 0 & -\gamma_{p-2} & 0 & 1 \end{bmatrix}, \\ &\sim \begin{bmatrix} 2 & 0 & -1 & 0 & & \\ 0 & 2 & 0 & -1 & & \\ -1 & 0 & 2 & 0 & -1 & \\ \vdots & & & \ddots & & \\ 0 & \dots & & -1 & 0 & 2 \end{bmatrix}. \end{aligned} \tag{4.3}$$

Moreover, an easy calculation shows, [25],

$$H_{1,I} = \operatorname{diag}\left[\frac{(2j-3)(2j+1)}{2} \right]_{j=2}^p . \tag{4.4}$$

Here and in the following, the relation $A \sim B$ means that $c_1 (A\underline{v}, \underline{v}) \le (B\underline{v}, \underline{v}) \le c_2 (A\underline{v}, \underline{v}) \quad \forall \underline{v}$, where the constants c_1 and c_2 do not depend on the polynomial degree p. Using a permutation P of rows and columns, one obtains

$$L_{2,I} \sim P^T \begin{bmatrix} L_2 & \\ & L_2 \end{bmatrix} P \quad \text{and} \quad H_{1,I} = P^T \begin{bmatrix} H_1 & \\ & H_1 \end{bmatrix} P, \tag{4.5}$$

where L_2 is a tridiagonal matrix with 2's on the main diagonal and -1's on the first sub-diagonal. Obviously, $\{\hat{\ell}_i\}_{i=2}^p$ is stable in $H_1(I)$. However, we are not able to prove that $\{\hat{\ell}_i\}_{i=2}^p$ is stable in $L_2(I)$. In order to find a stable basis, we use interpretations of the matrices L_2 and H_1 as discretization matrices on the unit interval $(0,1)$ using piecewise linear ansatz functions.
Let $k \in \mathbb{N}$, $n = 2^k$ and $p = 2n-1$. Moreover, let $\tau_i = \left[\frac{i}{n}, \frac{i+1}{n}\right]$, $i = 0, \ldots, n-1$, be an equidistant mesh on the interval $[0,1]$ and $\Theta = \{\theta_i\}_{i=1}^{n-1}$ be the basis of the usual hat functions

$$\theta_i(x) = \begin{cases} nx - (i-1) & \text{on} \quad \tau_{i-1} \\ (i+1) - nx & \text{on} \quad \tau_i \\ 0 & \text{else} \end{cases}, \quad i = 1, \ldots, n-1. \tag{4.6}$$

We introduce the matrices

$$M^\Theta = \left[\int_{-1}^1 \theta_i(x)\theta_j(x)x^2 \,\mathrm{d}x\right]_{i,j=1}^{n-1} \quad \text{and} \quad T^\Theta = \left[\int_{-1}^1 \theta_i'(x)\theta_j'(x) \,\mathrm{d}x\right]_{i,j=1}^{n-1}.$$

In [5], we have proved the following result.

Lemma 4.1 *The spectral equivalence relations $H_1 \sim n^3 M^\Theta$ and $L_2 = \frac{1}{n} T^\Theta$ are valid.*

Concerning the weighted mass matrix M^Θ and the unweighted stiffness matrix T^Θ, it is known from the wavelet theory, [9], that it can be constructed a multi-level basis in which M^Θ and T^Θ are spectrally equivalent to diagonal matrices. Let $J = (j,l)$, where $0 \le l \le k$ and $j = 1, 3, 5, \ldots, 2^l - 1$, $\mathrm{card}(J) = n-1$. The diagonal matrices

$$D_{1,0} = \mathrm{diag}\left[2^{2l}\right]_{J=(j,l)} \quad \text{and} \quad D_{2,0} = \mathrm{diag}\left[j^2 2^{-2l}\right]_{J=(j,l)} \tag{4.7}$$

are introduced.

Theorem 4.2 *There are multi-level bases $\tilde{\Psi} = \{\tilde{\psi}_J\}_J$ with $\tilde{\Psi} = \Theta Q$ such that the matrices $T^\Psi = Q^T T^\Theta Q$ and $M^\Psi = Q^T M^\Theta Q$ satisfy the spectral equivalence relations $T^\Psi \sim D_{1,0}$ and $M^\Psi \sim D_{2,0}$. The transformation $\underline{v} = Q^T \underline{w}$ can be performed in $\mathcal{O}(n)$ operations.*

Examples for wavelets bases are given in [9].
Now, let

$$W = P^T \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}, \tag{4.8}$$

where P denotes the permutation matrix in (4.5) and Q denotes the fast wavelet transform in Theorem 4.2.

Theorem 4.3 *The matrices $L_{2,I}$ and $H_{1,I}$ satisfy the relations*

$$L_{2,I} \sim \frac{1}{n} W^{-T} \begin{bmatrix} D_{1,0} & \mathbf{0} \\ \mathbf{0} & D_{1,0} \end{bmatrix} W^{-1} \quad \textit{and} \quad H_{1,I} \sim n^3 W^{-T} \begin{bmatrix} D_{2,0} & \mathbf{0} \\ \mathbf{0} & D_{2,0} \end{bmatrix} W^{-1}. \tag{4.9}$$

Proof: Using (4.5), Lemma 4.1 and Theorem 4.2, we have

$$\begin{aligned} L_{2,I} \sim P^T \begin{bmatrix} L_2 & \mathbf{0} \\ \mathbf{0} & L_2 \end{bmatrix} P &= \frac{P^T}{n} \begin{bmatrix} T^\Theta & \mathbf{0} \\ \mathbf{0} & T^\Theta \end{bmatrix} P \\ &= \frac{1}{n} P^T \begin{bmatrix} Q^{-T} T^\Psi Q^{-1} & \mathbf{0} \\ \mathbf{0} & Q^{-T} T^\Psi Q^{-1} \end{bmatrix} P \\ &\sim \frac{P^T}{n} \begin{bmatrix} Q^{-T} & \mathbf{0} \\ \mathbf{0} & Q^{-T} \end{bmatrix} \begin{bmatrix} D_{1,0} & \mathbf{0} \\ \mathbf{0} & D_{1,0} \end{bmatrix} \begin{bmatrix} Q^{-1} & \mathbf{0} \\ \mathbf{0} & Q^{-1} \end{bmatrix} P. \end{aligned}$$

Since P is a orthogonal matrix, the first assertion follows from (4.8). The second assertion can be proved in the same way. □
Via the matrix W (4.8), the basis

$$\Psi = [\psi_1(x), \ldots, \psi_{p-1}(x)] = \left[\hat{\ell}_2(x), \ldots, \hat{\ell}_p(x)\right] W \tag{4.10}$$

is introduced.

Corollary 4.4 *The basis Ψ is stable in $L_2(I)$ and $H^1(I)$. Moreover, the operation $\underline{v} = W\underline{w}$ can be performed in $\mathcal{O}(p)$ arithmetical operations.*

Thus, we have found an algorithm which transforms a function $g(x) = \sum_{j=2}^{p} g_j \hat{\ell}_j(x) \in \mathbb{P}_{p,00}$ from the basis of the integrated Legendre polynomials into the basis $\Psi \subset \mathbb{P}_{p,00}$ which is stable in $L_2(I)$ and $H^1(I)$ and requires $\mathcal{O}(p)$ operations.

4.1.2 The 2D case

We consider the discretization (3.5) of problem (3.1)-(3.3) in the case of the p-version of the FEM with $\omega = (-1,1)$, $\Omega = (-1,1)^2$. We use the integrated Legendre polynomials (4.1) for the discretization in $x-$ and $y-$ direction, i.e.

$$\begin{aligned} \xi_j(x) &= \hat{\ell}_j(x), \quad j = 2, \ldots, p, \quad \xi_1(x) = \frac{1-x}{2} \quad \text{and} \\ \chi_j(y) &= \hat{\ell}_{j+1}(y), \quad j = 1, \ldots, p-1. \end{aligned}$$

Using the notation (3.15), an easy calculation shows, cf. [25],

$$\begin{aligned} \underline{a}_1 = -\left[\sqrt{\frac{5}{12}}, \sqrt{\frac{7}{60}}, 0 \ldots, 0\right]^T, \quad \alpha_1 = \tfrac{2}{3}, \quad A_{1,0} = B_2 = L_{2,I}, \\ \underline{a}_2 = -[0 \ldots, 0]^T, \quad \alpha_2 = \tfrac{1}{2}, \quad A_{2,0} = B_1 = H_{1,I}. \end{aligned} \tag{4.11}$$

We use Algorithm 3.5 in order to compute a nearly optimal discrete harmonic extension. Since Corollary 4.4, the basis Ψ (4.10) is stable in $L_2(I)$ and $H_1(I)$. Thus, the assumptions of Theorem 3.6 are satisfied.
Written in vectors of the Euklidian space and matrices, the following operations have to be done in the basis $\{\hat{\ell}_i(x)\hat{\ell}_j(y)\}_{i=1,j=2}^{p}$. Let

$$D_1 = \frac{1}{n}\begin{bmatrix} D_{1,0} & \mathbf{0} \\ \mathbf{0} & D_{1,0} \end{bmatrix} \quad \text{and} \quad D_2 = n^3 \begin{bmatrix} D_{2,0} & \mathbf{0} \\ \mathbf{0} & D_{2,0} \end{bmatrix}. \tag{4.12}$$

Using Theorem 4.3, we have $B_i \sim W^{-T} D_i W^{-1}$, $i = 1, 2$. Thus, we set

$$C = A_1 \otimes W^{-T} D_1 W^{-1} + A_2 \otimes W^{-T} D_2 W^{-1}.$$

Inserting relations (4.11), $C_{22}^{-1} = (I \otimes W)(L_{2,I} \otimes D_1 + H_{1,I} \otimes D_2)(I \otimes W^T)$ and $C_{21} = (I \otimes W^{-T})(\underline{a_1} \otimes D_1)W^{-1}$, one concludes

$$\underline{u} = -C_{22}^{-1} C_{21} \underline{g} = -(I \otimes W)(L_{2,I} \otimes D_1 + H_{1,I} \otimes D_2)^{-1}(\underline{a_1} \otimes D_1) W^{-1} \underline{g}. \tag{4.13}$$

Theorem 4.5 *Let $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix} \leftrightarrow u$ be defined via (4.13). Then,*

$$\| u \|_{H^1((-1,1)^2)} \leq c_H c_E \| g \|_{H^{0.5}(\{-1\}\times(-1,1))} \tag{4.14}$$

with $c_E^2 \leq \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}}$. The computation of (4.13) requires $\mathcal{O}(p^2)$ operations.

Proof: We use Theorem 3.6. In [3], see also [4], it has been proved that for all $\hat{g} \in \mathbb{P}_{p,00}$ there exists a $\hat{u} \in \mathbb{Q}_p$ with

$$\| \hat{u} \|_{H^1(\Omega)} \leq c_H \| \hat{g} \|_{H^{0.5}(\partial\Omega)}$$

under the constraints $\hat{u}(-1,y) = \hat{g}(y)$, $\hat{u}(1,y) = 0$ and $\hat{u}(x,\pm 1) = 0$. The constant does not depend on the polynomial degree. Thus, the first assumption of Theorem 3.6, Assumption 3.1, is satisfied. The second assumption follows from Corollary 4.4. This proves the first assertion.
Since the multiplication $\underline{w} = Q\underline{v}$ requires $\mathcal{O}(p)$ operations, the cost of $\underline{w} = (I \otimes W)\underline{v}$ is of order p^2. Due to (4.3) and (4.4), the matrix $\beta L_{2,I} + \gamma H_{1,I}$ is a block diagonal matrix of two tridiagonal matrices after reordering the unknowns, $\beta, \gamma \in \mathbb{R}, \beta, \gamma > 0$. Using Cholesky decomposition, the cost of $(\beta L_{2,I} + \gamma H_{1,I})^{-1}\underline{v}$ is $\mathcal{O}(p)$. For the operation $(L_{2,I} \otimes D_1 + H_{1,I} \otimes D_2)^{-1}(\underline{a_1} \otimes D_1)\underline{g}$, where D_1 and D_2 are diagonal matrices, $p-1$ systems with a linear combination of the matrices $L_{2,I}$ and $H_{1,I}$ have to be solved, cf. Lemma 2.6. So, the total cost is of order p^2. For the operation $W^{-1}\underline{g}$, an inverse wavelet transform is required, where $W \in \mathbb{R}^{p-1 \times p-1}$. □

4.1.3 The 3D case

This case is similar to the 2D case. We consider the discretization (3.5) of problem (3.1)-(3.3) in the case of the p-version of the FEM with $\omega = (-1,1)^2$, $\Omega = (-1,1)^3$. We use the integrated Legendre polynomials (4.1) for the discretization in $x-$ and tensor products of integrated Legendre polynomials for the discretization in ω, i.e. in $y-$ direction. More precisely, let

$$\begin{aligned} \xi_j(x) &= \hat{\ell}_j(x), \quad j = 2,\ldots,p, \quad \xi_1(x) = \frac{1-x}{2} \quad \text{and} \\ \chi_j(y) &= \hat{\ell}_k(y_1)\hat{\ell}_l(y_2), \quad k,l = 2,\ldots,p, j = (k-2)(p-1)+l-1, \quad y = (y_1,y_2). \end{aligned}$$

We use the Algorithm 3.5 in order to compute a nearly optimal discrete harmonic extension. Since Corollary 4.4, the basis Ψ (4.10) is stable in $L_2(I)$ and $H^1(I)$. Thus, the tensor product basis $\Psi \times \Psi = \{\psi_i(x)\psi(j(y)\}_{i,j=1}^{p-1}$ is stable in $L_2(I \times I) = L_2(I) \times L_2(I)$ and $H^1(I \times I) = H^1(I) \times L_2(I) + L_2(I) \times H^1(I)$. Thus, the assumptions of Theorem 3.6 are satisfied and the discrete harmonic extension can be computed. As in sub-subsection 4.1.2, a similar calculation shows

$$\underline{u} = -C_{22}^{-1}C_{21}\underline{g} = -(I \otimes W \otimes W)(L_{2,I} \otimes D_3 + H_{1,I} \otimes D_4)^{-1}(\underline{a_1} \otimes D_3)(W^{-1} \otimes W^{-1})\underline{g}, \quad (4.15)$$

where $D_3 = D_1 \otimes D_2 + D_2 \otimes D_1$ and $D_4 = D_1 \otimes D_1$, cf. (4.3), (4.4), (4.12) and (4.8). We summarize the results in the following theorem.

Theorem 4.6 *Let* $\underline{x} = \begin{bmatrix} \underline{g} \\ \underline{u} \end{bmatrix} \leftrightarrow u$ *be defined via* (4.15). *Then,*

$$\| u \|_{H^1((-1,1)^3)} \leq c \| g \|_{H^{0.5}(-1 \times (-1,1)^2)} .$$

The constant c is independent of the polynomial degree and the computation of $\underline{u}$ requires $\mathcal{O}(p^3)$ operations.

Proof: The proof is similar to the proof of Theorem 4.5. The existence of an optimal harmonic extension has to be ensured, cf. [33] for the tetrahedral case and [27]. So, Assumption 3.1 is satisfied. □

4.2 h-version of the FEM in 2D and 3D

We consider the discretization (3.5) of problem (3.1)-(3.3) in the case of the h-version of the FEM with $\omega = (-1, 1)$, $\Omega = (-1, 1)^2$. Let $k \in \mathbb{N}$, $n = 2^{k+1} - 1$, $s = 2^k$.
Moreover, let $\tau_i = \left[\frac{i}{s}, \frac{i+1}{s}\right]$, $i = -s, \dots, s-1$, be an equidistant mesh on the interval I and $\Theta = \{\theta_i\}_{i=-s+1}^{s-1}$ be the basis of the usual hat functions (4.6). Then, we choose $\chi_j(y) = \theta_{j+s}(y)$ for $j = -s+1, \dots, s-1$. Moreover, let $\tilde{\tau}_i$, $i = 1, \dots, m$ be an arbitrary 1D finite element mesh of $(-1, 1)$. Then, Ξ is the basis of the usual hat functions according to the mesh $\tilde{\tau}_i$. We apply Theorem 3.6.
It is known from the theory of multi-resolution bases, [16], [15], [35], that several wavelet bases on the interval are stable in H^1 and L_2. Moreover, is has been shown in [20] that it exists a piecewise linear harmonic extension u of the piecewise linear function $g = \sum_{j=-s+1}^{s-1} g_j \theta_j$ into the interior of the domain Ω. Hence, Assumption 3.1 is satisfied.
So, the assumptions of Theorem 3.6 are satisfied and the optimal extension can be computed via Algorithm 3.5. For the computation of $\underline{u}$, n systems of linear algebraic equations with a linear combination $\mathfrak{M}_j$, $j = 1 \dots, n$, of the one dimensional mass matrix and one dimensional stiffness matrix with respect to the basis Ξ have to be solved, cf. Remark 3.7. Since Ξ is the basis of the usual hat-functions, the system matrices $\mathfrak{M}_j$ are tridiagonal matrices. Thus, one system solve requires $\mathcal{O}(m)$ operations and the total cost is $\mathcal{O}(mn)$.
The approach can be extended to the 3D-case, where ω in a surface in $\mathbb{R}^3$. For details concerning the assumptions to ω, we refer to [22].

5 Numerical examples

In this section, we compute the smallest possible constant c_E in (4.14) for the proposed extension operators in the case of the p-version of the finite element method in two dimensions. In all experiments, the wavelet basis described in [8] is used. Table 1 displays the quality of the extension (4.13). Moreover, the best constants $c_{1,1}$, $c_{1,2}$, $c_{2,1}$ and $c_{2,2}$ in (3.12) and (3.13) are computed. For all p, the constant c_E^2 lies between 1.4 and 1.9 and is bounded by a constant independent

p	3	7	15	31	63	127
c_E^2 in (4.14)	1.45	1.89	1.85	1.79	1.76	1.75
$c_{1,1}$	0.22	0.20	0.18	0.17	0.17	0.16
$c_{1,2}$	0.94	0.37	0.25	0.21	0.19	0.18
$c_{2,1}$	2.23	2.35	2.39	2.41	2.42	2.43
$c_{2,2}$	3.94	3.94	3.42	3.23	3.15	3.11

Table 1: Smallest possible constant of (3.16) for the p-version extension operator.

of p. The result is much better as we can expect from the estimate in Theorem 3.6, where $c_E^2 = \frac{\max\{c_{2,1}, c_{2,2}\}}{\min\{c_{1,1}, c_{1,2}\}} \approx \frac{3}{0.2} = 15$. In this example, we have used explicitely given numbers for the definition of the matrices D_1 and D_2, cf. relation (4.7).
In a second example, we use a better diagonal scaling for the matrices $W^T B_i W$ in (4.12). We replace the diagonal matrices D_1 and D_2 by the matrices $D_i = \text{diag}(W^T B_i W)$, $i = 1, 2$, where $\text{diag}(U)$ denotes the diagonal part of the matrix U. These entries can explicitely be computed in $\mathcal{O}(p)$ operations. Table 2 displays the constants c_E, $c_{i,j}$, $i, j = 1, 2$ if the matrices D_i in (4.12) are replaced by the matrices $\text{diag}(W^T B_i W)$. Now, the constants c_E and $c_{1,1}$, $c_{2,1}$ and $c_{2,2}$ are closer to one. Thus, the second way in order to define the extension is better than the first one.

p	3	7	15	31	63	127
c_E^2 in (4.14)	1.00	1.13	1.26	1.34	1.37	1.39
$c_{1,1}$	0.36	0.36	0.36	0.36	0.36	0.36
$c_{1,2}$	1.00	0.33	0.25	0.21	0.19	0.18
$c_{2,1}$	1.63	1.68	1.77	1.83	1.88	1.92
$c_{2,2}$	1.00	1.96	2.16	2.27	2.35	2.40

Table 2: Smallest possible constant of (4.14) for the p-version extension operator with diagonal scaled matrices.

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