On geometry of Diophantine approximation

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7th International Conference on UDT

Wednesday, 24 February 2021

Diophantine Approximation

I like Diophantine approximation because up to nowadays

- it is still some kind of "romantic" mathematics
- it is often counterintuitive

Diophantine Approximation

It is unusual to find a mathematical theory which is in a state as primitive and naive as the present one, and there is of course some delight in catching it in that state.

(Serge Lang, *Introduction to Diophantine Approximation*, Foreword to the First Edition, 1966)

Diophantine Approximation

- Lang, *Introduction to Diophantine Approximation* (1966)

- A year earlier
Report on Dioph. Appr., Bull. Soc. Math. France 93 (1965)
Lang conjectured that for complex $\omega_1, \dots, \omega_d$ a certain Diophantine condition implies that $\omega_1, \dots, \omega_d$ are algebraically dependent

- Five years later
Transcendental numb. and Dioph. Appr., Bull. AMS 77 (1971)
Lang mentioned that a counterexample to the conjecture above is given in Cassels' book
An introduction to Dioph. Appr., 1954
It is related to *singular vectors* discovered by Khintchine
Über eine Klasse lin. dioph. Appr., Rend. Circ. Palermo, 50 (1926)

Diophantine Approximation

In my lecture I will tell few stories about

- one -

and

- multi-dimensional

results in Diophantine Approximation which may look unnatural,

and discuss some open problems

Five stories

- (Non) primitive approximation;
- two irrationality measure functions;
- function associated with Minkowski "diagonal"continued fraction;
- multidimensional Diophantine approximation: geometry and Diophantine exponents;
- badly approximable numbers

$||x|| = \min_{a\in\mathbb{Z}} |x - a|$ — the distance to the nearest integer

Story No 1. (Non)primitive inhomogeneous approximation to one number

- Khintchine (1935) $\varepsilon > 0$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}, \eta \in \mathbb{R}$

then $\exists$ inf. many $q \in \mathbb{Z}_+, r \in \mathbb{Z}$ such that

$$|q\alpha - \eta - r| = ||q\alpha - \eta|| \leq \frac{1+\varepsilon}{\sqrt{5} \cdot q}$$

- Chalk and Erdős (1959) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\eta \in \mathbb{R}$, there exists an absolute constant C such that

$$|q\alpha - \eta - r| \leq \frac{C}{q} \cdot \left(\frac{\log q}{\log \log q} \right)^2$$

is satisfied by infinitely many coprime integers $(q, r), q \geq 1$.

(Non)primitive inhomogeneous approximation to one number

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- Modern history: M. Laurent, A. Nogueira, A. Haynes ...

- Jitomirskaya, Liu (2019) For any constant C, there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and η such that the inequality

$$q\,|q\alpha - \eta - r| \leq C$$

only has finitely many coprime integer solutions $(q, r), q \geq 1$.

(Non)primitive inhom. appr. to one number: new result

- Jitomirskaya, Liu (2019) For any constant C, there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and η such that the inequality

$$|q\alpha - \eta - r| \leq \frac{C}{q}$$

only has finitely many coprime integer solutions $(q, r), q \geq 1$.

- Theorem (N.M.,2021) *There exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and η such that*

$$\inf_{(q,r)\in\mathbb{Z}^2,\ q>100,\ (q,r)=1} q\,\frac{\log\log q}{\sqrt{\log q}}\,|q\alpha - \eta - r| > 0.$$

(Non)primitive inhom. appr. to one number: backgroung

Erdős (1958)

1) For every $\varepsilon > 0$ there exist arbitrarily large positive integer x and $y \geq x$ such that

$$\mathrm{g.c.d.}(x+i, y+j) > 1$$

for all pairs i, j with

$$0 \leq i, j \leq (1-\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{1/2}.$$

2) For a certain positive constant c for any positive integers $x \leq y$ there exist a pair of integers i, j with

$$0 \leq i, j \leq c\,\frac{\log x}{\log\log x}$$

such that

$$\mathrm{g.c.d.}(x+i, y+j) = 1.$$

(Non)primitive inhom. appr. to one number: a problem

WHAT IS OPTIMAL?

- Chalk and Erdős (1959) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\eta \in \mathbb{R}$, there exists an absolute constant C such that

$$|q\alpha - \eta - r| \leq \frac{C}{q} \cdot \left(\frac{\log q}{\log \log q} \right)^2$$

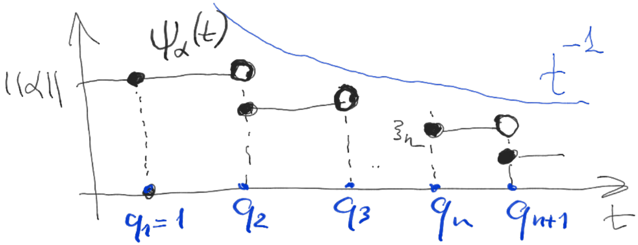
is satisfied by infinitely many coprime integers (q, r), $q \geq 1$.

- (N.M.,2021) There exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and η such that

$$\inf_{(q,r) \in \mathbb{Z}^2,\ q > 100,\ (q,r)=1} q \frac{\log \log q}{\sqrt{\log q}} |q\alpha - \eta - r| > 0.$$

Story No 2. (Ordinary) irrationality measure function

α - real irrational



$$\psi_\alpha(t) = \min_{1 \le q \le t,\, q \in \mathbb{Z}} ||q\alpha||, \quad t \ge 1$$

continued fractions (Lagrange Theorem):

$$\alpha = [a_0; a_1, a_2, \ldots, a_n, \ldots], \quad \frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n], \quad \xi_n = |q_n\alpha - p_n|$$

$$\psi_\alpha(t) = \xi_n \text{ for } q_n \le t < q_{n+1}.$$

for any t we have

$$\psi_\alpha(t) < t^{-1}$$

Irrationality measure functions for two numbers

Theorem 1. (I.D. Kan + N.M., Unif. Distr. Th. 5:2 (2010), 79–86)
Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\alpha \pm \beta \notin \mathbb{Z}$. Then the difference function

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes its sign infinitely many times as $t \to +\infty$

Theorem 2. (N.M, 2019)
Under the conditions of Theorem 1 $\exists t_\nu \to \infty$ such that

$$|\psi_\alpha(t_\nu) - \psi_\beta(t_\nu)| \geq K \cdot \min(\psi_\alpha(t_\nu), \psi_\beta(t_\nu)), \;\; K = \sqrt{\frac{\sqrt{5}+1}{2}} - 1.$$

Moreover the constant K here is optimal.

Corollary. (A. Dubickas, JNT 177 (2017))

$$\limsup_{t \to +\infty} \left| \frac{1}{\psi_\alpha(t)} - \frac{1}{\psi_\beta(t)} \right| = +\infty.$$

Irrationality measure functions for two numbers: a problem

Theorem 1. (I.D. Kan + N.M., Unif. Distr. Th. 5:2 (2010))
Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\alpha \pm \beta \notin \mathbb{Z}$. Then the difference function

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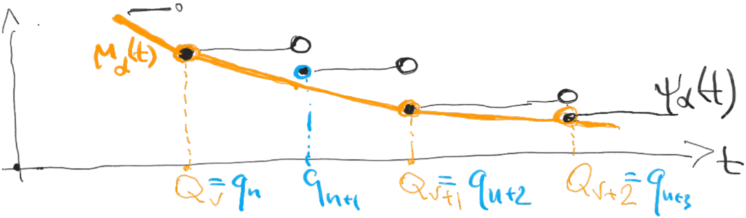
changes its sign infinitely many times as $t \to +\infty$

Is there objects that behave themselves in a similar way?

Story No 3. Minkowski irrationality measure function



Those denominators of convergent for which $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2}$:

$$Q_1, Q_2, ..., Q_n, ...$$

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \le t \le Q_{n+1}$$

Minkowski: $\mu_\alpha(t)$ is convex.

Example. $\mu_{\sqrt{2}}(t) < \mu_{\frac{1+\sqrt{5}}{2}}(t), \quad \forall t > 0$

Classical Diophantine spectra

$$\psi_\alpha(t) = \min_{1 \le q \le t} ||q\alpha||$$

$$\lambda(\alpha) = \liminf_{t\to\infty} t \cdot \psi_\alpha(t), \quad d(\alpha) = \limsup_{t\to\infty} t \cdot \psi_\alpha(t)$$

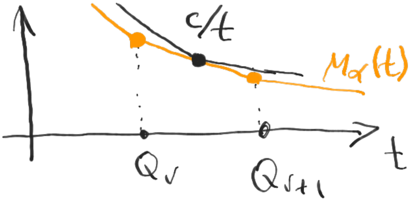
$$\mathbb{L} = \{\lambda : \ \exists\, \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } \lambda = \lambda(\alpha)\} - \text{Lagrange spectrum}$$

$$\mathbb{L} : \quad \underbrace{[0, H]}_{\text{Hall's ray}} \quad \underbrace{\ldots\ldots\ldots\ \frac{1}{3}}_{\text{smth}} \quad \underbrace{\cdots \ \frac{1}{\sqrt{221}/5}\ \frac{1}{\sqrt{8}}\ \frac{1}{\sqrt{5}}}_{\text{Markoff numbers}}$$

$$\mathbb{D} = \{d : \ \exists\, \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } d = d(\alpha)\} - \text{Dirichlet spectrum}$$

$$\mathbb{D} : \quad \underbrace{\frac{1}{2} + \frac{1}{2\sqrt{5}} \ \cdots}_{\text{Lesca numbers}} \quad \underbrace{\ldots\ldots\ldots}_{\text{smth}} \quad \underbrace{[Z, 1]}_{\text{ray}}$$

Spectrum for $\mu_\alpha(t)$



$$\mathfrak{m}(\alpha) = \limsup_{t\to+\infty} t \cdot \mu_\alpha(t).$$

$$\mathbb{M} = \{m \in \mathbb{R} : \ \exists \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ such that } m = \mathfrak{m}(\alpha)\}.$$

Theorem. (N.M., 2011)

$$\min \mathbb{M} = \frac{1}{4}, \quad \max \mathbb{M} = \frac{1}{2}.$$

Problem. Is it true that $\mathbb{M} = \left[\frac{1}{4}, \frac{1}{2}\right]$?

Story No 4. Multidimensional setting, one linear form

$$1, \alpha_1, ..., \alpha_d \ - \ \text{linearly independent over } \mathbb{Q}, \quad d \geq 2$$

$$\psi_{\boldsymbol{\alpha}}(t) = \min_{\mathbf{q}=(q_1,...,q_d)\in\mathbb{Z}^d:\, 0<|\mathbf{q}|\leq t} ||q_1\alpha_1 + ... + q_d\alpha_d||, \ \ |\mathbf{q}| = \max_{1\leq i\leq m} |q_i|.$$

Minkowski convex body theorem: $\psi_{\boldsymbol{\alpha}}(t) < t^{-d}, \ \ \forall t$

- Khintchine 's singular vectors (1926)

$\forall\, \psi(t) \quad \exists$ algebraically independent $\alpha_1, ..., \alpha_d \in \mathbb{R}$ such that

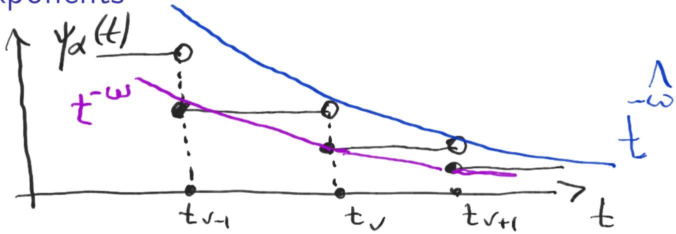
$$0 < \psi_{\boldsymbol{\alpha}}(t) < \psi(t).$$

- Corollary If $\boldsymbol{\alpha}$ is singular and $\boldsymbol{\beta}$ is badly approximable

that is $\exists c > 0$ s.t. $\psi_{\boldsymbol{\beta}}(t) > ct^{-d}, \ \ \forall t$ then

$$0 < \psi_{\boldsymbol{\alpha}}(t) < \psi_{\boldsymbol{\beta}}(t), \quad \text{for all large enough } t$$

Diophantine exponents



The ordinary Diophantine exponent

$$\omega = \omega(\boldsymbol{\alpha}) = \sup\{\gamma : \liminf_{t\to\infty} t^{\gamma}\psi_{\boldsymbol{\alpha}}(t) < \infty\}$$

The uniform Diophantine exponent

$$\hat\omega = \hat\omega(\boldsymbol{\alpha}) = \sup\{\gamma : \limsup_{t\to\infty} t^{\gamma}\psi_{\boldsymbol{\alpha}}(t) < \infty\}$$

Trivial inequalitiy:

$$d \le \hat\omega \le \omega \le +\infty$$

Irrationality measure function and best approximation

$$\psi_{\boldsymbol{\alpha}}(t) = \min_{\mathbf{q}\in\mathbb{Z}^d:\, 0<|\mathbf{q}|\le t} ||q_1\alpha_1 + \dots + q_d\alpha_d||,$$

$\psi_{\boldsymbol{\alpha}}(t)$ is piecewise constant function with a sequence of jumps

$$t_0 = 1 < t_1 < t_2 < \dots < t_\nu < t_{\nu+1} < \dots$$

best approximation vectors

$$\boldsymbol{z}_\nu = (q_{0,\nu}, q_{1,\nu}, \dots, q_{d,\nu}) \in \mathbb{Z}^{d+1}, \quad t_\nu = \max_{1\le i\le m} |q_{i,\nu}|$$

$$L_\nu = ||q_{1,\nu}\alpha_1 + \dots + q_{d,\nu}\alpha_d|| = |q_{0,\nu} + q_{1,\nu}\alpha_1 + \dots + q_{d,\nu}\alpha_d|$$

ordinary Diophantine exponent : $L_\nu \le t_\nu^{-\omega+\varepsilon}$ infinitely often

uniform Diophantine exponent : $L_\nu \le t_{\nu+1}^{-\hat{\omega}+\varepsilon}$ always - $\forall\, t$

Ideal situation

$\boldsymbol{z}_\nu = (q_{0,\nu}, q_{1,\nu}, ..., q_{d,\nu}) \in \mathbb{Z}^{d+1}$, $k \le \nu \le k+d$ are linearly independent

$$1 \le |\det \begin{pmatrix} q_{0,k} & q_{1,k} & \cdots & q_{d,k} \\ q_{0,k+1} & q_{1,k+1} & \cdots & q_{d,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ q_{0,k+d} & q_{1,k+d} & \cdots & q_{d,k+d} \end{pmatrix}| =$$

$$= |\det \begin{pmatrix} \pm L_k & q_{1,k} & \cdots & q_{d,k} \\ \pm L_{k+1} & q_{1,k+1} & \cdots & q_{d,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \pm L_{k+d} & q_{1,k+d} & \cdots & q_{d,k+d} \end{pmatrix}| \ll L_k t_{k+1} t_{k+2} ... t_{k+d}$$

If $L_k \le t_{k+1}^{-\hat{\omega}+\varepsilon}$ and $t_{\nu+1} \le t_\nu^g$ for any $\nu \in \{k+1, ..., k+d-1\}$,
then $1 \ll t_{k+1}^{1-\hat{\omega}+\varepsilon+g+g^2+...+g^{d-1}}$
and $1 - \hat{\omega} + g + g^2 + ... + g^{d-1} \ge 0$

Ideal situation: the upshot

If $\hat{\omega}(\boldsymbol{\alpha}) = \hat{\omega}$ and

$$g + g^2 + \dots + g^{d-1} < \hat{\omega} - 1$$

then

$$\exists j \in \{k+1, \dots, k+d\} : \quad t_{j+1} \geq t_j^g \quad \text{and of course} \quad L_j \leq t_{j+1}^{-\hat{\omega}+\varepsilon}$$

and

$$L_j = ||q_{1,j}\alpha_1 + \dots + q_{d,j}\alpha_j|| \ll t_{j+1}^{-\hat{\omega}+\varepsilon} \ll t_j^{(-\hat{\omega}+\varepsilon)g}, t_j = \max_{1 \leq i \leq d} |q_{i,j}|.$$

So

$$\omega \geq \hat{\omega} \times G_d \tag{1}$$

where $G_G \geq 1$ is the unique root of the polynomial

$$g + g^2 + \dots + g^{d-1} = \hat{\omega} - 1$$

Schmidt-Summerer conjecture: (1) in valid in general situation ($d = 2$ proved by Jarník in 1940th, in general dimension proved recently by Marnat-M.(2018))

Best approximations: degeneracy of dimension

Jarník (1940th) For $d = 2$ there exist infinitely many triples

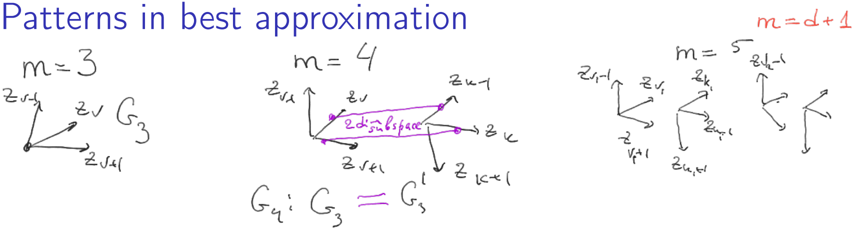
$$\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_{\nu}, \boldsymbol{z}_{\nu+1}$$

of independent consecutive best approximation vectors.

Theorem. (N.M. 1998) $d \geq 3$. *There exist algebraically independent $\alpha_1, ..., \alpha_d$ and a linear subspace $\mathfrak{L} \subset \mathbb{R}^{d+1}$ of dimension 3 such that*

$$\boldsymbol{z}_{\nu} = (q_{0,\nu}, q_{1,\nu}, ..., q_{d,\nu}) \in \mathfrak{L}, \quad \forall \nu$$

Patterns in best approximation



If we know the value of $\hat{\omega}$ and something about local behavior of best approximation vectors, we can improve the trivial bound $\omega \geq \hat{\omega}$

Problem. Given a pattern to determine the local bound for $\omega/\hat{\omega}$ provided by this pattern. (Important i.g. for appr. on Veronrse curve - D. Roy, 2008)

Simultaneous approximation

$1, \alpha_1, ..., \alpha_d$ are linearly independent over $\mathbb{Q}$

$$\psi_{\boldsymbol{\alpha}}(t) = \min_{1 \le q \le t} \max_{1 \le j \le d} ||q\alpha_j|| \le t^{-1/d}$$

$\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, ..., a_{d,\nu})$ - best approximation vectors
Diophantine exponents $\omega \ge \hat{\omega}$, $\frac{1}{d} \le \hat{\omega} \le 1$

$n = 1$: $\det \begin{pmatrix} q_\nu & a_{1,\nu} \\ q_{\nu+1} & a_{1,\nu+1} \end{pmatrix} = \pm 1$ - continued fractions

$n > 1$: $\exists$ infinitely many ν s.t. $\boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}, \boldsymbol{z}_{\nu+2}$ are independent
V. Jarník used this fact to prove inequality

$$\omega \ge \hat{\omega} \cdot \frac{\hat{\omega}}{1 - \hat{\omega}}$$

Degeneracy of dimension, example: $\exists$ $\alpha_1, ..., \alpha_d$ such that

$$M_\nu = \begin{pmatrix} a_\nu & a_{1,\nu} & \dots & a_{d,\nu} \\ \dots & \dots & \dots & \dots \\ q_{\nu+d} & q_{1,\nu+d} & \dots & a_{d,\nu+d} \end{pmatrix}, \quad \mathrm{rk}\, M_\nu \le 3 \ \ \forall \nu$$

Schmidt-Summerer's conjecture

Conjecture. *In the case of simultaneous approximation (as well as in the case of one linear form) the optimal lower bound for* $\frac{\omega}{\hat{\omega}}$ *comes from the case* $\operatorname{rk} M_\nu = d+1$ *for infinitely many* ν, *that is*

$$\omega \geq \hat{\omega} \cdot G(\hat{\omega}),$$

where $G(\hat{\omega}) > 1$ *the root of equation*

$$(1-\hat{\omega})x^d - x^{d-1} + \hat{\omega} = 0$$

Solution: Marnat-M. (2018, both for simultaneous approx. and one linear form)

Another proof: Ngyuen-Poëls-Roy (for simultaneous approx.) - used another "artificial" pattern invented by Ngyuen (2013).

Story No 5. Badly approximable real numbers

Definition. α is called badly approximable if $\inf_{q\in\mathbb{Z}_+} q\,||q\alpha|| > 0$

Proposition. α is a badly approximable number if and only if the partial quotients in continued fraction expansion

$$[a_0; a_1, a_2, ..., a_\nu, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_\nu + ...}}}$$

$$a_0 \in \mathbb{Z}, \quad a_j \in \mathbb{Z}_+, j = 1, 2, 3, ...$$

are bounded, that is $\sup_{\nu\geq 1} a_\nu < \infty$

Badly approximable real numbers: properties

Definition. α is called badly approximable if

$$\inf_{q\in\mathbb{Z}_+} q\,||q\alpha|| > 0 \Longleftrightarrow \sup_{\nu\geq 1} a_\nu < \infty$$

q_ν - denominator of convergent to α , $\xi_\nu = ||q_\nu\alpha||$
From $q_{\nu+1} = a_{\nu+1}q_\nu + q_{\nu-1}$ and $\xi_{\nu+1} = \xi_{\nu-1} - a_{\nu+1}\xi_\nu$ we have

$$a_{\nu+1} = \left[\frac{q_{\nu+1}}{q_\nu}\right] = \left[\frac{\xi_{\nu-1}}{\xi_\nu}\right],$$

Proposition 1. *Irrational number α is badly approximable if and only if*

$$\sup_{\nu\geq 1} \frac{q_{\nu+1}}{q_\nu} < \infty$$

and if and only if

$$\inf_{\nu\geq 1} \frac{\xi_{\nu+1}}{\xi_\nu} > 0.$$

Simultaneous approximation and one Linear form

$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$: $\quad 1, \alpha_1, ..., \alpha_d$ are linearly ind. over $\mathbb{Q}$.

Definition. $\boldsymbol{\alpha}$ is called badly approximable if

$$\inf_{q \in \mathbb{Z}_+} q^{1/d} \max_{1 \le j \le d} ||q\alpha_j|| > 0$$

Perron-Khintchine's transference theorem: **B.A.** $\iff$

$$\inf_{\boldsymbol{a}=(q_1,...,q_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \left(\max_{1 \le j \le n} |q_j| \right)^d ||q_1\alpha_1 + ... + q_d\alpha_d|| > 0.$$

if $\boldsymbol{\theta}$ is badly approximable then

- for simultaneous approximation $\hat{\omega} = \omega = \frac{1}{d}$
- for one linear form $\hat{\omega} = \omega = d$

Simultaneous approximation and Linear form: best approximations

simultaneous approximation:

$\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, \ldots, a_{d,\nu}), \quad \nu = 1, 2, 3, \ldots$

$$\xi_\nu = \max_{1\le j\le n} ||q_\nu \alpha_j|| = \max_{1\le j\le n} |q_\nu\alpha_j - a_{j,\nu}| < \max_{1\le j\le d} ||q\alpha_j||, \quad \forall q < q_\nu$$

$$q_1 < q_2 < \ldots < q_\nu < q_{\nu+1} < \ldots \ , \ \xi_1 > \xi_2 > \ldots > \xi_\nu > \xi_{\nu+1} > \ldots$$

linear form:

$\boldsymbol{q}_\nu = (q_{0,\nu}, q_{1,\nu}, \ldots, q_{d,\nu}),\ \nu = 1, 2, 3, \ldots;\ t_\nu = \max_{1\le j\le d} |q_{j,\nu}|$

$$L_\nu = ||q_{1,\nu}\alpha_1 + \ldots + q_{n,\nu}\alpha_n|| = |q_{0,\nu} + q_{1,\nu}\alpha_1 + \ldots + q_{d,\nu}\alpha_n|$$

$$L_\nu < ||q_1\alpha_1 + \ldots + q_d\alpha_d||, \quad \forall \boldsymbol{q} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\} \text{ with } \max_{1\le j\le n} |q_j| < t_\nu,$$

$$t_1 < t_2 < \ldots < t_\nu < t_{\nu+1} < \ldots \ , \ L_1 > L_2 > \ldots > L_\nu > L_{\nu+1} > \ldots$$

Theorems (N.M.+ Renat Akhunzhanov)

Theorem 1. *Suppose that* $\alpha_1, ..., \alpha_d, 1$ *are linearly independent over* $\mathbb{Q}$. *Then the following three statements are equivalent:*
(i) $\boldsymbol{\alpha}$ *is badly approximable;*
(ii) $\sup_j \frac{q_{j+1}}{q_j} < \infty$;
(iii) $\inf_j \frac{L_{j+1}}{L_j} > 0$.

Theorem 2. *There exists* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ *such that*
- $1, \alpha_1, \alpha_2$ *are linearly independent over* $\mathbb{Q}$;
- $\inf_\nu \frac{\xi_{\nu+1}}{\xi_\nu} > 0$;
- $\boldsymbol{\alpha}$ *is not badly approximable.*

Theorem 3. *There exists* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ *such that*
- $1, \alpha_1, \alpha_2$ *are linearly independent over* $\mathbb{Q}$;
- $\sup_\nu \frac{t_{\nu+1}}{t_\nu} < \infty$;
- $\boldsymbol{\alpha}$ *is not badly approximable.*

A comment

Theorem 2. *There exists* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ *such that*

- $1, \alpha_1, \alpha_2$ *are linearly independent over* $\mathbb{Q}$*;*
- $\inf_\nu \frac{\xi_{\nu+1}}{\xi_\nu} > 0$*;*
- $\boldsymbol{\alpha}$ *is not badly approximable.*

The idea of the construction from the proof of Theorem 2 is quite simple. One should construct a vector $\boldsymbol{\alpha} \in \mathbb{R}^2$ such that the best approximation vectors to it for long times lie in two-dimensional subspaces. Moreover, for the integer approximations from these two-dimensional subspaces we should ensure some kind of "one-dimensional badly approximability".

On Diophantine exponents

Theorem 2 shows that for $d \geq 2$ the condition

$$\inf_{\nu} \frac{\xi_{\nu+1}}{\xi_{\nu}} > 0 \qquad (2)$$

may be satisfied for $\boldsymbol{\alpha}$ which is not badly approximable. Moreover the construction from the proof of Theorem 2 gives $\boldsymbol{\alpha}$ with $\hat{\omega}(\boldsymbol{\alpha}) = \frac{1}{2}$ and $\omega(\boldsymbol{\alpha}) = +\infty$. We would like to give a comment on this, and formulate the following statement.

Proposition. *Suppose that among the numbers $\alpha_1, ..., \alpha_d$ there exist at least two numbers linearly independent together with 1 over $\mathbb{Q}$, and suppose that $\boldsymbol{\alpha}$ satisfies condition (2). Then*

$$\hat{\omega}(\boldsymbol{\alpha}) \leq \frac{1}{2}.$$

Problem. To determine the set of admissible values $\hat{\omega}(\boldsymbol{\alpha})$ here (for $d = 2$ it is just $\{1/2\}$).

THANK YOU!