New reconstructions from cone Radon transform

Victor Palamodov
Tel Aviv University
March 30, 2017
Trajectories of single-scattered photons with fixed income and outcome energies in Compton camera form a cone of rotation:

![Scheme of the Compton camera](attachment:image.png)

Scheme of the Compton camera
A spherical cone in an Euclidean space $E^3$ with apex at the origin can be written in the form

$$C(\lambda) = \{x \in E^3 : \lambda x_1 = s\}, \quad s = \sqrt{x_2^2 + x_3^2}.$$ 

The line $s = 0$ is the axis and $\lambda = \tan \psi$ where $\psi$ is the opening of the cone. In particular $C(\infty) = \{x : x_1 = 0\}$. 

The integral $\int_{C(\lambda)} f(y + x) w(x) \, dx_2 \, dx_3$, $y \in E^3$ is called weighted cone Radon or Compton transform. If $w(x) = jx^k$ we call this integral regular in the case $k = 0, 1$ and singular if $k = 2$. Any regular integral is well defined for any continuous $f$ defined on $E^3$ vanishing for $x_1 > m$ for some $m$. The singular integral is not well defined if $f(y) \neq 0$. Analytic inversion of the regular and singular monochrome (one opening) cone Radon transforms is in the focus of this talk.
A spherical cone in an Euclidean space $E^3$ with apex at the origin can be written in the form

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$$g_C(y) = \cos \psi \int_{x \in C(\lambda)} f(y + x) w(x) \, dx_2 dx_3, \ y \in E^3$$

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Analytic inversion of the regular and singular monochrome (one opening) cone Radon transforms is in the focus of this talk.
The realistic model (SPSF) for single-scattering optical tomography based on the photometric law of scattered radiation modeled by the singular cone transform.

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New reconstructions from cone Radon transform

March 30, 2017
Polychrome reconstructions

(many openings)

- Cree and Bones 1994 proposed reconstruction formulae from data of regular cone transform with apices restricted to a plane orthogonal to the axis.
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- Haltmeier 2014, Terzioglu 2015, Moon 2016, Jung and Moon 2016 gave inversion formulae for arbitrary dimension \( n \).
- Jung and Moon 2016 proposed the scheme for collecting non-redunded data from a line of detectors and rotating axis.
Basko \textit{et al} 1998 proposed a numerical method based on developing $f$ in spherical harmonics from cone integrals with swinging axis.
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Katsevich and Krylov 2013 studied reconstruction of the attenuation coefficient from of broken ray transform with curved lines of detectors.

Gouia-Zarrad and Ambartsoumian 2014 found the reconstruction formula for the regular cone transform in the half-space with free apex.
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Cone transform with free apex

- Cone Radon integral equation can be written in the convolution form

\[ g = |x|^{-k} \delta_{-C} \ast f, \]  

where

\[ \delta_{-C} (\varphi) = \int_{C} \varphi dS = \cos^{-1} \psi \int \int \varphi (-\lambda s, x_2, x_3) \, dx_2 \, dx_3, \]

\[ s = \sqrt{x_2^2 + x_3^2}. \]

is a tempered distribution in \( E^3 \).
Cone transform with free apex

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- The solution \( f \) of (1) defined on \( \{x_1 \geq 0\} \) is unique if it vanishes for \( x_1 > m \) for some \( m > 0 \).
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- We focus on the case \( n = 3 \) and use the notations
  \[ \Delta_0 = \delta_{-C}, \quad \Delta_1 = |x|^{-1} \delta_{-C}. \]
Support of the convolution

For a function $f$ on $E^n$ vanishing for $x_1 > m$ for some $m$, the convolution $g = \Delta_k * f$ is well defined and $\text{supp} \Delta_k * f \subset \text{supp} f - C$. 

\[ \text{supp } f \]
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Inversion of regular transforms

- **Case** $k = 0$. The solution of

\[ \Delta_0 \ast f_0 = g_0, \]

\[ f_0(x) = \frac{1}{2\pi \cos^3 \psi} \Box^2 \Delta_1 \ast \Theta_1 \ast g_0 \]

\[ = \frac{1}{2\pi \cos^3 \psi} \Box^2 \int_{t \in C} \left( \int_{x_1}^{\infty} g_0(y - t_1, x_2 - t_2, x_3 - t_3) \, dy \right) \frac{dS}{|t|} \]

and
Inversion of regular transforms

- **Case** $k = 0$. The solution of

$$\Delta_0 * f_0 = g_0,$$

*can be found in the form*

$$f_0(x) = \frac{1}{2\pi \cos^3 \psi} \Box^2 \Delta_1 * \Theta_1 * g_0$$

(2)

$$= \frac{1}{2\pi \cos^3 \psi} \Box^2 \int_{t \in C} \left( \int_{x_1}^{\infty} g_0(y - t_1, x_2 - t_2, x_3 - t_3) \, dy \right) \frac{dS}{|t|}$$

*and*

$$\Box = \frac{\partial^2}{\partial x_1^2} - \lambda^2 \left( \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right).$$
Case $k = 1$. The solution of

$$\Delta_1 \ast f_1 = g_1$$

(3)

reads

$$f_1(x) = \frac{1}{2\pi \cos^3 \psi} \bigtriangleup^2 \Delta_0 \ast \Theta_1 \ast g_1$$

(4)

$$= \frac{1}{2\pi \cos^3 \psi} \bigtriangleup^2 \int_{t \in \mathcal{C}} \int_{x_1}^{\infty} g_1(y - t_1, x_2 - t_2, x_3 - t_3) \, dy \, dS.$$
**Conclusion:** Inversion of any of two regular cone transform is given by the another cone transform followed (or preceded) by the 4 order differential operator and additional integration from \( x_1 \) to \( \infty \) in the vertical variable. No Fourier transform etc. is necessary.
Corollary For any function $f$ with support in $E_m$ for some $m$, we have
\[ \text{supp} f \subset \text{supp} \Delta_k \ast f - V, \quad k = 0, 1 \]
where $V$ is the convex hull of $C$. 
Proofs

Distributions $\Delta_0$ and $\Delta_1$ are homogeneous of order 2 and 1. Fourier transforms are equal to (V.P. 2016, P.140)

$$\hat{\Delta}_0 (p) = -\frac{1}{2\pi \cos^2 \psi} |p_1| \left( p_1^2 - \lambda^2 (p_2^2 + p_3^2) \right)^{-3/2},$$

$$\hat{\Delta}_1 (p) = -\frac{2i}{\cos \psi} \text{sgn} p_1 \left( p_1^2 - \lambda^2 (p_2^2 + p_3^2) \right)^{-1/2}$$

for $p_1^2 > \lambda^2 (p_2^2 + p_3^2)$. 
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for $p_1^2 > \lambda^2 (p_2^2 + p_3^2)$.

- Both have analytical continuation at $H_+ = \{ p \in \mathbb{C}^3 : \text{Im} \ p_1 \geq 0 \}$.
The above calculations result in

\[ 2\pi i \cos^3 \psi (p_1 + i0)^{-1} (p_1^2 - \lambda^2 (p_2^2 + p_3^2))^2 \hat{\Delta}_0(p) \hat{\Delta}_1(p) = 1 \]

since function \((p_1 + i0)^{-1}\) admits holomorphic continuation at \(H_+\). This equation holds for all \(p \in \mathbb{R}^3\).
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Calculating the inverse Fourier transform we obtain

\[ F^{-1} (p_1^2 - \lambda^2 (p_2^2 + p_3^2)) = -\frac{1}{4\pi^2} \Box \delta_0, \]

and

\[ F^{-1} (p_1 + i0)^{-1} = -2\pi i \Theta_1, \]

where \(\Theta_1 = \theta (x_1) \delta_0 (x_2, x_3), \theta (t) = 1 \text{ for } t < 0 \text{ and } \theta (t) = 0 \text{ for } t > 0.\)
The above calculations results

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Finally

\[ \cos^3 \psi \Box^2 \delta_0 * \Theta_1 * \Delta_1 * \Delta_0 = \delta_0, \]  

(5)

where the convolutions of distributions \(\Theta_1, \Delta_1 \text{ and } \Box^2 \delta_0\) are well defined and commute.
Applying (5) to $f_0$ gives

$$f_0 = \cos^3 \psi \Delta^2 \ast \Theta_1 \ast \Delta_0 \ast f_0 = \cos^3 \psi \Delta^2 \ast \Theta_1 \ast g_0$$

which is equivalent to (2).
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Commuting factors in (5) yields

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Remark 1. Constant attenuation can be included in this method.
Remark 2. Solution of (1) could be done in form $\hat{g}(p) / \hat{\Delta}_k(p)$ in the frequency domain. Implementation of this method supposes cutting out the "plumes" of $g$ which causes the artifacts in the reconstruction as in the following picture.
Remark 2. Solution of (1) could be done in form $\hat{g}(p) / \hat{\Delta}_k(p)$ in the frequency domain. Implementation of this method supposes cutting out the "plumes" of $g$ which causes the artifacts in the reconstruction as in the following picture.

which is due to the courtesy of Gouia-Zarrad, Ambartsoumian 2014.
Inversion of the singular cone transform

- Fix $\lambda > 0$ and consider the singular integral transform

$$G(q, \theta) = \int_{C_\lambda(\theta)} f(q + x) \frac{dS}{|x|^2}, \; \theta \in S^2, \; q \in E^3,$$

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where $C_\lambda(\theta)$ means for the spherical cone with apex $x = 0$, axis $\theta \in S^2$ and opening $\lambda$. 
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- The integral is well defined if $f$ is smooth and $f(q) = 0$. 

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- **Theorem** For any $\lambda > 0$ and any set $Q \subset E^3$, an arbitrary function $f \in C^2$ with compact support can be recovered from data of integrals (6) for $q \in Q$ provided
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  1. any plane $H$ which meets $\text{supp} f$ has a common point with $Q$,
  2. for any point $q \in Q$, there exists a unit vector $\theta(q)$ such that $\text{supp} f \subset q + C_\lambda(\theta(q))$. 

Compton cones with swinging axis

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Proof

- **Step 1.** The singular ray transform

\[
Xf(q, \zeta) = \int_0^\infty f(q + r\zeta) \frac{dr}{r}, \quad \zeta \in S^2, q \in Q
\]

is well defined since \( f \) vanishes on \( Q \) since of (ii).
Proof

Step 1. The singular ray transform

\[ Xf(q, \xi) = \int_0^\infty f(q + r\xi) \frac{dr}{r}, \quad \xi \in S^2, q \in Q \quad (7) \]

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By Fubini’s theorem

\[ G(q, \theta) = \int_{S_\lambda(\theta)} \int_0^\infty f(q + \xi(\sigma) r) \frac{dr}{r} d\sigma = \int_{S_\lambda(\theta)} Xf(q, \xi(\sigma)) d\sigma, \]

where \( \xi(\sigma) \) runs over the circle \( S_\lambda(\theta) = C_\lambda(\theta) \cap S^2 \).
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Circles \( S_\lambda(\theta) \) have the same radius \( r = \lambda \left(1 + \lambda^2\right)^{-1/2} \).
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where \( \zeta(\sigma) \) runs over the circle \( S_\lambda(\theta) = C_\lambda(\theta) \cap S^2 \).

Circles \( S_\lambda(\theta) \) have the same radius \( r = \lambda (1 + \lambda^2)^{-1/2} \).

The planes containing these circles are tangent to the central ball \( B \) of radius \( \rho = (1 + \lambda^2)^{-1/2} \).
Step 2: Nongeodesic Funk transform
Theorem  For any $0 \leq \rho < 1$, $\alpha \in E$, $|\alpha| \leq 1$, an arbitrary function $g \in C^2 \left( S^2 \right)$ can be reconstructed from data of integrals

$$\Gamma (\theta) = \int_{\xi \in S^2, \langle \xi - \alpha, \theta \rangle = \rho} g (\xi) \, d\sigma, \quad \theta \in S^2$$  \hspace{1cm} (8)
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(8)

by

$$g(\xi) = -\frac{|\xi - \alpha|^2}{2\pi^2 \left(|\xi - \alpha|^2 - \rho^2\right)^{1/2}} \int_{S^2} \frac{\Gamma(\theta)}{(\langle \xi - \alpha, \theta \rangle - \rho)^2} \, dS$$

(9)

Provided there exists a vector $\theta_0 \in S^2$ such that $\text{supp}g \subset \{\xi \in S^2 : \langle \xi - \alpha, \theta_0 \rangle \geq \rho\}$. 
Theorem. For any $\rho, 0 \leq \rho < 1$, $\alpha \in E$, $|\alpha| \leq 1$, an arbitrary function $g \in C^2(S^2)$ can be reconstructed from data of integrals

$$\Gamma(\theta) = \int_{\tilde{\xi} \in S^2, \langle \tilde{\xi} - \alpha, \theta \rangle = \rho} g(\tilde{\xi}) \, d\sigma, \ \theta \in S^2$$

by

$$g(\tilde{\xi}) = -\frac{|\tilde{\xi} - \alpha|^2}{2\pi^2 \left(|\tilde{\xi} - \alpha|^2 - \rho^2\right)^{1/2}} \int_{S^2} \frac{\Gamma(\theta)}{(\langle \tilde{\xi} - \alpha, \theta \rangle - \rho)^2} \, dS$$

provided there exists a vector $\theta_0 \in S^2$ such that $\text{supp} g \subset \{\tilde{\xi} \in S^2 : \langle \tilde{\xi} - \alpha, \theta_0 \rangle \geq \rho\}$.

The singular integral is regularized as follows

$$\int_{S^2} \frac{\Gamma(\theta)}{(\langle \tilde{\xi} - \alpha, \theta \rangle - \rho)^2} \, dS = -\Delta(\theta) \int_{S^2} \Gamma(\theta) \log (\langle \tilde{\xi} - \alpha, \theta \rangle - \rho) \, dS.$$
Hermann Minkowski 1905 stated uniqueness of an even functions on $S^2$ with given big circle integrals. For $n = 2, \rho = 0, \alpha = 0$, The analytic reconstruction of an even function is due to Paul Funk’s 1913 (student of David Hilbert).
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The case \( \rho = 0, |\alpha| = 1 \) of the above theorem follows from Radon’s reconstruction by means of the stereographic projection.
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The reconstruction from spherical integrals (8) on $S^n$ was stated in V.P. 2016 for arbitrary $n$, $0 \leq \rho < 1, |\alpha| \leq 1$. 

References

Victor Palamodov (Tel Aviv University)

New reconstructions from cone Radon transform

March 30, 2017

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Step 3. By (i) formula (9) can be applied to $\Gamma(q, \theta)$ for $\alpha = 0$, $\rho = (1 + \lambda^2)^{-1/2}$ which provides the reconstruction of $g(q, \xi) = Xf(q, \xi)$ for any $q \in Q$ and all $\xi$. 
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For any \( x \in E^3 \) and any unit orthogonal vectors \( \omega, \xi \), we have
\[
\langle \omega, \nabla_\xi \rangle^2 f(q + r\xi) = r^2 \langle \omega, \nabla_q \rangle^2 f(q + r\xi)
\]
which yields (by Grangeat’s method) for any \( p \),
\[
\int_{\langle \omega, \xi \rangle = 0} \langle \omega, \nabla_\xi \rangle^2 Xf(q, \xi) \, d\varphi = \int \langle \omega, \nabla_\xi \rangle^2 \int_0^\infty f(q + r\xi) \frac{dr}{r} \, d\varphi
\]
\[
= \int \int_0^\infty \langle \omega, \nabla_q \rangle^2 f(q + r\xi) r \, dr \, d\varphi = \frac{\partial^2}{\partial p^2} \int_{\langle \omega, q \rangle = p} f(q) \, dS,
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Step 3. By (i) formula (9) can be applied to \( \Gamma (q, \theta) \) for \( \alpha = 0 \), \( \rho = (1 + \lambda^2)^{-1/2} \) which provides the reconstruction of \( g (q, \zeta) = Xf (q, \zeta) \) for any \( q \in Q \) and all \( \zeta \).

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\]

where the left hand side can be calculated from \( Xf \).
Step 4 By (ii) we can use the Lorentz-Radon formula for any $x \in \text{supp} f$,

$$f(x) = -\frac{1}{8\pi^2} \int_{\omega \in S^2} \frac{\partial^2}{\partial p^2} \int_{\langle \omega, q-x \rangle = 0} f(q) \, dq \, d\Omega$$

$$= -\frac{1}{8\pi^2} \int_{\omega \in S^2} \int_{\langle \omega, \xi \rangle = 0} \langle \omega, \nabla_\xi \rangle^2 Xf(q(\omega), \xi) \, d\varphi \, d\Omega$$

if we choose for any $\omega \in S^2$, a point $q = q(\omega) \in Q$ such that $\langle q(\omega) - x, \omega \rangle = 0$. 
Step 4 By (ii) we can use the Lorentz-Radon formula for any \( x \in \text{supp} f \),

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\]

if we choose for any \( \omega \in S^2 \), a point \( q = q(\omega) \in Q \) such that

\[
\langle q(\omega) - x, \omega \rangle = 0.
\]

This completes the reconstruction of \( f \).
Other reconstructions from the singular cone beam transform

- Let $\Gamma = \{ y = y(s) \}$ be a closed $C^2$ smooth curve.
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- Let $\Gamma = \{y = y(s)\}$ be a closed $C^2$ smooth curve.
- Let $\sigma : \Gamma \times S^2 \to \mathbb{R} \times S^2; \sigma(y, \xi) = (\langle y, \xi \rangle, \xi)$. All critical points of the map $\sigma$ are supposed of Morse type.
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$$\sum_{y : \langle y, \xi \rangle = p} \langle y', \xi \rangle \varepsilon(y, \xi) = 1, \ (p, \xi) \in \text{Im} \sigma.$$
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  \]

**Theorem** For any function $f \in C^2_0 (E^3)$ and any $x \in \text{supp} f \setminus \Gamma$ such that any plane $P$ through $x$ meets $\Gamma$, the equation holds
\[
f(x) = -\frac{1}{32\pi^4} \int_{y \in \Gamma} \int_{\langle y - x, \xi \rangle = 0} \partial_s^2 \varepsilon(y, \xi) ds\]
\[
\times \int_{\langle \xi, \nabla_v \rangle = 0} \langle \xi, \nabla_v \rangle^2 \partial_s g(y, v) d\theta d\varphi.
\]
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