# Auxiliary Space AMG for $\boldsymbol{H}(\mathrm{curl})$ Problems

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## 1 Introduction

The search for efficient preconditioners for $\boldsymbol{H}(\mathrm{curl})$ problems on unstructured meshes has intensified in the last few years. The attempts to directly construct AMG (algebraic multigrid) methods had some success, see [10, 1, 6]. Exploiting available multilevel methods on auxiliary mesh for the same bilinear form led to efficient auxiliary mesh preconditioners to unstructured problems as shown in [7, 4]. A computationally more attractive approach was recently announced in [5]. Their method borrows the main tool from the above mentioned auxiliary mesh preconditioners, namely, the interpolation operator $\boldsymbol{\Pi}_h$ that maps functions from $\boldsymbol{H}(\mathrm{curl})$ into the lowest order Nédélec finite element space $\boldsymbol{V}_h$. The method of [5] and its motivation are outlined in Section 2. In particular, we describe briefly their Nédélec space decomposition, which is the basis of the auxiliary space AMG preconditioners.

In the present paper we consider several options for constructing unstructured mesh AMG preconditioners for $\boldsymbol{H}(\mathrm{curl})$ problems and report a summary of computational results from [8, 9]. In contrast to [5], we apply AMG directly to variationally constructed coarse-grid operators, and therefore no additional Poisson matrices are needed on input. We also consider variable coefficient problems, including some that lead to a singular matrix. Both types of problems are of great practical importance, and are not covered by the theory of [5].

The main Section 3 consists of an extensive set of numerical experiments that illustrate the behavior of various auxiliary space AMG preconditioners.

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## 2 The Auxiliary Spaces and Operators

We are interested in solving the following variational problem stemming from the definite Maxwell equations:

$$\text{Find } \mathbf{u} \in \boldsymbol{V}_h \; : \quad (\alpha \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) + (\beta\, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in \boldsymbol{V}_h \,. \quad (1)$$

Here we consider $\alpha > 0$ and $\beta \geq 0$ which are scalar coefficients, but extensions to (semi)definite tensors are possible. We allow $\beta$ to be zero in part or the whole domain (in which case the resulting matrix is only semidefinite, and for solvability the right-hand side should be chosen to satisfy compatibility conditions). Let $\boldsymbol{A}_h$ be the stiffness matrix corresponding to (1), where $\boldsymbol{V}_h$ is the (lowest order) Nédélec space associated with a triangulation $\mathcal{T}_h$.

Let $S_h$ be the space of continuous piecewise linear finite elements associated with the same mesh $\mathcal{T}_h$ as $\boldsymbol{V}_h$, and $\boldsymbol{S}_h$ be its vector counterpart. Let $G_h$ and $\boldsymbol{\Pi}_h$ be the matrix representations of the mapping $\varphi \in S_h \mapsto \nabla\varphi \in \boldsymbol{V}_h$ and the nodal interpolation from $\boldsymbol{S}_h$ to $\boldsymbol{V}_h$, respectively. Note that $G_h$ has as many rows as the number of edges in the mesh, with each row having two nonzero entries: $+1$ and $-1$ in the columns corresponding to the edge vertices. The sign depends on the orientation of the edge. Furthermore, $\boldsymbol{\Pi}_h$ can be computed based only on $G_h$ and on the coordinates of the vertices of the mesh.

The auxiliary space AMG preconditioner for $\boldsymbol{A}_h$ is a “three-level” method utilizing the subspaces $\boldsymbol{V}_h$, $G_h S_h$, and $\boldsymbol{\Pi}_h \boldsymbol{S}_h$. Its additive form reads (cf. [11])

$$\Lambda_h^{-1} + G_h B_h^{-1} G_h^T + \boldsymbol{\Pi}_h \boldsymbol{B}_h^{-1} \boldsymbol{\Pi}_h^T \,, \quad (2)$$

where $\Lambda_h$ is a smoother for $\boldsymbol{A}_h$, while $B_h$ and $\boldsymbol{B}_h$ are efficient preconditioners for $G_h^T \boldsymbol{A}_h G_h$ and $\boldsymbol{\Pi}_h^T \boldsymbol{A}_h \boldsymbol{\Pi}_h$ respectively. Since these matrices come from elliptic forms, the preconditioner of choice is AMG (especially for unstructured meshes).

If $\beta$ is identically zero, one can skip the subspace correction associated with $G_h$, in which case we get a two-level method.

The motivation for (2) is that any finite element function $\mathbf{u}_h \in \boldsymbol{V}_h$ allows for decomposition of the form (cf., [5]) $\mathbf{u}_h = \mathbf{v}_h + \boldsymbol{\Pi}_h \mathbf{z}_h + \nabla\varphi_h$ with $\mathbf{v}_h \in \boldsymbol{V}_h$, $\mathbf{z}_h \in \boldsymbol{S}_h$ and $\varphi_h \in S_h$ such that the following stability estimates hold,

$$h^{-1}\|\mathbf{v}_h\|_0 + \|\mathbf{z}_h\|_1 \leq C \, \|\operatorname{curl} \mathbf{u}_h\|_0 \quad \text{and} \quad \|\nabla\varphi_h\|_0 \leq C \, \|\mathbf{u}_h\|_0. \quad (3)$$

## 3 Numerical Experiments

In this section we present results from numerical experiments with different versions of the auxiliary space AMG method used as a preconditioner in PCG.

We set $\Lambda_h^{-1}$ to be a sweep of symmetric Gauss-Seidel, and consider the following preconditioners:

{1} Multiplicative version of (2) with $B_h$ and $\boldsymbol{B}_h$ implemented as one AMG V-cycle for $G_h^T \boldsymbol{A}_h G_h$ and $\boldsymbol{\Pi}_h^T \boldsymbol{A}_h \boldsymbol{\Pi}_h$ respectively.
{2} Additive preconditioner using the same components as {1} and extra smoothing.
{3} Multiplicative preconditioner with $B_h$ and $\boldsymbol{B}_h$ implemented using a sweep of geometric multigrid for Poisson problems, as described in [5].
{4} Additive preconditioner using the same components as {3} and extra smoothing.
{5} The preconditioner {3} using AMG instead of geometric multigrid.

The AMG algorithm we used is a serial version of the BoomerAMG solver from the *hypre* library. For more details see [2].

We report the number of preconditioned conjugate gradient iterations with the above preconditioners and relative tolerance $10^{-6}$, i.e. the iterations were stopped after the preconditioned residual norm was reduced by six orders of magnitude. In a few of the tests we also tried the corresponding two-level methods (using exact solution in the subspaces) and listed the iteration counts in parentheses following the V-cycle results.

### 3.1 Constant Coefficients

First we consider several simple constant coefficients problems with $\alpha = \beta = 1$. We test both two-dimensional triangular and three-dimensional tetrahedral meshes. The results are listed in Tables 1–6, where the following notation was used: $\ell$ is the refinement level of the mesh, $N$ is the size of the problem, and $n_1$ to $n_5$ give the iteration count for each of the auxiliary space AMG preconditioners {1} to {5}. When available, the error in $L^2$ is also reported.

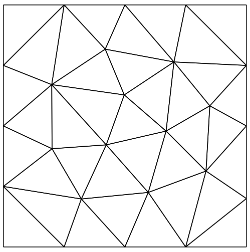
The experiments show that all considered solvers result in uniform and small number of iterations, which is in agreement with the theoretical results from [5, 9]. One can also observe that the multilevel results are very close in terms of number of iterations to the two-level ones.

Note that the first two methods (based on the original form) appear to work the same, independently of how complicated the geometry is. This is particularly interesting in the case of the third problem, where the assumption that the boundary is connected (needed to establish the decomposition in [5]) is violated. In contrast, the third and forth methods (based on Poisson subspace solvers) consistently result in higher number of iterations, and perform much worse on the third problem.

### 3.2 Variable Coefficients

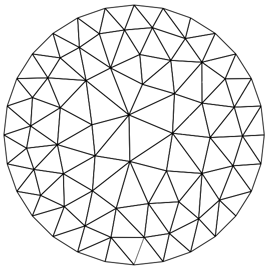
In Tables 7–8 we report results from a test where $\alpha$ and $\beta$ are piecewise constant coefficients. Note that this case is not covered by the theory in [5]. However, the modifications to the Poisson-based preconditioners are straightforward, namely they assemble matrices corresponding to the bilinear forms

**Table 1.** Initial mesh and numerical results for the problem on a square.



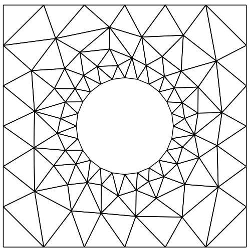
| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $\|e\|_{L^2}$ |
|---|---|---|---|---|---|---|
| 2 | 896 | 4 (3) | 9 (9) | 10 (9) | 16 (15) | 0.011898 |
| 3 | 3520 | 4 (3) | 10 (9) | 11 (10) | 17 (16) | 0.005953 |
| 4 | 13952 | 4 (3) | 10 (9) | 12 (11) | 18 (15) | 0.002977 |
| 5 | 55552 | 4 (3) | 10 (9) | 13 (11) | 18 (16) | 0.001489 |
| 6 | 221696 | 4 (3) | 10 (8) | 13 (11) | 18 (16) | 0.000744 |
| 7 | 885760 | 5 | 10 | 13 | 18 | 0.000372 |
| 8 | 3540992 | 5 | 11 | 13 | 19 | 0.000186 |

**Table 2.** Initial mesh and numerical results for the problem on a disk.



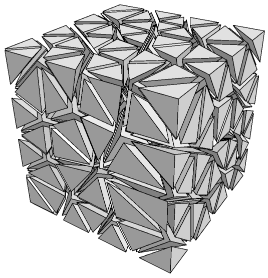
| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 1 | 736 | 4 (2) | 9 (8) | 7 (7) | 11 (11) |
| 2 | 2888 | 4 (3) | 10 (9) | 7 (7) | 12 (11) |
| 3 | 11440 | 4 (3) | 10 (9) | 7 (7) | 12 (11) |
| 4 | 45536 | 5 (3) | 11 (9) | 7 (7) | 12 (11) |
| 5 | 181696 | 5 (3) | 11 (8) | 8 (7) | 12 (11) |
| 6 | 725888 | 5 | 11 | 8 | 12 |
| 7 | 2901760 | 5 | 12 | 8 | 11 |

**Table 3.** Initial mesh and numerical results for the problem on a square with a circular hole.



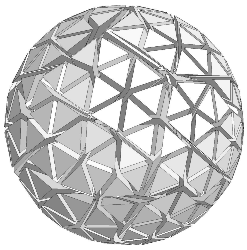
| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 2 | 972 | 6 (3) | 11 (9) | 21 (20) | 33 (31) |
| 3 | 14976 | 6 (3) | 12 (9) | 23 (21) | 33 (31) |
| 4 | 59520 | 7 (3) | 12 (9) | 23 (17) | 35 (23) |
| 5 | 237312 | 6 | 13 | 24 | 35 |
| 6 | 947712 | 7 | 13 | 25 | 35 |
| 7 | 3787776 | 7 | 14 | 25 | 35 |

**Table 4.** Initial mesh and numerical results for the problem on a cube.



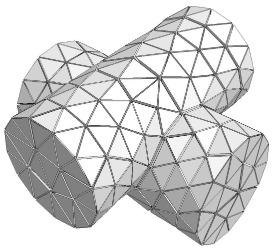
| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $\|e\|_{L^2}$ |
|---|---|---|---|---|---|---|
| 0 | 722 | 3 (3) | 9 (7) | 6 (6) | 11 (11) | 0.6777 |
| 1 | 5074 | 4 (3) | 10 (9) | 9 (9) | 16 (15) | 0.3776 |
| 2 | 37940 | 5 (4) | 11 (10) | 12 (11) | 20 (19) | 0.2152 |
| 3 | 293224 | 5 (4) | 11 (10) | 14 (12) | 22 (20) | 0.1096 |
| 4 | 2305232 | 5 | 11 | 15 | 23 | 0.0549 |

**Table 5.** Initial mesh and numerical results for the problem on a ball.



| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 0 | 704 | 3 (3) | 9 (7) | 5 (5) | 9 (9) |
| 1 | 4669 | 4 (3) | 10 (9) | 7 (7) | 13 (12) |
| 2 | 37940 | 5 (4) | 12 (10) | 8 (8) | 15 (14) |
| 3 | 255700 | 6 (5) | 13 (12) | 9 (8) | 15 (14) |
| 4 | 1990184 | 7 | 14 | 10 | 16 |

**Table 6.** Initial mesh and numerical results for the problem on a union of two cylinders.



| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 0 | 1197 | 3 (3) | 10 (8) | 7 (7) | 13 (13) |
| 1 | 8248 | 4 (4) | 11 (10) | 10 (9) | 17 (16) |
| 2 | 60940 | 5 (5) | 12 (11) | 12 (11) | 19 (18) |
| 3 | 467880 | 6 (5) | 12 (11) | 13 (12) | 21 (19) |
| 4 | 3665552 | 6 | 13 | 14 | 22 |

$(\beta\nabla u, \nabla v)$ and $(\alpha\nabla \mathbf{u}, \nabla \mathbf{v}) + (\beta \mathbf{u}, \mathbf{v})$. Here we concentrated only on the multiplicative AMG methods.

For the problem illustrated in Table 7 (where the jumps are simple), we observe stable number of iterations with respect to both the mesh size and the magnitude of the jumps. Note that this setup was reported to be problematic for geometric multigrid in [3]. As before, the method based on the original form outperforms the one based on AMG Poisson subspace solvers.

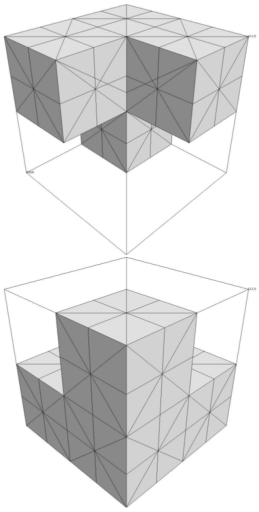
### 3.3 Singular Problems

Tables 9–10 present results for the problem corresponding to $\alpha = 1$, $\beta = 0$, i.e., to the bilinear form $(\text{curl}\,\mathbf{u}, \text{curl}\,\mathbf{v})$. In this case the matrix is singular, and the right-hand side, as well as the solution, belong to the space of discretely divergence free vectors (the kernel of $G_h^T$). Since $\beta = 0$, the solvers were modified to skip the correction in the space $G_h S_h$. This leads to a simpler preconditioner, which in additive form reads

$$\Lambda_h^{-1} + \boldsymbol{\Pi}_h \boldsymbol{B}_h^{-1} \boldsymbol{\Pi}_h^T. \qquad (4)$$

The results in Tables 9–10 are quite satisfactory and comparable to those from Tables 3 and 6. This is not surprising, since (3) implies that any $[\mathbf{u}_h]$ in the factor space $\boldsymbol{V}_h/\nabla S_h$, has a representative $\tilde{\mathbf{u}}_h \in [\mathbf{u}_h]$, such that $\tilde{\mathbf{u}}_h = \mathbf{u}_h - \nabla\varphi_h = \mathbf{v}_h + \boldsymbol{\Pi}_h \mathbf{z}_h$ and

**Table 7.** Numerical results for the problem on a cube with $\alpha$ and $\beta$ having different values in the shown regions (cf. [3]). Multiplicative preconditioner with AMG V-cycles in the subspaces.



| $\ell$ | $N$ | $p$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $-8$ | $-4$ | $-2$ | $-1$ | 0 | 1 | 2 | 4 | 8 |
| $n_1$ for $\alpha = 1, \beta \in \{1, 10^p\}$ | | | | | | | | | | |
| 1 | 716 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 | 4 |
| 2 | 5080 | 4 | 4 | 4 | 4 | 4 | 4 | 5 | 6 | 6 |
| 3 | 38192 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 4 | 296032 | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 |
| 5 | 2330816 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6 | 6 |
| $n_1$ for $\beta = 1, \alpha \in \{1, 10^p\}$ | | | | | | | | | | |
| 1 | 716 | 6 | 6 | 5 | 4 | 3 | 4 | 4 | 4 | 4 |
| 2 | 5080 | 6 | 6 | 6 | 5 | 4 | 5 | 5 | 5 | 5 |
| 3 | 38192 | 7 | 7 | 7 | 5 | 5 | 5 | 6 | 6 | 6 |
| 4 | 296032 | 8 | 8 | 7 | 6 | 5 | 6 | 6 | 6 | 6 |
| 5 | 2330816 | 8 | 9 | 7 | 6 | 5 | 6 | 6 | 6 | 6 |

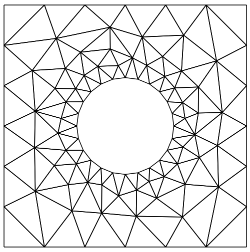
**Table 8.** Numerical results for the problem from Table 7 using multiplicative preconditioner with Poisson subspace solvers based on algebraic multigrid.

| $\ell$ | $N$ | $p$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $-8$ | $-4$ | $-2$ | $-1$ | 0 | 1 | 2 | 4 | 8 |
| $n_5$ for $\alpha = 1, \beta \in \{1, 10^p\}$ | | | | | | | | | | |
| 1 | 716 | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 6 (6) | 6 (6) | 5 (5) | 5 (5) |
| 2 | 5080 | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 9 (9) | 9 (9) | 9 (9) |
| 3 | 38192 | 11 (11) | 11 (11) | 11 (11) | 11 (11) | 11 (11) | 11 (11) | 10 (10) | 11 (11) | 12 (11) |
| 4 | 296032 | 12 (12) | 12 (12) | 12 (12) | 12 (12) | 12 (12) | 12 (12) | 12 (12) | 13 (13) | 14 (13) |
| 5 | 2330816 | 14 | 14 | 14 | 14 | 14 | 13 | 13 | 14 | 15 |
| $n_5$ for $\beta = 1, \alpha \in \{1, 10^p\}$ | | | | | | | | | | |
| 1 | 716 | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 7 (7) | 6 (6) |
| 2 | 5080 | 10 (9) | 9 (9) | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 10 (10) | 9 (9) |
| 3 | 38192 | 11 (11) | 11 (11) | 12 (12) | 12 (12) | 11 (11) | 12 (12) | 12 (12) | 12 (12) | 12 (12) |
| 4 | 296032 | 13 (13) | 13 (13) | 14 (14) | 14 (14) | 12 (12) | 15 (14) | 15 (15) | 15 (15) | 14 (14) |
| 5 | 2330816 | 15 | 15 | 16 | 16 | 14 | 16 | 17 | 17 | 17 |

$$h^{-1}\|\mathbf{v}_h\|_0 + \|\mathbf{z}_h\|_1 \leq C \,\|\,\mathrm{curl}\,\tilde{\mathbf{u}}_h\|_0 = C\,\|\,\mathrm{curl}\,[\mathbf{u}_h]\|_0 \,.$$

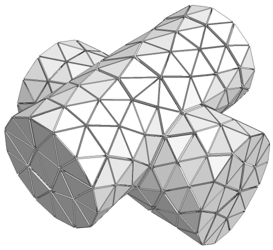
In Table 11 we also consider the important practical case when $\beta$ is zero only in part of the region. For this test we used a preconditioner based on (2) instead of (4). Even though the problem is singular and $\beta$ has jumps, the iterations counts are comparable to the case of constant coefficients. For example, the number of iterations for $\alpha = 1$, $\beta = 1$ given to the right of the table is almost identical to those when $\alpha = 1$, $\beta = 0$.

**Table 9.** Initial mesh and numerical results for the singular problem on a square with circular hole.



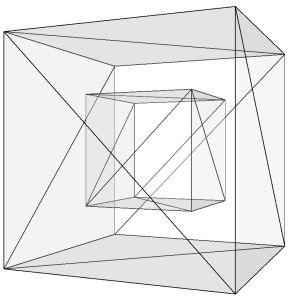
| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 2 | 972 | 7 | 12 | 19 | 28 |
| 3 | 14976 | 6 | 12 | 19 | 28 |
| 4 | 59520 | 7 | 12 | 19 | 28 |
| 5 | 237312 | 7 | 12 | 20 | 29 |
| 6 | 947712 | 7 | 11 | 20 | 29 |
| 7 | 3787776 | 7 | 12 | 21 | 29 |

**Table 10.** Initial mesh and numerical results for the singular problem on a union of two cylinders.



| $\ell$ | $N$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ |
|---|---|---|---|---|---|
| 0 | 1197 | 5 | 11 | 9 | 17 |
| 1 | 8248 | 6 | 13 | 12 | 19 |
| 2 | 60940 | 7 | 15 | 13 | 22 |
| 3 | 467880 | 7 | 15 | 14 | 23 |
| 4 | 3665552 | 8 | 15 | 15 | 23 |

**Table 11.** Initial mesh and numerical results for the problem on a cube with $\beta = 0$ outside the interior cube. Multiplicative preconditioner with AMG V-cycles in the subspaces.



| $\ell$ | $N$ | $p$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $-8$ | $-4$ | $-2$ | $-1$ | 0 | 1 | 2 | 4 | 8 | |
| $n_1$ for $\alpha = 1, \beta \in \{0, 10^p\}$ | | | | | | | | | | | |
| 1 | 485 | 2 | 3 | 3 | 2 | 4 | 2 | 3 | 3 | 3 | (4) |
| 2 | 3674 | 5 | 5 | 5 | 6 | 5 | 6 | 6 | 6 | 7 | (6) |
| 3 | 28692 | 8 | 7 | 8 | 7 | 7 | 8 | 8 | 10 | 10 | (7) |
| 4 | 226984 | 7 | 7 | 7 | 7 | 7 | 9 | 8 | 9 | 9 | (7) |
| 5 | 1806160 | 8 | 8 | 8 | 8 | 8 | 8 | 9 | 10 | 11 | (8) |

## References


[1] P.B. Bochev, C.J. Garasi, J.J. Hu, A.C. Robinson, and R.S. Tuminaro. An improved algebraic multigrid method for solving Maxwell's equations. *SIAM J. Sci. Comput.*, 25(2):623–642, 2003.

[2] V.E. Henson and U.M. Yang. BoomerAMG: a parallel algebraic multigrid solver and preconditioner. *Appl. Numer. Math.*, 41(1):155–177, 2002.

[3] R. Hiptmair. Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 36(1):204–225, 1999.

[4] R. Hiptmair, G. Widmer, and J. Zou. Auxiliary space preconditioning in $H_0(\text{curl};\ \Omega)$. *Numer. Math.*, 103(3):435–459, 2006.

[5] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $H(curl)$ and $H(div)$ spaces. Technical Report 2006-09, ETH, Switzerland, 2006.

[6] J. Jones and B. Lee. A multigrid method for variable coefficient Maxwell's equations. *SIAM J. Sci. Comput.*, 27(5):1689–1708, 2006.

[7] Tz.V. Kolev, J.E. Pasciak, and P.S. Vassilevski. $H(curl)$ auxiliary mesh preconditioning, 2006. In preparation.

[8] Tz.V. Kolev and P.S. Vassilevski. Parallel $H^1$-based auxiliary space AMG solver for $H(curl)$ problems. Technical Report UCRL-TR-222763, LLNL, 2006.

[9] Tz.V. Kolev and P.S. Vassilevski. Some experience with a $H^1$-based auxiliary space AMG for $H(curl)$ problems. Technical Report UCRL-TR-221841, LLNL, 2006.

[10] S. Reitzinger and J. Schöberl. An algebraic multigrid method for finite element discretizations with edge elements. *Numer. Linear Algebra Appl.*, 9(3):223–238, 2002.

[11] J. Xu. The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids. *Computing*, 56(3):215–235, 1996.