Domain Decomposition Algorithms for an Indefinite Hypersingular Integral Equation in Three Dimensions

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1 Introduction

In this paper we report on a non-overlapping and an overlapping domain decomposition method as preconditioners for the boundary element approximation of an indefinite hypersingular integral equation on a surface. The equation arises from an integral reformulation of the Neumann screen problem with the Helmholtz equation in the exterior of a screen in $\mathbb{R}^3$.

It is well-known that the linear algebraic system arising from the boundary element approximation to this integral equation is indefinite, and an iterative method like GMRES can be used to solve the system. Preconditioners by domain decomposition methods can be used to reduce the number of iterations. A non-overlapping preconditioner for the hypersingular integral equation reformulation of the 2D problem is studied in [10]. In this paper we study both non-overlapping and overlapping methods for the 3D problem. We prove that the convergence rate depends logarithmically on H/h for the non-overlapping method, and on H/δ for the overlapping method, where H and h are respectively the size of the coarse mesh and fine mesh, and δ is the overlap size. We note that domain decomposition methods with finite element approximations for the Helmholtz equation have been studied by many authors; see e.g. [2, 3, 5].

2 The Neumann Screen Problem and Boundary Integral Equation

Let Γ be a planar surface piece in $\mathbb{R}^3$ with polygonal boundary. The problem to be studied consists in finding U satisfying

$$\begin{aligned} \Delta U + k^2 U &= 0, && \text{in } \Omega_\Gamma := \mathbb{R}^3 - \overline{\Gamma}, \\ \frac{\partial U}{\partial \boldsymbol{n}} &= g, && \text{on } \Gamma, \\ \frac{\partial U}{\partial \boldsymbol{n}} - ikU &= o(1/r), && \text{as } r := |x| \to \infty, \end{aligned} \tag{1}$$

where k is a nonzero constant and g a given function. The condition at infinity is the well-known radiation condition.

The solution U can be expressed as a double-layer potential

$$U(x) = \frac{1}{4\pi} \int_\Gamma u(y) \frac{\partial}{\partial \boldsymbol{n}_y} \frac{e^{ik|x-y|}}{|x-y|} ds_y, \quad x \in \Omega_\Gamma,$$

where $u = [U]$ is the jump of U across Γ. It is shown in [9] that solving (1) is equivalent to solving

$$D_k u(x) = g(x), \quad x \in \Gamma, \tag{2}$$

where the operator D_k is defined as

$$D_k \phi(x) := -\frac{1}{4\pi} \int_\Gamma \phi(y) \frac{\partial}{\partial \boldsymbol{n}_x} \frac{\partial}{\partial \boldsymbol{n}_y} \frac{e^{ik|x-y|}}{|x-y|} ds_y, \quad x \in \Gamma. \tag{3}$$

The Sobolev spaces $\widetilde{H}^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ and their duals $H^{-1/2}(\Gamma)$ and $\widetilde{H}^{-1/2}(\Gamma)$ (respectively) are defined as usual; see [7]. It is shown in [9] that the operator D_0 defined as in (3) with $k = 0$ is a continuous and surjective mapping from $\widetilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$. Moreover, D_k can be written as

$$D_k = D_0 + K, \tag{4}$$

where K is a bounded operator from $\tilde{H}^{1/2}(\Gamma)$ into $L^2(\Gamma)$. Let

$$b(v, w) := \langle D_k v, w \rangle \quad \forall v, w \in \widetilde{H}^{1/2}(\Gamma),$$

(where $\langle D_k v, w \rangle$ denotes the duality pairing which coincides with the L_2 inner product on Γ if $D_k v, w \in L_2(\Gamma)$) then the bilinear form $b(\cdot, \cdot)$ can be written as

$$b(v, w) = a(v, w) + c(v, w),$$

where $a(v, w) = \langle D_0 v, w \rangle$ and $c(v, w) = \langle Kv, w \rangle$. The bilinear form $a(\cdot, \cdot)$ is a positive-definite and symmetric bilinear form on $\widetilde{H}^{1/2}(\Gamma)$ satisfying

$$a(v, v) \simeq \|v\| \widetilde{H}^{1/2}(\Gamma)^2 \quad \forall v \in \widetilde{H}^{1/2}(\Gamma), \tag{5}$$

whereas $b(\cdot, \cdot)$ is indefinite and satisfies

$$\mathrm{Re}(b(v, v)) \geq \gamma \|v\| \tilde{H}^{1/2}(\Gamma)^2 - \eta \|v\| L^2(\Gamma)^2 \quad \forall v \in \widetilde{H}^{1/2}(\Gamma),$$

for some $\gamma > 0$ and $\eta > 0$ independent of v.

A weak form of (2) is the problem of finding

$$u \in \widetilde{H}^{1/2}(\Gamma): \quad b(u, v) = \langle g, v \rangle \quad \forall v \in \widetilde{H}^{1/2}(\Gamma). \tag{6}$$

The problem (6) will be approximated by first constructing a finite-dimensional subspace $\mathcal{S} \subset \widetilde{H}^{1/2}(\Gamma)$, and then finding

$$u_{\mathcal{S}} \in \mathcal{S}: \quad b(u_{\mathcal{S}}, v) = \langle g, v \rangle \quad \forall v \in \mathcal{S}. \tag{7}$$

3 Additive Schwarz Algorithm

3.1 General Framework

Additive Schwarz methods provide fast solutions to (7) by solving (at the same time) problems of smaller size. Let $\mathcal{S}$ be decomposed as

$$\mathcal{S} = \mathcal{S}_0 + \cdots + \mathcal{S}_J, \tag{8}$$

where $\mathcal{S}_i$, $i = 0, \ldots, J$, are subspaces of $\mathcal{S}$. Let $Q_i : \mathcal{S} \to \mathcal{S}_i$ be projections defined by

$$b_i(Q_i v, w) = b(v, w) \quad \forall v \in \mathcal{S}, \forall w \in \mathcal{S}_i, \tag{9}$$

where the bilinear forms $b_i(\cdot, \cdot)$, $i = 0, \ldots, J$, are to be defined later. Then the additive Schwarz method for (7) consists in solving the equation

$$Q u_{\mathcal{S}} = \tilde{g},$$

where $Q = Q_0 + \cdots + Q_J$ is the additive Schwarz operator and $\tilde{g}$ is given by $\tilde{g} = g_0 + \cdots + g_J$ with $g_i \in \mathcal{S}_i$ being solutions of

$$b_i(g_i, w) = \langle g, w \rangle \quad \forall w \in \mathcal{S}_i.$$

This equation is solved iteratively by the GMRES method. Starting with an initial guess u_0 and the initial residual $r_0 = \tilde{g} - Q u_0$, we compute the mth iterate u_m as $u_m = u_0 + z_m$ where z_m is chosen to minimize the residual norm $\|\tilde{g} - Q(u_{m-1} + z)\| a$, where $\|v\| a = a(v, v)$. It is proved in [4] that

$$\|r_m\| a \leq \left(1 - \frac{C_1^2}{C_2}\right)^{m/2} \|r_0\| a,$$

where $r_m = \tilde{g} - Q u_m$ and

$$C_1 = \inf_{v \in \mathcal{S}} \frac{a(v, Qv)}{a(v, v)}, \quad C_2 = \sup_{v \in \mathcal{S}} \frac{a(Qv, Qv)}{a(v, v)}. \tag{10}$$

We now define two different subspace decomposition of the form (8) which result in two different preconditioners: a non-overlapping method and an overlapping method.

3.2 Non-overlapping Algorithm

Boundary Element Space

We first define the finite-dimensional space $\mathcal{S}$ in (7) on a two-level grid.

The coarse grid. Assume that Γ is partitioned into subdomains Γ_i, $i = 1, \ldots, N$, where each subdomain Γ_i is the image of the reference square $\hat{R} = (-1, 1)^2$ under a smooth bijective mapping $\mathcal{F}_i : \hat{R} \to \Gamma_i$. Denoting by H the diameter of the subdomains, we assume that

$$\|J_{\mathcal{F}_i}\| L_\infty(\hat{R}) \preceq H \quad \text{and} \quad \|J_{\mathcal{F}_i^{-1}}\| L_\infty(\hat{R}) \preceq H^{-1},$$

where $J_{\mathcal{F}_i}$ denotes the Jacobian matrix of the transformation and the norm is a matrix norm. The partition is assumed to be conforming in the sense that the non-empty intersection of a pair of distinct subdomains is a single common vertex or edge of both subdomains, and that each vertex of the domain Γ coincides with at least one subdomain vertex.

We define on this coarse grid the space $\mathcal{V}_0$ of continuous piecewise bilinear functions, vanishing on the boundary of Γ.

The fine grid. Each subdomain Γ_i is further divided into disjoint quadrilateral or triangular elements, giving a locally uniform mesh of element of size h_i in Γ_i. We denote by h the maximum value of h_i, $i = 1, \ldots, N$.

The finite-dimensional space $\mathcal{S}$ is defined as the space of continuous piecewise-bilinear functions (in the case of quadrilateral elements) or piecewise-linear functions (in the case of triangular elements) on the fine grid, vanishing on the boundary of Γ. We also define subspaces $\mathcal{V}_j = \mathcal{S} \cap \widetilde{H}^{1/2}(\Gamma_j)$ of functions in $\mathcal{S}$ supported in $\overline{\Gamma}_j$.

We denote by $\mathcal{N} = \{\boldsymbol{x}_k : k \in \mathcal{I}\}$ the set of all vertices of elements in the fine grid which are not on the boundary of Γ (where $\mathcal{I}$ is some index set), by $\mathcal{N}_w = \{\boldsymbol{x}_k \in \mathcal{N} : \boldsymbol{x}_k$ lies on a subdomain boundary$\}$ the wirebasket, and by $\phi_k \in \mathcal{S}$ the nodal basis function at $\boldsymbol{x}_k$, i.e., $\phi_k(\boldsymbol{x}_l) = \delta_{kl}$.

Subspace Decomposition

The non-overlapping method is defined by the subspace decomposition (8) where

$$\begin{aligned}
&\mathcal{S}_0 = \Pi_F \mathcal{V}_0, && \text{(coarse space)}\\
&\mathcal{S}_1 = \operatorname{span}\{\{\}\phi_k : \boldsymbol{x}_k \in \mathcal{N}_w\}, && \text{(wirebasket space)}\\
&\mathcal{S}_i = \mathcal{V}_{i-1} \quad \forall i = 2, \ldots, N+1, && \text{(interior spaces)},
\end{aligned}$$

in which Π_F is the interpolation operator which interpolates continuous functions into functions in $\mathcal{S}$. (Note that $J = N + 1$.)

The bilinear forms $b_i(\cdot, \cdot)$ on $\mathcal{S}_i$ (see (9)) are defined as follows:

$$\begin{aligned}
b_0(v, w) &= b(\Pi_C v, \Pi_C w) \quad \forall v, w \in \mathcal{S}_0,\\
b_1(v, w) &= \sum_{j=1}^{N} \sum_{\boldsymbol{x}_k \in \partial\Gamma_j} h_j v(\boldsymbol{x}_k) w(\boldsymbol{x}_k), \quad \forall v, w \in \mathcal{S}_1,\\
b_i(v, w) &= a(v, w) \quad \forall v, w \in \mathcal{S}_i,\ i = 2, \ldots, J.
\end{aligned}$$

Here Π_C is the interpolation operator that interpolates continuous functions into functions in $\mathcal{V}_0$.

Algorithm

The preconditioning technique is in practice performed by computing the action of the inverse of the preconditioner B on a residual $r \in \mathcal{S}$ when GMRES is used to solve (7) iteratively. This consists of the solution of independent problems on each of the subspaces involved in the decomposition.

1. Coarse space correction:

$$u_0 \in \mathcal{S}_0 : \quad b_0(u_0, v) = \langle r, v \rangle \quad \forall v \in \mathcal{S}_0$$

2. Wirebasket space correction:

$$u_1 \in \mathcal{S}_1 : \quad b_1(u_1, v) = \langle r, v \rangle \quad \forall v \in \mathcal{S}_1$$

3. Interior space corrections:

$$u_i \in \mathcal{S}_i : \quad b_i(u_i, v) = \langle r, v \rangle \quad \forall v \in \mathcal{S}_i,\ i = 2, \ldots, J.$$

4. Preconditioned residual:

$$B^{-1} r = \sum_{j=0}^{J} u_j.$$

Matrix Representation

Let $\boldsymbol{\Psi}$ be the set of nodal basis functions. We use the bilinear form $a(\cdot, \cdot)$ (respectively, $b(\cdot, \cdot)$) to compute the stiffness matrix $\boldsymbol{A}_a$ (respectively, $\boldsymbol{A}_b$). The coefficient vector $\boldsymbol{v}$ of a function $v \in \mathcal{S}$ is given as $v = \boldsymbol{\Psi}^T \boldsymbol{v}$, where T denotes transpose. Let $\boldsymbol{\Phi}_0$ be the vector composed of the nodal basis functions for the subspace $\mathcal{S}_0$. Then we denote by $\boldsymbol{R}_0$ the rectangular matrix that represents $\boldsymbol{\Phi}_0$ in the basis $\boldsymbol{\Psi}$, i.e., $\boldsymbol{\Phi}_0 = \boldsymbol{R}_0 \boldsymbol{\Psi}$. We also define $\boldsymbol{R}_i$, $i = 1, \ldots, J$, to be matrices of entries 0 and 1 such that $\boldsymbol{R}_i \boldsymbol{\Psi}$ forms the nodal bases for $\mathcal{S}_i$. If $v = B^{-1} r$ then $\boldsymbol{v} = \sum_{i=0}^{J} \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{M} \boldsymbol{r}$ where, noting the bilinear form used in each subspace,

$$\boldsymbol{A}_0 = \boldsymbol{R}_0 \boldsymbol{A}_b \boldsymbol{R}_0^T, \quad \boldsymbol{A}_1 = \boldsymbol{R}_1 \boldsymbol{D} \boldsymbol{R}_1^T, \quad \boldsymbol{A}_i = \boldsymbol{R}_1 \boldsymbol{A}_a \boldsymbol{R}_i^T,\ i = 2, \ldots, J.$$

The size of $\boldsymbol{A}_1$ is large; however, the matrix $\boldsymbol{D}$ computed with the bilinear form $b_1(\cdot, \cdot)$ is a diagonal matrix.

3.3 Overlapping Algorithm

Overlapping Subdomains

As in [11], we extend each subdomain Γ_j in the following way. First we define, for some $\delta > 0$ called the overlap size,

$$\tilde{\mathcal{V}}_j = \text{span}\{\{\}\phi_k : \boldsymbol{x}_k \notin \overline{\Gamma}_j,\ \text{dist}(\boldsymbol{x}_k, \partial\Gamma_j) \leq \delta\},$$

and denote

$$\tilde{\Gamma}_j = \text{supp}\{\phi_k : \phi_k \in \tilde{\mathcal{S}}_j\},$$

which is the shaded area in Figure 1. (Here the distance is defined with the max norm $\|\boldsymbol{x}\| = \max\{|x_1|, |x_2|\}$ where $\boldsymbol{x} = (x_1, x_2)$.) The extended subdomain Γ_j' is then defined as $\Gamma_j' = \overline{\Gamma}_j \cup \tilde{\Gamma}_j$. We note that Γ_j' need not be a quadrilateral domain. Also, if δ is chosen such that $\delta \in (0, H]$, then

$$\text{diam}(\Gamma_i') \simeq H. \tag{11}$$

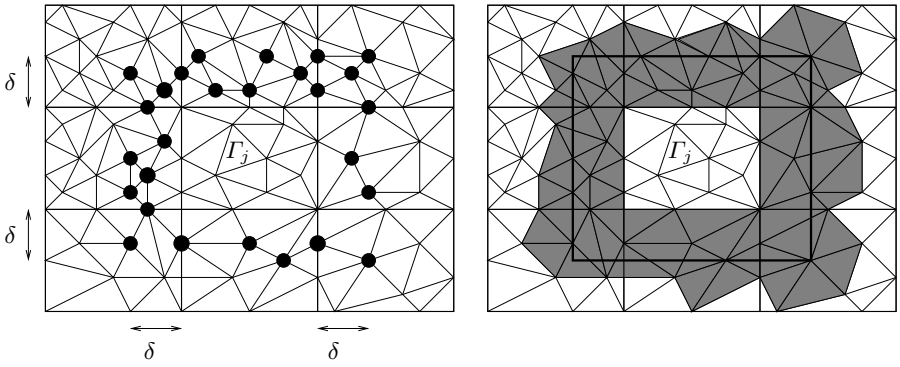

Fig. 1. • vertex at a distance δ to $\overline{\Gamma}_j$, $\tilde{\Gamma}_j$: shaded region, $\Gamma'_j = \Gamma_j \cup \tilde{\Gamma}_j$: overlapping subdomain.

Subspace Decomposition

The decomposition (8) is performed with subspaces $\mathcal{S}_j$, $j = 0, \ldots, J = N$, defined as

$$\begin{aligned} &\mathcal{S}_0 = \Pi_F \mathcal{V}_0, \\ &\mathcal{S}_j = \mathcal{V}_j \cup \tilde{\mathcal{V}}_j = \mathcal{S} \cap \widetilde{H}^{1/2}(\Gamma'_j) \quad \forall j = 1, \ldots, J. \end{aligned}$$

The bilinear forms $b_i(\cdot, \cdot)$ on $\mathcal{S}_i$ (see (9)) are defined as follows:

$$\begin{aligned} &b_0(v, w) = b(\Pi_C v, \Pi_C w) \quad \forall v, w \in \mathcal{S}_0, \\ &b_i(v, w) = a(v, w) \quad \forall v, w \in \mathcal{S}_i,\ i = 1, \ldots, J. \end{aligned}$$

Algorithm

The overlapping preconditioner is performed in the same manner as the non-overlapping version, with subspace corrections being

$$u_i \in \mathcal{S}_i : \quad b_i(u_i, v) = \langle r, v \rangle \quad \forall v \in \mathcal{S}_i,\ i = 0, \ldots, J.$$

Matrix Representation

As in the case of non-overlapping method, the updated residual vector is given by $\boldsymbol{v} = \sum_{i=0}^{J} \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{M} \boldsymbol{r}$ where

$$\boldsymbol{A}_0 = \boldsymbol{R}_0 \boldsymbol{A}_b \boldsymbol{R}_0^T, \quad \boldsymbol{A}_i = \boldsymbol{R}_1 \boldsymbol{A}_a \boldsymbol{R}_i^T,\ i = 2, \ldots, J.$$

3.4 Convergence

The preconditioned GMRES method using the non-overlapping and overlapping preconditioners converges with constants C_1 and C_2 (see (10)) slightly dependent on the mesh sizes H and h and the overlap size δ, as given in the following theorem.

Theorem 1.

- **Bound for** C_1**:** *There exists* $H_0 > 0$ *such that for all* $H \in (0, H_0]$ *and* $u \in \mathcal{S}$ *there hold*
$$\left(1 + \log^2 \frac{H}{h}\right)^{-1} a(u,u) \preceq a(u, Qu)$$
for the non-overlapping method, and
$$\left(1 + \log^2 \frac{H}{\delta}\right)^{-1} a(u,u) \preceq a(u, Qu)$$
for the overlapping method.
- **Bound for** C_2**:** *There exists* $H_1 > 0$ *such that for all* $H \in (0, H_1]$ *and* $u \in \mathcal{S}$ *there holds, for both methods,*
$$a(Qu, Qu) \preceq a(u,u)\,.$$

Proof. Sketch of the proof: First we note that

$$a(Qu,Qu) \simeq \|Qu\|\widetilde{H}^{1/2}(\Gamma)^2 \preceq \sum_{i=0}^{J} \|Q_i u\|\widetilde{H}^{1/2}(\Gamma_i)^2 \simeq \sum_{i=0}^{J} a(Q_i u, Q_i u).$$

Using this result, the boundedness of Q_0, and the definition of the projections Q_i, we can prove the bound for C_2.

The proof of the bound for C_1 is more complicated and involves the operator $P = P_0 + \cdots + P_J$ where P_i is defined as Q_i but with the bilinear form $a(\cdot,\cdot)$ in the place of $b(\cdot,\cdot)$. This operator P is in fact the additive Schwarz operator for the positive definite operator D_0 (see (4)). It is proved in [6] and [1] for the nonoverlapping method that

$$\left(1 + \log^2 \frac{H}{h}\right)^{-1} a(v,v) \preceq a(Pv, v),$$

and in [11] for the overlapping method that

$$\left(1 + \log^2 \frac{H}{\delta}\right)^{-1} a(v,v) \preceq a(Pv, v).$$

The difference in P and Q is due to the bounded operator K in (4), and further analysis to obtain similar estimates for Q involves this operator. For a detailed proof, see [8].

4 Numerical Experiments

We solve equation (2) with $k = 5$ and $g(x) \equiv 1$ on a uniform triangular mesh, by using the non-overlapping and overlapping preconditioners. In Table 1 we report on the number of iterations and CPU times (in seconds) when the equation is solved without any preconditioner, and when the non-overlapping preconditioner is used with various values of H/h. In Table 2 we report on the number of iterations and CPU times when the overlapping preconditioner is used with various values of H/δ. Choosing a suitable mesh size ratio H/h, we observe that the non-overlapping as well as the overlapping preconditioned method clearly outperform the non-preconditioned method in iteration numbers and CPU times. Here we use the GMRES without restart and stop if the relative residual is less than 10^{-10}. The local problems in computing the correction steps are solved by the GMRES or, if appropriate, by CG.

Table 1. Number of iterations and CPU times (in parentheses). WP: without preconditioner

DoF	WP	Non-overlapping			
		$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$
9	6 (0.01)	6 (0.01)			
49	17 (0.02)	17 (0.01)	17 (0.02)		
225	23 (0.02)	20 (0.03)	20 (0.02)	21 (0.04)	
961	31 (0.15)	21 (0.24)	21 (0.12)	23 (0.20)	23 (0.62)
3969	44 (3.02)	21 (4.39)	21 (1.62)	21 (1.75)	26 (4.16)
16129	63 (84.94)	21 (93.72)	21 (32.18)	21 (29.86)	24 (41.06)

Table 2. Number of iterations and CPU times (in parentheses) of overlapping method

DoF	$\delta = h$				$\delta = 2h$			
	$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$	$H/h = 2$	$H/h = 4$	$H/h = 8$	$H/h = 16$
9	6 (0.02)				6 (0.02)			
49	19 (0.01)	18 (0.02)			17 (0.03)	20 (0.02)		
225	28 (0.04)	24 (0.03)	22 (0.05)		22 (0.09)	26 (0.08)	26 (0.09)	
961	30 (0.39)	27 (0.23)	26 (0.34)	25 (0.86)	26 (0.65)	29 (0.48)	27 (0.57)	27 (1.17)
3969	30 (6.53)	28 (2.50)	27 (2.84)	28 (6.12)	31 (8.34)	31 (3.92)	28 (4.13)	28 (7.99)
16129	30 (135.47)	28 (43.76)	27 (41.14)	29 (58.46)	35 (166.31)	31 (53.10)	29 (49.80)	29 (69.04)

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