# Completions to Sparse Shape Functions for Triangular and Tetrahedral $p$-FEM

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## 1 Introduction

In this paper, we investigate the following boundary value problem: Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain and let $\mathcal{A}$ be a matrix which is symmetric and uniformly positive definite in $\Omega$. Find $u \in H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ such that

$$a_\triangle(u, v) := \int_\Omega (\nabla u)^T \mathcal{A} \nabla v = \int_\Omega fv + \int_{\Gamma_2} f_1 v := \langle f, v\rangle_\Omega + \langle f_1, v\rangle_{\Gamma_2} \tag{1}$$

holds for all $v \in H^1_{\Gamma_1}(\Omega)$. Problem (1) will be discretized by means of the $hp$-version of the finite element method using triangular/tetrahedral elements $\triangle_s$, $s = 1, \ldots, nel$, see e.g. [5, 8]. Let $\hat{\triangle}_d$, $d = 2, 3$ be the reference triangle (tetrahedron) and $F_s : \hat{\triangle} \to \triangle_s$ be the (possibly nonlinear) isoparametric mapping to the element $\triangle_s$. We define the finite element space $\mathbb{M} := \{u \in H^1_{\Gamma_1}(\Omega), u \mid_{\triangle_s} = \tilde{u}(F_s^{-1}(x, y, z)), \tilde{u} \in \mathbb{P}_p\}$, where $\mathbb{P}_p$ is the space of all polynomials of maximal total degree $p$. By $\Psi = (\psi_1, \ldots, \psi_N)$, we denote a basis for $\mathbb{M}$. The Galerkin projection of (1) onto $\mathbb{M}$ leads to the linear system of algebraic finite element equations

$$\mathcal{K}_\Psi \underline{u} = \underline{f}, \quad \text{where} \quad \mathcal{K}_\Psi = [a_\Delta(\psi_j, \psi_i)]_{i,j=1}^N, \quad \underline{f}_p = [\langle f, \psi_i\rangle + \langle f_1, \psi_i\rangle_{\Gamma_2}]_{i=1}^N. \tag{2}$$

The global stiffness matrix $\mathcal{K}_\Psi$ can be expressed by the local stiffness matrices on the elements, i.e.

$$\mathcal{K}_\Psi = \sum_{s=1}^{nel} R_s^T K_s R_s, \tag{3}$$

where $K_s$ is the stiffness matrix on the element $\triangle_s$ and $R_s$ denotes the connectivity matrix for the numbering of the shape functions on $\triangle_s$ and $\Omega$. In the 2D and 3D case, the choice of a basis which is optimal due to condition number and sparsity of $\mathcal{K}_\Psi$ is not so clear. In [7], a new basis for triangular and tetrahedral elements has been proposed. This basis is optimal w.r.t. the number of nonzero entries of the element stiffness matrix, see [6]. A proof for the sparsity of the element stiffness matrix with $\mathcal{O}(p^d)$ nonzero entries is not known in the literature. In [4], another basis for the triangular case is proposed. Moreover, it is proved that the element stiffness matrix has $\mathcal{O}(p^2)$ nonzero entries. This paper is a completion to the papers [4] and [3]. We will prove the sparsity for the Karniadakis-Sherwin basis, [7]. This proof is similar to the proof for the basis in [4]. However, the proof requires some additional relations for Jacobi polynomials which makes the proof more technical.

The outline of the paper is the following. In Section 2, we summarize the most important properties for Jacobi polynomials. In Section 3, the 2D case is investigated. In Section 4, the 3D case is investigated.

## 2 Properties of Jacobi Polynomials

For the definition of our basis functions on the reference element, Jacobi polynomials are required, see [1, 2, 9] for more details.

Let

$$p_n^\alpha(x) = \frac{1}{2^n n!(1-x)^\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left((1-x)^\alpha (x^2-1)^n\right) \quad n \in \mathbb{N}_0, \alpha > -1 \tag{4}$$

be the $n$th Jacobi polynomial with respect to the weight function $(1-x)^\alpha$. $p_n^\alpha(x)$ is a polynomial of degree $n$, i.e. $p_n^\alpha \in \mathbb{P}_n((-1,1))$, where $\mathbb{P}_n$ is the space of all polynomials of degree $n$ on the interval. Moreover, let

$$\hat{p}_n^\alpha(x) = \int_{-1}^{x} p_{n-1}^\alpha(y)\, \mathrm{d}y \quad n \geq 1, \quad \hat{p}_0^\alpha(x) = 1 \tag{5}$$

be the $n$th integrated Jacobi polynomial.

**Lemma 1.** *Let $p_n^\alpha$ be defined via* (4)*. Moreover, let $j, l \in \mathbb{N}_0$ and $\alpha > -1$. Then, we have*

$$p_n^{\alpha-1}(x) = \frac{1}{\alpha+2n}\left[(\alpha+n)p_n^\alpha(x) - n p_{n-1}^\alpha(x)\right]. \tag{6}$$

*Moreover, the integral relations*

$$\int_{-1}^{1} (1-x)^\alpha p_j^\alpha(x) p_l^\alpha(x)\, \mathrm{d}x = \rho_j^\alpha \delta_{jl}, \quad \textit{where } \rho_j^\alpha = \frac{2^{\alpha+1}}{2j+\alpha+1}, \tag{7}$$

$$\int_{-1}^{1} (1-x)^\alpha p_j^\beta(x) q_l(x)\, \mathrm{d}x = 0 \quad \forall q_l \in \mathbb{P}_l, \alpha-\beta \in \mathbb{N}_0, j > l+\alpha-\beta \tag{8}$$

*are valid.*

*Proof.* A proof can be found in [4].

The next lemma considers properties of the integrated Jacobi polynomials (5).

**Lemma 2.** *Let $l, j \in \mathbb{N}_0$. Let $p_n^\alpha$ and $\hat{p}_n^\alpha$ be defined via* (4) *and* (5)*. Then, the identities*

$$\hat{p}_n^\alpha(x) = \frac{2n+2\alpha}{(2n+\alpha-1)(2n+\alpha)} p_n^\alpha(x) + \frac{2\alpha}{(2n+\alpha-2)(2n+\alpha)} p_{n-1}^\alpha(x)$$

$$-\frac{2n-2}{(2n+\alpha-1)(2n+\alpha-2)} p_{n-2}^{\alpha}(x), \quad n \geq 2, \tag{9}$$

$$\hat{p}_n^{\alpha}(x) = \frac{2}{2n+\alpha-1}\left(p_n^{\alpha-1}(x) + p_{n-1}^{\alpha-1}(x)\right), \quad n \geq 1 \tag{10}$$

*are valid.*

*Proof.* The proof can be found in [4].

Finally, we present two properties of the Jacobi polynomials which have not been presented in [4].

**Lemma 3.** *Let $l, j \in \mathbb{N}_0$. Let $p_n^{\alpha}$ and $\hat{p}_n^{\alpha}$ be defined via* (4) *and* (5). *Then, the following assertions are valid for $\alpha > -1, j > 1$:*

$$(\alpha-1)\hat{p}_j^{\alpha}(y) = (1-y)p_{j-1}^{\alpha}(y) + 2p_j^{\alpha-2}(y), \tag{11}$$

$$\begin{aligned}(1-y)&\left((2-2j)p_{j-2}^{\alpha}(y) + \alpha p_{j-1}^{\alpha}(y)\right) \\ &+(\alpha+2j-2)(\alpha-1)\hat{p}_j^{\alpha}(y) = 4(\alpha+j-2)p_{j-1}^{\alpha-2}(y) + (2\alpha-4)p_j^{\alpha-2}(y).\end{aligned} \tag{12}$$

*Proof.* The proof can be found in [3].

## 3 The Triangular Case

In this section, we consider the case $d = 2$. Let $\hat{\triangle}_2$ be the reference triangle with the vertices $(-1,-1)$, $(1,-1)$ and $(0,1)$. We introduce

$$\phi_{ij}(x,y) = \hat{p}_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^i \hat{p}_j^{2i}(y), \quad i+j \leq p, i \geq 2, j \geq 1, \tag{13}$$

as the interior bubble functions on $\hat{\triangle}_2$. This is the basis proposed in [7], whereas the basis with $\hat{p}_j^{2i-1}(y)$ instead of $\hat{p}_j^{2i}(y)$ in (13) has been investigated in [4]. The vertex functions and edge bubbles are taken from [4]. Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and let

$$\hat{K}_2 = [a_{ij,kl}]_{(i,j);(k,l)} = \left[\int_{\hat{\triangle}_2} (\nabla\phi_{ij}(x,y))^T A \nabla\phi_{kl}(x,y)\, \mathrm{d}(x,y)\right]_{(i,j);(k,l)} \tag{14}$$

be the stiffness matrix on $\hat{\triangle}_2$ with respect to the basis (13).

**Theorem 1.** *Let $\hat{K}_2$ be defined via* (13)-(14). *Then, the matrix $\hat{K}_2$ has $\mathcal{O}(p^2)$ nonzero matrix entries. More precisely, $a_{ij,kl} = 0$ if $|i-k| > 2$ or $|i-k+j-l| > 1$.*

*Proof.* First, we compute $\nabla\phi_{ij}$. A simple computation shows that

$$\nabla\phi_{ij} = \begin{bmatrix} p_{i-1}^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1}\hat{p}_j^{2i}(y) \\ \frac{1}{2}p_{i-2}^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i-1}\hat{p}_j^{2i}(y) + \hat{p}_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^{i} p_{j-1}^{2i}(y) \end{bmatrix}, \tag{15}$$

see [4]. Let

$$a_{ij,kl}^{(y)} = \int_{\hat{\triangle}_2} \frac{\partial\phi_{ij}}{\partial y}(x,y)\frac{\partial\phi_{kl}}{\partial y}(x,y)\, \mathrm{d}(x,y). \tag{16}$$

Using (15) and the Duffy transform $z = \frac{2x}{1-y}$, we obtain

$$\begin{aligned}
a_{ij,kl}^{(y)} &= \frac{1}{4}\int_{-1}^{1} p_{i-2}^{0}(z)p_{k-2}^{0}(z)\,\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k-1} \hat{p}_j^{2i}(y)\hat{p}_l^{2k}(y)\,\mathrm{d}y \\
&+ \int_{-1}^{1} \hat{p}_i^{0}(z)\hat{p}_k^{0}(z)\,\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k+1} p_{j-1}^{2i}(y)p_{l-1}^{2k}(y)\,\mathrm{d}y \\
&+ \frac{1}{2}\int_{-1}^{1} p_{i-2}^{0}(z)\hat{p}_k^{0}(z)\,\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k} \hat{p}_j^{2i}(y)p_{l-1}^{2k}(y)\,\mathrm{d}y \\
&+ \frac{1}{2}\int_{-1}^{1} \hat{p}_i^{0}(z)p_{k-2}^{0}(z)\,\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k} p_{j-1}^{2i}(y)\hat{p}_l^{2k}(y)\,\mathrm{d}y \\
&=: a_{ij,kl}^{(y,1)} + a_{ij,kl}^{(y,2)} + a_{ij,kl}^{(y,3)} + a_{ij,kl}^{(y,4)}.
\end{aligned}$$

With (9) for $\alpha = 0$ we arrive at $a_{ij,kl}^{(y)} = 0$ if $|i-k| \notin \{0,2\}$.

Let $k = i-2$. Then $a_{i,j,i-2,l}^{(y,1)}$ and $a_{i,j,i-2,l}^{(y,4)}$ vanish by repeated application of (7) and (9). The remaining integrals can be simplified using (9) and by (7) be evaluated to

$$\begin{aligned}
\int_{-1}^{1} \hat{p}_i^{0}(z)\hat{p}_{i-2}^{0}(z)\,\mathrm{d}z &= -\frac{2}{(2i-1)(2i-3)(2i-5)}, \\
\int_{-1}^{1} p_{i-2}^{0}(z)\hat{p}_{i-2}^{0}(z)\,\mathrm{d}z &= \frac{2}{(2i-3)(2i-5)}.
\end{aligned}$$

We insert now these relations into the expressions for $a_{ij,kl}^{(y,2)}$ and $a_{ij,kl}^{(y,3)}$ and use relation (11). Then, we obtain

$$\begin{aligned}
a_{i,j,i-2,l}^{(y,2)} + a_{i,j,i-2,l}^{(y,3)} &= 2c_0 \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i-1} p_{j-1}^{2i}(y)p_{l-1}^{2i-4}(y)\,\mathrm{d}y \\
&\quad -(2i-1)c_0 \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i-2} \hat{p}_j^{2i}(y)p_{l-1}^{2i-4}(y)\,\mathrm{d}y \\
&= c_0 \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i-2} \left[(1-y)p_{j-1}^{2i}(y) - (2i-1)\hat{p}_j^{2i}(y)\right] p_{l-1}^{2i-4}(y)\,\mathrm{d}y \\
&= 2c_0 \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i-2} p_j^{2i-2}(y)p_{l-1}^{2i-4}(y)\,\mathrm{d}y
\end{aligned}$$

with $c_0^{-1} = -(2i-1)(2i-3)(2i-5)$. Now, we apply (8) and obtain $a_{i,j,i-2,l} \neq 0$ if $-3 \le j-l \le -1$.

The case $k = i+2$ can be proved by the same arguments. For $i = k$, we investigate each term $a_{ij,kl}^{y,s}$, $s = 1,2,3,4$ separately. Using (6)-(10), the assertion can be proved. This proof is similar to the proof given in [4].

Next, we consider

$$a_{ij,kl}^{(xy)} = \int_{\hat{\triangle}_2} \frac{\partial \phi_{ij}}{\partial x}(x,y)\frac{\partial \phi_{kl}}{\partial y}(x,y)\,\mathrm{d}(x,y). \tag{17}$$

Using (15) and the Duffy transform $z = \frac{2x}{1-y}$ again, we obtain

$$a_{ij,kl}^{(xy)} = \frac{1}{2}\int_{-1}^{1} p_{i-1}^{0}(z)p_{k-2}^{0}(z)\;\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k-1} \hat{p}_j^{2i}(y)\hat{p}_l^{2k}(y)\;\mathrm{d}y$$
$$+\int_{-1}^{1} p_{i-1}^{0}(z)\hat{p}_{k}^{0}(z)\;\mathrm{d}z \int_{-1}^{1}\left(\frac{1-y}{2}\right)^{i+k} \hat{p}_j^{2i}(y)p_{l-1}^{2k}(y)\;\mathrm{d}y.$$

Using (7) and (9), $a_{ij,kl}^{(xy)} = 0$ if $|i-k| \neq 1$. Let $k = i+1$. Then, we have

$$\begin{aligned} a_{i,j,i+1,l}^{(xy)} &= \frac{1}{4i^2-1}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i} \hat{p}_j^{2i}(y) \\ &\qquad \times \left[(2i+1)\hat{p}_l^{2i+2}(y) - (1-y)p_{l-1}^{2i+2}(y)\right]\;\mathrm{d}y \\ &= -\frac{2}{4i^2-1}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2i} \hat{p}_j^{2i}(y)p_l^{2i}(y)\;\mathrm{d}y. \end{aligned}$$

Finally, we apply (7) and (9) to obtain $a_{i,j,i+1,l}^{(xy)} \neq 0$ if $-2 \leq j-l \leq 0$. The case $k = i-1$ follows by the same arguments.

*Remark 1.* Since $a_{i,j,i-2,l}^{(y,2)} \neq 0$ for all $j > l$, the sparsity of $\hat{K}$ cannot be proved with a direct evaluation of $a_{i,j,i-2,l}^{(y,2)}$ and $a_{i,j,i-2,l}^{(y,3)}$. Only for $i = k$, the terms $a_{i,j,i,l}^{(y,s)}$, $s = 1,2,3,4$, can be considered separately.

## 4 The Tetrahedral Case

Let $\hat{\triangle}_3$ be the reference tetrahedron with the vertices $(-1,-1,-1)$, $(1,-1,-1)$, $(0,1,-1)$ and $(0,0,1)$. The interior bubbles are

$$\begin{aligned} \phi_{ijk}(x,y,z) = \hat{p}_i^{0}\left(\frac{4x}{1-2y-z}\right)\left(\frac{1-2y-z}{4}\right)^{i} \hat{p}_j^{2i}\left(\frac{2y}{1-z}\right) \\ \times\left(\frac{1-z}{2}\right)^{j}\hat{p}_k^{2i+2j}(z), \quad i+j+k \leq p, i \geq 2, j,k \geq 1. \end{aligned} \tag{18}$$

The vertex functions, edge bubbles and face bubbles are taken from [3]. Let $\hat{\mathcal{A}}_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in {}^{3\times 3}$ be a diffusion matrix with constant coefficients. We introduce

$$\hat{K}_3 = [a_{ijk,i'j'k'}]_{i,i'=2,j,j',k,k'=1}^{i+j+k=p,i'+j'+k'=p} = \left[\int_{\hat{\triangle}_3} (\nabla\phi_{ijk})^T \hat{\mathcal{A}}_3 \nabla\phi_{i'j'k'}\right]_{i,i',j,j',k,k'} \tag{19}$$

as the part of the stiffness matrix that corresponds to the interior bubbles (18).

**Theorem 2.** *Let $\hat{K}_3$ be defined via* (19)*. Then, the matrix has $\frac{(p-1)(p-2)(p-3)}{6}$ rows and columns. Moreover, the entry $a_{ijk,i'j'k'}$ of the matrix $\hat{K}_3$ is zero if $|i-i'| > 2$, or $|i+j-i'-j'| > 3$, or $|i+j+k-i'-j'-k'| > 2$.*

*Proof.* The proofs for $\int_{\hat{\triangle}_3} \frac{\partial \phi_{ijk}}{\partial x} \frac{\partial \phi_{i'j'k'}}{\partial x}$ and $\int_{\hat{\triangle}_3} \frac{\partial \phi_{ijk}}{\partial y} \frac{\partial \phi_{i'j'k'}}{\partial y}$ are similar to the triangular case. For the computation of $\int_{\hat{\triangle}_3} \frac{\partial \phi_{ijk}}{\partial z} \frac{\partial \phi_{i'j'k'}}{\partial z}$, 16 different integrals have to be considered, see [3]. For $i = i'$, all $y$-integrals can be considered separately, whereas for $|i - i'| = 2$ we collect several integrals and use relation (12). To illustrate this procedure we consider the following three integrands,

$$\begin{aligned}
\hat{\mathcal{I}}^{(14)} &= -(j-1)c_1\, \hat{p}_i^0(x)\hat{p}_{i'}^0(x)\; w_{\gamma_y}(y)\, p_{j-2}^{2i}(y) p_{j'-1}^{2i'}(y) \\
&\quad \times w_{\gamma_z}(z)\, \hat{p}_k^{2i+2j}(z)\hat{p}_{k'}^{2i'+2j'}(z), \\
\hat{\mathcal{I}}^{(15)} &= i\, i'\; c_1 \hat{p}_i^0(x)\hat{p}_{i'}^0(x)\; w_{\gamma_y}(y)\, p_{j-1}^{2i}(y) p_{j'-1}^{2i'}(y)\; w_{\gamma_z}(z)\, \hat{p}_k^{2i+2j}(z)\hat{p}_{k'}^{2i'+2j'}(z), \\
\hat{\mathcal{I}}^{(17)} &= -i' c_1\; p_{i-2}^0(x)\hat{p}_{i'}^0(x)\; w_{\gamma_y-1}(y)\, \hat{p}_j^{2i}(y) p_{j'-1}^{2i'}(y) \\
&\quad \times w_{\gamma_z}(z)\, \hat{p}_k^{2i+2j}(z)\hat{p}_{k'}^{2i'+2j'}(z),
\end{aligned}$$

where $w_\gamma(\zeta) = \left(\frac{1-\zeta}{2}\right)^\gamma$ is the weight function for Jacobi polynomials $p_n^\gamma(\zeta)$ and $\gamma_y = i + i' + 1$, resp. $\gamma_z = i + j + i' + j'$. The constant $c_1$ is given by $c_1^{-1} = 4(i+j-1)(i'+j'-1)$. The numbering of the terms $\hat{\mathcal{I}}^{(14)}$, $\hat{\mathcal{I}}^{(15)}$, and $\hat{\mathcal{I}}^{(17)}$ corresponds to the numbering in [3]. These integrands are obtained after taking the partial derivative in $z$-direction and performing the corresponding Duffy transformations, compare [3]. The $x$-integrals can be evaluated as in the triangular case and for $i' = i - 2$ one obtains for $h(y, z) = \int_{-1}^1 \hat{\mathcal{I}}^{(14)} + \hat{\mathcal{I}}^{(15)} + \hat{\mathcal{I}}^{(17)}\, \mathrm{d}x$, the equation

$$\begin{aligned}
h(y,z) =& c_2\; w_{2i-2}(y) \Big[(1-y)\Big((j-1)p_{j-2}^{2i}(y) - i p_{j-1}^{2i}(y)\Big) \\
&\quad -(2i-1)(i+j-1)\hat{p}_j^{2i}(y)\Big]\, p_{j'-1}^{2i-4}(y) w_{\gamma_z}(z)\hat{p}_k^{2i+2j}(z)\hat{p}_{k'}^{2i'+2j'}(z) \\
=:& c_2\, h_1(y)\, w_{\gamma_z}(z)\hat{p}_k^{2i+2j}(z)\hat{p}_{k'}^{2i'+2j'}(z),
\end{aligned}$$

where $c_2^{-1} = 4(2i-5)(2i-3)(2i-1)(i+j-3)(i+j-1)/(i-2)$. The straightforward approach is to evaluate these integrals by exploiting the orthogonality relation (7). To do so, one has to rewrite all polynomials in terms of Jacobi polynomials with parameter $\alpha$ corresponding to the appearing weight, i.e. $\alpha = 2i - 1$ for the first two summands and $\alpha = 2i - 2$ for the third one. This can easily be achieved for $p_{j'-1}^{2i-4}(y)$ using identity (6) recursively. In order to expand $p_{j-2}^{2i}(y), p_{j-1}^{2i}(y)$ and $\hat{p}_j^{2i}(y)$ in the basis of Jacobi polynomials $p_m^{2i-1}(y)$ we need the coefficients $a_m, b_m$ of

$$p_n^{2i}(y) = \sum_{m=0}^n a_m\, p_m^{2i-1}(y), \qquad \text{resp.} \qquad \hat{p}_n^{2i}(y) = \sum_{m=0}^n b_m\, p_m^{2i-1}(y).$$

But for both transformations all $n + 1$ coefficients are nonzero. Hence we consider these three integrals together and rewrite the expression in angular brackets using identity (12) yielding,

$$\begin{aligned}
h_1(y) &= \Big[(1-y)\Big((j-1)\, p_{j-2}^{2i}(y) - i\, p_{j-1}^{2i}(y)\Big) \\
&\quad -(2i-1)(i+j-1)\, \hat{p}_j^{2i}(y)\Big]\, p_{j'-1}^{2i-4}(y) \\
&= \Big[2(2i+j-2)\, p_{j-1}^{2i-2}(y) + (2i-2)\, p_j^{2i-2}(y)\Big]\, p_{j'-1}^{2i-4}(y).
\end{aligned}$$

Using this substitution the $y$-integrand of $h(y, z)$, $h_1(y)$ has the following form,

$$h_1(y) = w_{2i-2}(y)\left[(2i+j-2)\,p_{j-1}^{2i-2}(y) + (i-1)\,p_j^{2i-2}(y)\right]p_{j'-1}^{2i-4}(y). \tag{20}$$

Finally we use identity (6) twice on $p_{j'-1}^{2i-4}(y)$,

$$\begin{aligned} p_{j'-1}^{2i-4}(y) =& \frac{(j'-2)(j'-1)}{2(i+j'-3)(2i+2j'-5)}\,p_{j'-3}^{2i-2}(y) \\ &- \frac{(j'-1)(2i+j'-4)}{2(i+j'-3)(i+j'-2)}\,p_{j'-2}^{2i-2}(y) \\ &+ \frac{(2i+j'-4)(2i+j'-3)}{2(i+j'-2)(2i+2j'-5)}\,p_{j'-1}^{2i-2}(y). \end{aligned}$$

Hence we have to evaluate integrals of the form

$$\int_{-1}^{1} w_{2i-2}(y)\,p_m^{2i-2}(y)p_{m'}^{2i-2}(y)\;\mathrm{d}y,$$

where the polynomial degrees range from $m \in \{j-1, j\}$ and $m' \in \{j'-3, j'-2, j'-1\}$. Now by orthogonality relation (7) it easily follows that the integral over (20) is nonzero only for $j' = j, j+1, j+2, j+3$.
The evaluation of the $z$-integrals can be done by the same procedure as in the triangular case. To finish the proof one has to consider also the off-diagonal terms, i.e. integrals of the form $\int_{\hat{\triangle}_3} \frac{\partial\phi_{ijk}}{\partial\eta}\frac{\partial\phi_{i'j'k'}}{\partial\zeta}$, $\eta, \zeta \in \{x, y, z\}$, $\eta \neq \zeta$. These integrals can be treated in complete analogy.

Our practical computations were performed using a program written in the environment of the computer algebra software Mathematica. A description of the applied algorithm can be found in [3].

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