A FETI-DP Method for Mortar Finite Element Discretization of a Fourth Order Problem

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Summary. In this paper we present a FETI-DP type algorithm for solving the system of algebraic equations arising from the mortar finite element discretization of a fourth order problem on a nonconforming mesh. A conforming reduced Hsieh-Clough-Tocher macro element is used locally in the subdomains. We present new FETI-DP discrete problems and later introduce new parallel preconditioners for two cases: where there are no crosspoints in the coarse division of subdomains and in the general case.

1 Introduction

The mortar methods are effective methods for constructing approximations of PDE problems on nonconforming meshes. They impose weak integral coupling conditions across the interfaces on the discrete solutions, cf. [1].

In this paper we present a FETI-DP method (dual primal Finite Element Tearing and Interconnecting, see [6, 9, 8]) for solving discrete problems arising from a mortar discretization of a fourth order model problem. The original domain is divided into subdomains and a local conforming reduced HCT (Hsieh-Clough-Tocher) macro element discretization is introduced in each subdomain. The discrete space is constructed using mortar discretization, see [10]. Then the degrees of freedom corresponding to the interior nodal points are eliminated as usually in all substructuring methods. The remaining system of unknowns is solved by a FETI-DP method.

Many variants of FETI-DP methods for solving systems arising from the discretizations on a single conforming mesh of second and fourth order problems are fully analyzed, cf. [9, 8].

Recently there have been a few FETI-DP type algorithms for mortar discretization of second order problems, cf. [11, 5, 4, 3], and [7].

To our knowledge there are no FETI type algorithms for solving systems of equations arising from a mortar discretization of a fourth order problem in the literature.

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The remainder of the paper is organized as follows. In Section 2 we introduce our differential and discrete problems. When there are no crosspoints in the coarse division of the domain, the FETI operator takes a much simpler form and therefore this case is presented separately together with a parallel preconditioner in Section 3, while Section 4 is dedicated to a short description of the FETI-DP operator and a respective preconditioner in the general case.

2 Differential and Discrete Problems

Let Ω be a polygonal domain in $\mathbb{R}^2$. Then our model problem is to find $u^* \in H_0^2(\Omega)$ such that

$$a(u^*, v) = f(v) \qquad v \in H_0^2(\Omega), \tag{1}$$

where u^* is the displacement, $f \in L^2(\Omega)$ is the body force,

$$a(u,v) = \int_\Omega \left[\triangle u \triangle v + (1-\nu)\left(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1}\right)\right] \, dx.$$

Here

$$H_0^2(\Omega) = \{v \in H^2(\Omega) : \ v = \partial_n v = 0 \ \text{ on } \ \partial\Omega\},$$

∂_n is the normal unit derivative outward to $\partial\Omega$, and $u_{x_ix_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}$ for $i,j = 1,2$. We assume that the Poisson ratio ν satisfies $0 < \nu < 1/2$. From the Lax-Milgram theorem and the continuity and ellipticity of the bilinear form $a(\cdot\,,\cdot)$ it follows that there exists a unique solution of this problem.

Next we assume that Ω is a union of disjoint polygonal substructures Ω_i which form a coarse triangulation of Ω, i.e. the intersection of the boundaries of two different subdomains $\partial\Omega_k \cap \partial\Omega_l, k \neq l$, is either the empty set, a vertex or a common edge. We also assume that this triangulation is shape regular in the sense of Section 2, p. 5 in [2].

An important role is played by the interface Γ, defined as the union of all open edges of substructures, which are not on the boundary of Ω.

In each subdomain Ω_k we introduce a quasiuniform triangulation $T_h(\Omega_k)$ made of triangles. Let $h_k = \max_{\tau \in T_h(\Omega_k)} \operatorname{diam} \tau$ be the parameter of this triangulation.

In each Ω_k we introduce a local conforming reduced Hsieh-Clough-Tocher (RHCT) macro finite element space $X_h(\Omega_k)$ as follows, cf. Figure 1:

$$\begin{aligned} X_h(\Omega_k) = \{v \in C^1(\Omega_k) : \ & v_{|\tau} \in P_3(\tau_i), \text{ for triangles } \tau_i, \ \ i = 1,2,3, \\ & \text{formed by connecting the vertices of } \tau \in T_h(\Omega_k) \text{ to} \\ & \text{its centroid, } \partial_n v \text{ is linear on each edge of } \partial\tau, \text{ and} \\ & v = \partial_n v = 0 \ \text{ on } \ \partial\Omega_k \cap \partial\Omega\}. \end{aligned} \tag{2}$$

The degrees of freedom of RHCT macro elements are given by

$$\{u(p_i), u_{x_1}(p_i), u_{x_2}(p_i)\}, \quad i = 1,2,3, \tag{3}$$

for the three vertices p_i of an element $\tau \in T_h(\Omega_k)$, cf. Figure 1.

We introduce next an auxiliary global space $X_h(\Omega) = \prod_{k=1}^N X_h(\Omega_k)$, and the so called broken bilinear form:

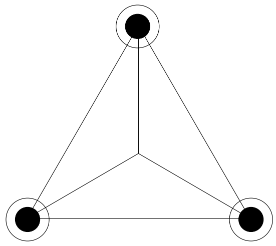

Fig. 1. Reduced HCT element

$$a_h(u,v) = \sum_{k=1}^{N} a_k(u,v),$$

where

$$a_k(u,v) = \int_{\Omega_k} \left[\triangle u \triangle v + (1-\nu)\left(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1}\right)\right] \, dx.$$

Then let $X(\Omega)$ be the subspace of $X_h(\Omega)$ consisting of all functions which have all degrees of freedom (dofs) of the RHCT elements continuous at the crosspoints – the vertices of the substructures.

The interface Γ_{kl} which is a common edge of two neighboring substructures Ω_k and Ω_l inherits two 1D independent triangulations: $T_{h,k}(\Gamma_{kl})$ – the h_k one from $T_h(\Omega_k)$ and $T_{h,l}(\Gamma_{kl})$ – the h_l one from $T_h(\Omega_l)$. Hence we can distinguish the sides (or meshes) of this interface. Let $\gamma_{m,k}$ be the side of Γ_{kl} associated with Ω_k and called master (mortar) and let $\delta_{m,l}$ be the side corresponding to Ω_l and called slave (nonmortar). Note that both the master and the slave occupy the same geometrical position of Γ_{kl}. The set of vertices of $T_{h,k}(\gamma_{m,k})$ on $\gamma_{m,k}$ is denoted by $\gamma_{m,k,h}$ and the set of nodes of $T_{h,l}(\delta_{m,l})$ on $\delta_{m,l}$ by $\delta_{m,l,h}$. In order to obtain our results we need a technical assumption of a uniform bound for the ratio $h_{\gamma_m}/h_{\delta_m}$ for any interface $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l}$.

An important role in our algorithm is played by four trace spaces onto the edges of the substructures. For each interface $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ let $W_{t,k}(\Gamma_{kl})$ be the space of C^1 continuous functions piecewise cubic on the 1D triangulation $T_{h,k}(\Gamma_{kl})$ and let $W_{n,k}(\Gamma_{kl})$ be the space of continuous piecewise linear functions on $T_{h,k}(\Gamma_{kl})$. The spaces $W_{t,l}(\Gamma_{kl})$ and $W_{n,l}(\Gamma_{kl})$ are defined analogously, but on the h_l triangulation $T_{h,l}(\Gamma_{kl})$ of Γ_{kl}.

Note that these four spaces are the tangential and normal trace spaces onto the interface $\Gamma_{kl} \subset \Gamma$ of functions from $X_h(\Omega_k)$ and $X_h(\Omega_l)$, respectively.

We also need to introduce two test function spaces for each slave $\delta_{m,l} = \Gamma_{kl}$. Let $M_t(\delta_{m,l})$ be the space of all C^1 continuous piecewise cubic on $T_{h,l}(\delta_{m,l})$ functions which are linear on the two end elements of $T_{h,l}(\delta_{m,l})$ and let $M_n(\delta_{m,l})$ be the space of all continuous piecewise linear on $T_{h,l}(\delta_{m,l})$ functions which are constant on the two end elements of $T_{h,l}(\delta_{m,l})$.

We now define the global space $M(\Gamma) = \prod_{\delta_{m,l} \subset \Gamma} M_t(\delta_{m,l}) \times M_n(\delta_{m,l})$ and the bilinear form $b(u,\psi)$ defined over $X(\Omega) \times M(\Gamma)$ as follows: let $u = (u_1, \ldots, u_N) \in X(\Omega)$ and $\psi = (\psi_m)_{\delta_m} = (\psi_{m,t}, \psi_{m,n})_{\delta_m} \in M(\Gamma)$, then let

$$b(u,\psi) = \sum_{\delta_m \subset \Gamma} b_{m,t}(u,\psi_{m,t}) + b_{m,n}(u,\psi_{m,n})$$

with

$$b_{m,t}(u,\psi_{m,t}) = \int_{\delta_m} (u_k - u_l)\psi_{m,y}\, ds \tag{4}$$

$$b_{m,n}(u,\psi_{m,n}) = \int_{\delta_m} (\partial_n u_k - \partial_n u_l)\psi_{m,n}\, ds. \tag{5}$$

Then our discrete problem is to find the pair $(u_h^*, \lambda^*) \in X(\Omega) \times M(\Gamma)$ such that

$$a_h(u_h^*, v) + b(v, \lambda^*) = f(v) \qquad \forall v \in X(\Omega) \tag{6}$$

$$b(u_h^*, \phi) = 0 \qquad \forall \phi \in M(\Gamma). \tag{7}$$

Note that if we introduce the discrete space

$$V^h = \{u \in X(\Omega) : b(u,\phi) = 0 \quad \forall \phi \in M(\Gamma)\}$$

then u_h^* is the unique function in V^h that satisfies

$$a_h(u_h^*, v) = f(v) \qquad \forall v \in V^h,$$

which is a standard mortar discrete problem formulation, cf. e.g. [10].

Note that we can split the matrix $K^{(l)}$ – the matrix representation of $a_l(u,v)$ in the standard nodal basis of $X_h(\Omega_l)$ as:

$$K^{(l)} := \begin{pmatrix} K_{ii}^{(l)} & K_{ic}^{(l)} & K_{ir}^{(l)} \\ K_{ci}^{(l)} & K_{cc}^{(l)} & K_{cr}^{(l)} \\ K_{ri}^{(l)} & K_{rc}^{(l)} & K_{rr}^{(l)} \end{pmatrix}, \tag{8}$$

where in the rows the indices i, c and r refer to the unknowns $u^{(i)}$ corresponding to the interior nodes, $u^{(c)}$ to the crosspoints, and $u^{(r)}$ to the remaining nodes, i.e. those related to the edges.

2.1 Matrix Form of the Mortar Conditions

Note that (7) is equivalent to two mortar conditions on each slave $\delta_{m,l} = \gamma_{m,k} = \Gamma_{kl}$:

$$b_{m,t}(u,\phi) = \int_{\delta_m} (u_k - u_l)\phi\, ds = 0 \qquad \forall \phi \in M_t(\delta_{m,l}) \tag{9}$$

$$b_{m,n}(u,\psi) = \int_{\delta_m} (\partial_n u_k - \partial_n u_l)\psi\, ds = 0 \qquad \forall \psi \in M_n(\delta_{m,l}). \tag{10}$$

Introducing the following splitting of two vectors representing the tangential and normal traces $u_{\delta_{m,l}}$ and $\partial_n u_{\delta_{m,l}}$ we get $u_{\delta_{m,l}} = u_{\delta_{m,l}}^{(r)} + u_{\delta_{m,l}}^{(c)}$ and $\partial_n u_{\delta_{m,l}} = \partial_n u_{\delta_{m,l}}^{(r)} + \partial_n u_{\delta_{m,l}}^{(c)}$ on a slave $\delta_{m,l} \subset \partial\Omega_l$, cf. (8). We can now rewrite (9) and (10) in a matrix form as

$$B_{t,\delta_{m,l}}^{(r)} u_{\delta_{m,l}}^{(r)} + B_{t,\delta_{m,l}}^{(c)} u_{\delta_{m,l}}^{(c)} = B_{t,\gamma_{m,k}}^{(r)} u_{\gamma_{m,k}}^{(r)} + B_{t,\gamma_{m,k}}^{(c)} u_{\gamma_{m,k}}^{(c)}, \tag{11}$$

$$B^{(r)}_{n,\delta_{m,l}}\partial_n u^{(r)}_{\delta_{m,l}} + B^{(c)}_{n,\delta_{m,l}}\partial_n u^{(c)}_{\delta_{m,l}} = B^{(r)}_{n,\gamma_{m,k}}\partial_n u^{(r)}_{\gamma_{m,k}} + B^{(c)}_{n,\gamma_{m,k}}\partial_n u^{(c)}_{\gamma_{m,k}},$$

where the matrices $B_{t,\delta_{m,l}} = (B^{(r)}_{t,\delta_{m,l}},\ B^{(c)}_{t,\delta_{m,l}})$ and $B_{n,\delta_{m,l}} = (B^{(r)}_{t,\gamma_{m,k}}, B^{(c)}_{t,\gamma_{m,k}})$ are mass matrices obtained by substituting the standard nodal basis functions of $W_{t,l}(\delta_{m,l}), W_{n,l}(\delta_{m,l})$ and $M_t(\delta_{m,l}), M_n(\delta_{m,l})$ into (9) and (10), respectively i.e.

$$B_{t,\delta_{m,l}} = \{(\phi_{x,s}, \psi_{y,r})\}_{\substack{x,y\in\delta_{m,l,h}\\ s,r=0,1}} \quad \phi_{x,s} \in W_t(\delta_{m,l}), \psi_{y,r} \in M_t(\delta_{m,l}), \tag{12}$$

$$B_{n,\delta_{m,l}} = \{(\phi_x, \psi_y)\}_{x,y\in\delta_{m,l,h}} \quad \phi_x \in W_n(\delta_{m,l}), \psi_y \in M_n(\delta_{m,l}), \tag{13}$$

where $\phi_{x,s}$, $(\psi_{x,s}\ )$ is a nodal basis function of $W_t(\delta_{m,l})$, $(M_t(\delta_{m,l}))$ associated with a vertex x of $T_{h,l}(\delta_{m,l})$ and is either a value if $s = 0$ or a derivative if $s = 1$, and $\phi_x \in W_{n,l}(\delta_{m,l})$ and $\psi_x, \in M_n(\delta_{m,l})$ are nodal basis function of these respective spaces equal to one at the node x and zero at all remaining nodal points on $\overline{\delta}_{m,l}$. The matrices $B_{t,\gamma_{m,k}} = (B^{(r)}_{n,\delta_{m,l}},\ B^{(c)}_{n,\delta_{m,l}})$, and $B_{n,\gamma_{m,k}} = (B^{(r)}_{n,\gamma_{m,k}},\ B^{(c)}_{n,\gamma_{m,k}})$ are defined analogously.

Note that $B^{(r)}_{t,\delta_{m,l}}, B^{(r)}_{n,\delta_{m,l}}$ are positive definite square matrices, see e.g. [10], but the other matrices in (11) are in general rectangular.

We also need the block-diagonal matrices

$$B_{\delta_{m,l}} = \begin{pmatrix} B_{t,\delta_{m,l}} & 0 \\ 0 & B_{n,\delta_{m,l}} \end{pmatrix} \quad B_{\gamma_{k,l}} = \begin{pmatrix} B_{t,\gamma_{k,l}} & 0 \\ 0 & B_{n,\gamma_{k,l}} \end{pmatrix}. \tag{14}$$

3 FETI-DP Problem – No Crosspoints Case

In this section we present a FETI-DP formulation for the case with no crosspoints, i.e. two subdomains are either disjoint or have a common edge, cf. Figure 2. In this case both the FETI-DP problem and the preconditioner are fully parallel and simple to describe and implement.

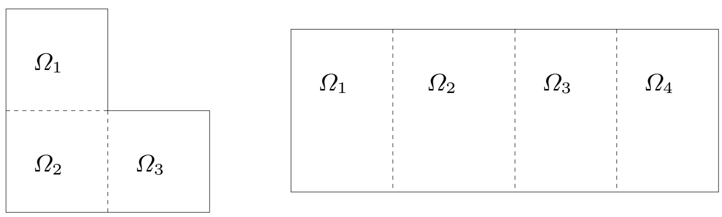

Fig. 2. Decompositions of Ω into subdomains with no crosspoints

3.1 Definition of the FETI Method

We now reformulate the system (6)–(7) as follows

$$K := \begin{pmatrix} K_{ii} & K_{ir} & 0 \\ K_{ri} & K_{rr} & B_r^T \\ 0 & B_r & 0 \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(r)} \\ \tilde{\lambda}^* \end{pmatrix} = \begin{pmatrix} f_i \\ f_r \\ 0 \end{pmatrix}, \tag{15}$$

where $B_r = \text{diag}\{B_{r,\delta_{m,l}}\}_{\delta_m}$ with $B_{r,\delta_{m,l}} = \left(I_{\delta_{m,l}}, \quad -(B^{(r)}_{\delta_{m,l}})^{-1} B^{(r)}_{\gamma_{m,k}}\right)$. Here K_{rr} and K_{ii} are block diagonal matrices of $K^{(l)}_{rr}$ and $K^{(l)}_{ii}$, respectively, cf. (8), and $\tilde{\lambda}^* = \{(B^{(r)}_{\delta_{m,l}})^T\}\lambda^*$.

Next the unknowns related to interior nodes and crosspoints, i.e. $u^{(i)}$ in (15), are eliminated, which yields a new system

$$\begin{aligned} Su^{(r)} + B_r^T \tilde{\lambda}^* &= g_r, \\ B_r u^{(r)} &= 0, \end{aligned} \tag{16}$$

where $S = K_{rr} - K_{ri} (K_{ii})^{-1} K_{ir}$ and $g_r = f_r - K_{ri} (K_{ii})^{-1} f_i$. We now eliminate $u^{(r)}$ and we end up with the following FETI-DP problem – find $\tilde{\lambda}^* \in M(\Gamma)$ such that

$$F(\tilde{\lambda}^*) = d, \tag{17}$$

where $d = B_r S^{-1} g_r$ and $F = B_r S^{-1} B_r^T$. Note that both S and B are block diagonal matrices due to the assumption that there are no crosspoints.

Next we introduce the following parallel preconditioner

$$M^{-1} = B_r S B_r^T. \tag{18}$$

3.2 Convergence Estimates

We say that the coarse triangulation is in Neumann-Dirichlet ordering if every subdomain has either all edges as slaves or all as mortars. In the case of no crosspoints it is always possible to choose the master-slave sides so as to obtain an N-D ordering of subdomains.

We have the following theorem in which a condition bound is established:

Theorem 1. *For any $\lambda \in M(\Gamma)$ it holds that*

$$c\ (1 + \log(H/\underline{h})^p \left\langle M\lambda, \lambda \right\rangle \le \left\langle F\lambda, \lambda \right\rangle \le C\ \left\langle M\lambda, \lambda \right\rangle,$$

where c and C are positive constants independent of any mesh parameters, $H = \max_k H_k$ and $\underline{h} = \min_k h_k$, $p = 0$ in the case of Neumann-Dirichlet ordering and $p = 2$ in general case.

4 General Case

Here we present briefly the case with crosspoints: the matrix formulation of (6)–(7) is as follows:

$$K := \begin{pmatrix} K_{ii} & K_{ic} & K_{ir} & 0 \\ K_{ci} & \tilde{K}_{cc} & K_{cr} & B_c^T \\ K_{ri} & K_{rc} & K_{rr} & B_r^T \\ 0 & B_c & B_r & 0 \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(c)} \\ u^{(r)} \\ \tilde{\lambda}^* \end{pmatrix} = \begin{pmatrix} f_i \\ f_c \\ f_r \\ 0 \end{pmatrix}, \tag{19}$$

where the global block matrices $B_c = \text{diag}\{B_{c,\delta_{m,l}}\}$ and $B_r = \text{diag}\{B_{r,\delta_{m,l}}\}$ are split into local ones defined over the vector representation spaces of traces on the interface $\Gamma_{kl} = \gamma_{m,k} = \delta_{m,l}$:

$$B_{c,\delta_{m,l}} = \left((B^{(r)}_{\delta_{m,l}})^{-1} B^{(c)}_{\delta_{m,l}}, \quad -(B^{(r)}_{\delta_{m,l}})^{-1} B^{(c)}_{\gamma_{m,k}} \right), \tag{20}$$

and $B_{r,\delta_{m,l}}$ is defined in (15). Here $\tilde{K}_{cc}$ is a block built of $K^{(l)}_{cc}$ taking into account the continuity of dofs at crosspoints, $\tilde{\lambda}^* = \{(B^{(r)}_{\delta_{m,l}})^T\}\lambda^*$, and K_{rr} and K_{ii} are block diagonal matrices as in (15).

Next we eliminate the unknowns related to the interior nodes and crosspoints i.e. $u^{(i)}$, $u^{(c)}$ in (19) and we get

$$\begin{aligned} \hat{S}u^{(r)} + \hat{B}^T\tilde{\lambda}^* &= \hat{f}_r, \\ \hat{B}u^{(r)} + \hat{S}_{cc}\tilde{\lambda}^* &= \hat{f}_c, \end{aligned} \tag{21}$$

where the matrices are defined as follows: $\hat{S} = K_{rr} - (K_{ri} \ \ K_{rc})\tilde{K}^{-1}_{i\&c} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix}$, $\hat{B} = B_r - (0 \ B_c)\tilde{K}^{-1}_{i\&c} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix}$, and $\hat{S}_{cc} = -(0 \ B_c)\tilde{K}^{-1}_{i\&c} \begin{pmatrix} 0 \\ B_c^T \end{pmatrix}$ with $\tilde{K}_{i\&c} = \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}$. We now eliminate $u^{(r)}$ and we end up with finding $\tilde{\lambda}^* \in M(\Gamma)$ such that

$$F(\tilde{\lambda}^*) = d, \tag{22}$$

where $d = f_c - \hat{B}\hat{S}^{-1}f_r$ and $F = \hat{S}_{cc} - \hat{B}\hat{S}^{-1}\hat{B}^T$.

Next we introduce the following parallel preconditioner: $M^{-1} = B_r S_{rr} B_r^T$ where $S_{rr} = \mathrm{diag}\{S^{(l)}_{rr}\}_{l=1}^N$ with $S^{(l)}_{rr} = (K^{(l)}_{rr} - K^{(l)}_{ri}(K^{(l)}_{ii})^{-1}K^{(l)}_{ir})$, i.e. $S^{(l)}_{rr}$ is the respective submatrix of the Schur matrix $S^{(l)}$ over Ω_l.

Then in the case of Neumann-Dirichlet ordering we have that the condition number $\kappa(M^{-1}F)$ is bounded by $(1 + \log(H/\underline{h})^2$ and in the general case by $(1 + \log(H/\underline{h})^4$.

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