On Multilevel BDDC

Jan Mandel[1] *, Bedřich Sousedík[1,2] †, and Clark R. Dohrmann[3]

[1] Department of Mathematical Sciences, University of Colorado at Denver and Health Sciences Center, Campus Box 170, Denver, CO 80217, USA. {jan.mandel,bedrich.sousedik}@cudenver.edu
[2] Department of Mathematics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7, 166 36 Prague 6, Czech Republic.
[3] Structural Dynamics Research Department, Sandia National Laboratories, Mail Stop 0847, Albuquerque NM 87185-0847, USA. crdohrm@sandia.gov

1 Introduction

The BDDC method [2] is the most advanced method from the BDD family [5]. Polylogarithmic condition number estimates for BDDC were obtained in [6, 7] and a proof that eigenvalues of BDDC and FETI-DP are same except for an eigenvalue equal to one was given in [7]. For important insights, alternative formulations of BDDC, and simplified proofs of these results, see [1] and [4].

In the case of many substructures, solving the coarse problem exactly is becoming a bottleneck. Since the coarse problem in BDDC has the same form as the original problem, the BDDC method can be applied recursively to solve the coarse problem approximately, leading to a multilevel form of BDDC in a straightforward manner [2]. Polylogarithmic condition number bounds for three-level BDDC (BDDC with two coarse levels) were proved in [10, 9]. This contribution is concerned with condition number estimates of BDDC with an arbitrary number of levels.

2 Abstract Multispace BDDC

All abstract spaces in this paper are finite dimensional. The dual space of a linear space U is denoted by U', and $\langle \cdot, \cdot \rangle$ is the duality pairing. We wish to solve the abstract linear problem

$$u \in U : a(u, v) = \langle f, v \rangle \,, \quad \forall v \in U, \tag{1}$$

for a given $f \in U'$, where a is a symmetric positive semidefinite bilinear form on some space $W \supset U$ and positive definite on U. The form $a(\cdot, \cdot)$ is called the energy inner product, the value of the quadratic form $a(u, u)$ is called the energy of u, and

* Partially supported by the NSF under grant CNS-0325314.
† Partially supported by the Grant Agency of the Czech Republic under grant 106/05/2731 and by the Program of the Information Society 1ET400300415.

the norm $\|u\|_a = a(u,u)^{1/2}$ is called the energy norm. The operator $A : U \mapsto U'$ associated with a is defined by

$$a(u,v) = \langle Au, v \rangle, \quad \forall u, v \in U.$$

Algorithm 1 (Abstract multispace BDDC) *Given spaces V_k and operators Q_k ($k = 1, \ldots, M$) such that*

$$U \subset V_1 + \cdots + V_M \subset W, \quad Q_k : V_k \to U,$$

define a preconditioner $B : r \in U' \longmapsto u \in U$ by

$$B : r \mapsto \sum_{k=1}^{M} Q_k v_k, \quad v_k \in V_k : \quad a(v_k, z_k) = \langle r, Q_k z_k \rangle, \quad \forall z_k \in V_k.$$

The following estimate can be proved from the abstract additive Schwarz theory [3]. The case when $M = 1$, which covers the existing two-level BDDC theory set in the spaces of discrete harmonic functions, was given in [8].

Lemma 1. *Assume that the subspaces V_k are energy orthogonal, the operators Q_k are projections, and*

$$\forall u \in U : u = \sum_{k=1}^{M} Q_k v_k \text{ if } u = \sum_{k=1}^{M} v_k, \ v_k \in V_k. \tag{2}$$

Then the abstract multispace BDDC preconditioner from Algorithm 1 satisfies

$$\kappa = \frac{\lambda_{\max}(BA)}{\lambda_{\min}(BA)} \leq \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2}.$$

Note that (2) is a type of decomposition of unity property.

3 BDDC for a 2D Model Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, decomposed into N nonoverlapping polygonal substructures Ω_i, $i = 1, ..., N$, which form a conforming triangulation. That is, if two substructures have a nonempty intersection, then the intersection is a vertex, or a whole edge. Let W_i be the space of Lagrangian $\mathbb{P}_1$ or $\mathbb{Q}_1$ finite element functions with characteristic mesh size h on Ω_i, and which are zero on the boundary $\partial\Omega$. Suppose that the nodes of the finite elements coincide on edges common to two substructures. Let

$$W = W_1 \times \cdots \times W_N,$$

$U \subset W$ be the subspace of functions that are continuous across the substructure interfaces, and

$$a(u,v) = \sum_{i=1}^{N} \int_{\Omega_i} \nabla u \nabla v, \quad u, v \in W.$$

We are interested in the solution of the problem (1).

Substructure vertices will be also called corners, and the values of functions from W on the corners are called *coarse degrees of freedom.* Let $\widetilde{W} \subset W$ be the space of all functions such that the values of any coarse degrees of freedom have a common value for all relevant substructures and vanish on $\partial\Omega$. Define $U_I \subset U \subset W$ as the subspace of all functions that are zero on all substructure boundaries $\partial\Omega_i$, $\widetilde{W}_\Delta \subset W$ as the subspace of all function such that their coarse degrees of freedom vanish, define $\widetilde{W}_\Pi$ as the subspace of all functions such that their coarse degrees of freedom between adjacent substructures coincide, and such that their energy is minimal. Then

$$\widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi, \quad \widetilde{W}_\Delta \perp_a \widetilde{W}_\Pi. \tag{3}$$

Functions that are a-orthogonal to U_I are called discrete harmonic. In [7] and [8], the analysis was done in spaces of discrete harmonic functions after eliminating U_I; this is not the case here, so $\widetilde{W}$ does not consist of discrete harmonic functions only.

Let $E : \widetilde{W} \to U$ be the operator defined by taking the average over the substructure interfaces.

Algorithm 2 (Original BDDC) *Define the preconditioner* $r \in U' \longmapsto u \in U$ *as follows. Compute the interior pre-correction:*

$$u_I \in U_I : a\left(u_I, z_I\right) = \langle r, z_I \rangle, \quad \forall z_I \in U_I, \tag{4}$$

updated the residual:

$$r_B \in U', \quad \langle r_B, v\rangle = \langle r, v\rangle - a\left(u_I, v\right), \quad \forall v \in U$$

compute the substructure correction and the coarse correction:

$$\begin{aligned} u_\Delta &= E w_\Delta, \quad w_\Delta \in \widetilde{W}_\Delta : a\left(w_\Delta, z_\Delta\right) = \langle r_B, E z_\Delta\rangle, \quad \forall z_\Delta \in \widetilde{W}_\Delta \\ u_\Pi &= E w_\Pi, \quad w_\Pi \in \widetilde{W}_\Pi : a\left(w_\Pi, z_\Pi\right) = \langle r_B, E z_\Pi\rangle, \quad \forall z_\Pi \in \widetilde{W}_\Pi \end{aligned} \tag{5}$$

and the interior post-correction:

$$v_I \in U_I : a\left(v_I, z_I\right) = a\left(u_\Delta + u_\Pi, z_I\right), \quad \forall z_I \in U_I.$$

Apply the interior post-correction and add the interior pre-correction:

$$u = u_I + \left(u_\Delta + u_\Pi - v_I\right). \tag{6}$$

Denote by P the energy orthogonal projection from U to U_I. Then $I - P$ is known as the projection onto discrete harmonic functions.

Lemma 2. *The original BDDC preconditioner from Algorithm 2 is the abstract multispace BDDC from Algorithm 1 with* $M = 3$ *and*

$$\begin{aligned} &V_1 = U_I, \quad V_2 = (I-P)\widetilde{W}_\Delta, \quad V_3 = (I-P)\widetilde{W}_\Pi, \\ &Q_1 = I, \quad Q_2 = Q_3\left(I-P\right)E, \end{aligned}$$

and the assumptions of Lemma 1 are satisfied.

Because of (3) and and the fact that $\|I\|_a = 1$, we only need an estimate of $\|(I - P)\, Ew\|_a$ on $\widetilde{W}$, which is well known [6].

Theorem 1. *The condition number of the original BDDC algorithm satisfies* $\kappa \leq \omega$, *where*

$$\omega = \sup_{w \in \widetilde{W}} \frac{\|(I - P)\, Ew\|_a^2}{\|w\|_a^2} \leq C \left(1 + \log \frac{H}{h}\right)^2 . \tag{7}$$

4 Multilevel BDDC and an Abstract Bound

The substructuring components from Section 3 will be denoted by an additional subscript $_1$, as Ω_1^i, $i = 1, \ldots N_1$, etc., and called level 1. The spaces and operators involved can be written concisely as a part of a hierarchy of spaces and operators:

$$\left.\begin{array}{ccccccccc}
 & & U & & & & & & \\
 & & \parallel & & & & & & \\
U_{I1} & \overset{P_1}{\underset{\subset}{\leftarrow}} & U_1 & \overset{E_1}{\underset{\subset}{\leftarrow}} & \widetilde{W}_1 & \subset & W_1 & & \\
 & & & & \parallel & & & & \\
 & \widetilde{U}_2 & \overset{I_2}{\leftarrow} & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & & & \\
 & & & \parallel & & & & & \\
 & U_{I2} & \overset{P_2}{\underset{\subset}{\leftarrow}} & U_2 & \overset{E_2}{\underset{\subset}{\leftarrow}} & \widetilde{W}_2 & \subset & W_2 & \\
 & & & & & \parallel & & & \\
 & & \widetilde{U}_3 & \overset{I_3}{\leftarrow} & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2} & & \\
 & & & & \parallel & & & & \\
 & & & & \vdots & & & & \\
 & & & & \parallel & & & & \\
 & & U_{I,L-1} & \overset{P_{L-1}}{\underset{\subset}{\leftarrow}} & U_{L-1} & \overset{E_{L-1}}{\underset{\subset}{\leftarrow}} & \widetilde{W}_{L-1} & \subset & W_{L-1} \\
 & & & & & & \parallel & & \\
 & & & \widetilde{U}_L & \overset{I_L}{\leftarrow} & \widetilde{W}_{\Pi,L-1} & \oplus & \widetilde{W}_{\Delta,L-1} &
\end{array}\right\} \tag{8}$$

We will call the coarse problem (5) the level 2 problem. It has the same finite element structure as the original problem (1) on level 1, so we have $U_2 = \widetilde{W}_{\Pi 1}$. Level 1 substructures are level 2 elements, level 1 coarse degrees of freedom are level 2 degrees of freedom. The shape functions on level 2 are the coarse basis functions in $\widetilde{W}_{\Pi 1}$, which are given by the conditions that the value of exactly one coarse degree of freedom is one and others are zero, and that they are energy minimal in $\widetilde{W}_1$. Note that the resulting shape functions on level 2 are in general discontinuous between level 2 elements. Level 2 elements are then agglomerated into nonoverlapping level 2 substructures, etc. Level k elements are level $k-1$ substructures, and the level k substructures are agglomerates of level k elements. Level k substructures are denoted by Ω_k^i, and they are assumed to form a quasiuniform conforming triangulation with characteristic substructure size H_k. The degrees of freedom of level k elements are given by level $k-1$ coarse degrees of freedom, and shape functions on level k are determined by minimization of energy on each level $k-1$ substructure separately, so $U_k = \widetilde{W}_{\Pi,k-1}$. The mapping I_k is an interpolation from the level k degrees of freedom to functions in another space $\widetilde{U}_k$. For the model problem, $\widetilde{U}_k$ will consist of functions which are (bi)linear on each Ω_k^i. The averaging operators on level k, $E_k : \widetilde{W}_k \to U_k$, are defined by averaging of the values of level k degrees of freedom between level k substructures Ω_k^i. The space U_{Ik} consists of functions in U_k that are zero on the boundaries of all level k substructures, and $P_k : U_k \to U_{Ik}$ is the $a-$orthogonal projection in U_k onto U_{Ik}. For convenience, let Ω_0^i be the original finite elements, $H_0 = h$, and $I_1 = I$.

Algorithm 3 (Multilevel BDDC) *Given $r \in U_1'$, find $u \in U_1$ by (4)–(6), where the solution coarse problem (5) is replaced by the right hand side preconditioned by the same method, applied recursively. At the coarsest level, (5) is solved by a direct method.*

Lemma 3. *The multilevel BDDC preconditioner in Algorithm 3 is the abstract multispace BDDC preconditioner (Algorithm 1) with $M = 2L - 2$ and the spaces and operators*

$$\begin{aligned}
&V_1 = U_{I1}, \quad V_2 = (I - P_1)\widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \quad V_4 = (I - P_2)\widetilde{W}_{\Delta,2}, \dots \\
&V_{2L-4} = (I - P_{L-2})\widetilde{W}_{\Delta,L-2}, \quad V_{2L-3} = U_{I,L-1}, \quad V_{2L-2} = (I - P_{L-1})\widetilde{W}_{L-1}, \\
&Q_1 = I, \quad Q_2 = Q_3 = (I - P_1)\, E_1, \dots \\
&Q_{2L-4} = Q_{2L-3} = (I - P_1)\, E_1 \cdots (I - P_{L-2})\, E_{L-2}, \\
&Q_{2L-2} = (I - P_1)\, E_1 \cdots (I - P_{L-1})\, E_{L-1},
\end{aligned}$$

satisfying the assumptions of Lemma 1.

The following bound follows from writing of multilevel BDDC as multispace BDDC in Lemma 3 and the estimate for multispace BDDC in Lemma 1.

Lemma 4. *If for some $\omega_k \geq 1$,*

$$\|(I - P_k)E_k w_k\|_a^2 \leq \omega_k \|w_k\|_a^2, \quad \forall w_k \in \widetilde{W}_k, \quad k = 1, \dots, L-1, \tag{9}$$

then the multilevel BDDC preconditioner satisfies $\kappa \leq \prod_{k=1}^{L-1} \omega_k$.

5 Multilevel BDDC Bound for the 2D Model Problem

To apply Lemma 4, we need to generalize the estimate (7) to coarse levels. From (7), it follows that for some $\widetilde{C}_k$ and all $w_k \in U_k$, $k = 1, \dots, L-1$,

$$\min_{u_{Ik} \in U_{Ik}} \|I_k E_k w_k - I_k u_{Ik}\|_a^2 \leq \widetilde{C}_k \left(1 + \log \frac{H_k}{H_{k-1}}\right)^2 \|I_k w_k\|_a^2. \tag{10}$$

Denote $|w|_{a,\Omega_k^i} = \left(\int_{\Omega_k^i} \nabla w \nabla w\right)^{1/2}$.

Lemma 5. *For all $k = 0, \dots, L-1$, $i = 1, \dots, N_k$,*

$$c_{k,1} |I_{k+1} w|_{a,\Omega_k^i}^2 \leq |w|_{a,\Omega_k^i}^2 \leq c_{k,2} |I_{k+1} w|_{a,\Omega_k^i}^2, \quad \forall w \in \widetilde{W}_{\Pi k}, \ \forall \, \Omega_k^i, \tag{11}$$

with $c_{k,2}/c_{k,1} \leq \overline{C}_k$, independently of $H_0, \dots, H_{k+1}$.

Proof. For $k = 0$, (11) holds because $I_1 = I$. Suppose that (11) holds for some $k < L-2$ and let $w \in \widetilde{W}_{\Pi,k+1}$. From the definition of $\widetilde{W}_{\Pi,k+1}$ by energy minimization,

$$|w|_{a,\Omega^i_{k+1}} = \min_{w_\Delta \in \widetilde{W}_{\Delta,k+1}} |w + w_\Delta|_{a,\Omega^i_{k+1}} . \tag{12}$$

From (12) and the induction assumption, it follows that

$$\begin{aligned} c_{k,1} &\min_{w_\Delta \in \widetilde{W}_{\Delta,k+1}} |I_{k+1}w + I_{k+1}w_\Delta|^2_{a,\Omega^i_{k+1}} \\ &\leq \min_{w_\Delta \in \widetilde{W}_{\Delta,k+1}} |w + w_\Delta|^2_{a,\Omega^i_k} \leq c_{k,2} \min_{w_\Delta \in \widetilde{W}_{\Delta,k+1}} |I_{k+1}w + I_{k+1}w_\Delta|^2_{a,\Omega^i_k} \end{aligned} \tag{13}$$

Now from [10, Lemma 4.2], applied to the piecewise linear functions of the form $I_{k+1}w$ on Ω^i_{k+1},

$$c_1 |I_{k+2}w|^2_{a,\Omega^i_{k+1}} \leq \min_{w_\Delta \in \widetilde{W}_{\Delta,k+1}} |I_{k+1}w + I_{k+1}w_\Delta|^2_{a,\Omega^i_{k+1}} \leq c_2 |I_{k+2}w|^2_{a,\Omega^i_{k+1}} \tag{14}$$

with c_2/c_1, bounded independently of $H_0, \ldots, H_{k+1}$. Then (12), (13) and (14) imply (11) with $\overline{C}_k = \overline{C}_{k-1}c_2/c_1$.

Theorem 2. *The multilevel BDDC with for the model problem with corner corner coarse degrees of freedom satisfies the condition number estimate*

$$\kappa \leq \textstyle\prod_{k=1}^{L-1} C_k \left(1 + \log \dfrac{H_k}{H_{k-1}}\right)^2 .$$

Proof. By summation of (11), we have

$$c_{k,1} \|I_k w\|^2_a \leq \|w\|^2_a \leq c_{k,2} \|I_k w\|^2_a , \quad \forall w \in U_k,$$

with $c_{k,2}/c_{k,1} \leq \overline{C}_k$, so from (10),

$$\|(I - P_k)E_k w_k\|^2_a \leq C_k \left(1 + \log \frac{H_k}{H_{k-1}}\right)^2 \|w_k\|^2_a , \quad \forall w_k \in \widetilde{W}_k,$$

with $C_k = \overline{C}_k \widetilde{C}_k$. It remains to use Lemma 4.

For $L = 3$ we recover the estimate by [10]. In the case of uniform coarsening, i.e. with $H_k/H_{k-1} = H/h$ and the same geometry of decomposition on all levels $k = 1, \ldots L-1$, we get

$$\kappa \leq C^{L-1} (1 + \log H/h)^{2(L-1)} . \tag{15}$$

6 Numerical Examples and Conclusion

A multilevel BDDC preconditioner was implemented in Matlab for the 2D Laplace equation on a square domain with periodic boundary conditions. For these boundary conditions, all subdomains at each level are identical and it is possible to solve very large problems on a single processor. The periodic boundary conditions result in a

Table 1. 2D Laplace equation results for $H/h = 2$. The number of levels is Nlev (Nlev = 2 for the standard approach), the number iterations is iter, the condition number estimate is κ, and the total number of degrees of freedom is ndof.

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	2	1.5625	1	1	16
3	8	1.8002	5	1.1433	64
4	11	2.4046	7	1.2703	256
5	14	3.4234	8	1.3949	1,024
6	17	4.9657	9	1.5199	4,096
7	20	7.2428	9	1.6435	16,384
8	25	10.5886	10	1.7696	65,536

Table 2. 2D Laplace equation results for $H/h = 4$.

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	9	2.1997	6	1.1431	256
3	14	4.0220	8	1.5114	4,096
4	21	7.7736	10	1.8971	65,536
5	30	15.1699	12	2.2721	1,048,576

Table 3. 2D Laplace equation results for $H/h = 8$.

Nlev	corners only		corners and faces		ndof
	iter	κ	iter	κ	
2	14	3.1348	7	1.3235	4,096
3	23	7.8439	10	2.0174	262,144
4	36	19.9648	13	2.7450	16,777,216

stiffness matrix with a single zero eigenvalue, but this situation can be accommodated in preconditioned conjugate gradients by removing the mean from the right hand side of $Ax = b$. The coarse grid correction at each level is replaced by the BDDC preconditioned coarse residual.

Numerical results are in Tables 1-3. As predicted by Theorem 2, the condition number grows slowly in the ratios of mesh sizes for a fixed number of levels L. However, for fixed H_i/H_{i-1} the growth of the condition number is seen to be exponential in L. With additional constraints by side averages, the condition number is seen to grow linearly. Our explanation is that a bound similar to Theorem 2 still applies, though possibly with (much) smaller constants, so the exponential growth of the condition number is no longer apparent.

References

[1] S.C. Brenner and L.-Y. Sung. BDDC and FETI-DP without matrices or vectors. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1429–1435, 2007.

[2] C.R. Dohrmann. A preconditioner for substructuring based on constrained energy minimization. *SIAM J. Sci. Comput.*, 25(1):246–258, 2003.

[3] M. Dryja and O.B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. *Comm. Pure Appl. Math.*, 48(2):121–155, 1995.

[4] J. Li and O.B. Widlund. FETI-DP, BDDC, and block Cholesky methods. *Internat. J. Numer. Methods Engrg.*, 66(2):250–271, 2006.

[5] J. Mandel. Balancing domain decomposition. *Comm. Numer. Methods Engrg.*, 9(3):233–241, 1993.

[6] J. Mandel and C.R. Dohrmann. Convergence of a balancing domain decomposition by constraints and energy minimization. *Numer. Linear Algebra Appl.*, 10(7):639–659, 2003.

[7] J. Mandel, C.R. Dohrmann, and R. Tezaur. An algebraic theory for primal and dual substructuring methods by constraints. *Appl. Numer. Math.*, 54(2):167–193, 2005.

[8] J. Mandel and B. Sousedík. Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods. *Comput. Methods Appl. Mech. Engrg.*, 196(8):1389–1399, 2007.

[9] X. Tu. Three-level BDDC in three dimensions. To appear in *SIAM J. Sci. Comput.*, 2007.

[10] X. Tu. Three-level BDDC in two dimensions. *Internat. J. Numer. Methods Engrg.*, 69(1):33–59, 2007.