Optimized Domain Decomposition Methods for Three-dimensional Partial Differential Equations

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Summary. Optimized Schwarz methods (OSM) have shown to be an efficient iterative solver and preconditioner in solving partial differential equations. Different investigations have been devoted to study optimized Schwarz methods and many applications have shown their great performance compared to the classical Schwarz methods. By simply making slight modifications of transmission conditions between subdomains, and without changing the size of the matrix, we obtain a fast and a robust family of methods. In this paper we give an extension of optimized Schwarz methods to cover three-dimensional partial differential equations. We present the asymptotic behaviors of optimal and optimized Schwarz methods and compare it to the performance of the classical Schwarz methods. We confirm the obtained theoretical results with numerical experiments.

1 Introduction

The classical Schwarz algorithm has a long history. In 1869, Jacob Schwarz introduced an alternating procedure to prove existence and uniqueness of solutions to Laplace's equation on irregular domains. More than a century later the Schwarz method was used as a computational method in [9]. The advent of computers with parallel architecture give a wide popularity to this method. Recently, [6, 7] gives a mathematical analysis of the Schwarz alternating method at the continuous level and presented different versions of the method, including the extension to many subdomains decomposition. The method was investigated as a preconditioner for discretized problems in [2]. The convergence properties of the classical Schwarz methods are well understood for a wide variety of problems, see e.g., [12, 11]. Recently a new class of Schwarz methods know as optimized Schwarz methods have been introduced to enhance the convergence properties of the classical Schwarz methods. They converge uniformly faster than the classical Schwarz methods due to the exchange of solution and its derivatives between subdomains. Many studies have been devoted to OSM more specifically in $1d$ and $2d$ spaces, see e.g., [5, 3]. A convergence analysis of OSM was done in [4], where a uniform convergence independently of the mesh parameter h has been proved. Those methods have been investigated for problems with discontinuity and anisotropy, see e.g., [8], they were also analyzed for systems of PDE's see [1]. Some industrial applications of OSM in the domain of weather predictions are shown in [10]. For a comparison of OSM with modern DDM like direct Schur methods, FETI and their variants see, e.g. [3, 8]. In this paper we give an extension of OSM to three-dimensional partial differential equations.

2 The Classical Schwarz Method

Throughout this paper we consider the following model problem

$$L(u) = (\eta - \Delta)(u) = f, \qquad \text{in} \quad \Omega = \mathbb{R}^3, \quad \eta > 0, \tag{1}$$

where we require the solution to be bounded at the infinity. We decompose Ω into $\Omega_1 = (-\infty, \ell) \times \mathbb{R}^2$, and $\Omega_2 = (0, \infty) \times \mathbb{R}^2$, where $\ell \geq 0$ is the size of the overlap. The Jacobi Schwarz method on this decomposition is given by

$$\begin{array}{ll} Lu_1^n = f, \text{ in } \Omega_1, & u_1^n(\ell, y, z) = u_2^{n-1}(\ell, y, z), \\ Lu_2^n = f, \text{ in } \Omega_2, & u_2^n(0, y, z) = u_1^{n-1}(0, y, z). \end{array} \tag{2}$$

By linearity we consider only the case $f = 0$ and analyze convergence to the zero solution. Taking a Fourier transform of the Schwarz algorithm (2) in y and z directions, we obtain

$$\begin{array}{ll} (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_1^n = 0, \ x < \ell, \ k \in \mathbb{R}, \ m \in \mathbb{R}, & \hat{u}_1^n(\ell, k, m) = \hat{u}_2^{n-1}(\ell, k, m), \\ (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_2^n = 0, \ x > 0, \ k \in \mathbb{R}, \ m \in \mathbb{R}, & \hat{u}_2^n(0, k, m) = \hat{u}_1^{n-1}(0, k, m), \end{array}$$

where k and m are the frequencies in y and z directions, respectively. Therefore the solutions in the Fourier domain take the form

$$\hat{u}_j^n(x, k, m) = A_j(k, m)e^{\lambda_1(k,m)x} + B_j(k, m)e^{\lambda_2(k,m)x}, \qquad j = 1, 2, \tag{3}$$

where $\lambda_1(k, m) = \kappa$ and $\lambda_2(k, m) = -\kappa$, with $\kappa = \sqrt{\eta + k^2 + m^2}$. Due to the condition on the iterates at the infinity and using transmission conditions, we find that

$$\hat{u}_1^{2n}(0, k, m) = e^{-2\ell\kappa}\hat{u}_1^0(0, k, m) \quad \text{and} \quad \hat{u}_2^{2n}(\ell, k, m) = e^{-2\ell\kappa}\hat{u}_2^0(\ell, k, m). \tag{4}$$

Thus the convergence factor of the classical Schwarz method is given by

$$\rho_{cla} = \rho_{cla}(\eta, k, m, \ell) := e^{-2\ell\kappa} \leq 1, \quad \forall k \in \mathbb{R}, \quad \forall m \in \mathbb{R}. \tag{5}$$

The convergence factor depends on the problem parameter η, the size of the overlap ℓ and on k and m. Figure 1 on the left shows the dependence of the convergence factor on k and m for an overlap $\ell = \frac{1}{100}$ and $\eta = 1$. This shows that the classical Schwarz method damp efficiently high frequencies, whereas for low frequencies the algorithm is very slow.

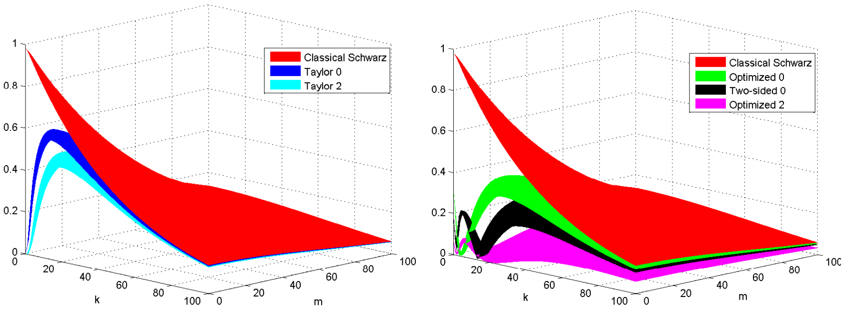

Fig. 1. Left: The convergence factor ρ_{cla} compared to ρ_{T0} and ρ_{T2}. **Right:** The convergence factor ρ_{cla} compared to ρ_{OO0} and ρ_{OO2} and to the convergence factor of two-sided optimized Robin method.

3 The Optimal Schwarz Method

We introduce the following modified algorithm

$$\begin{array}{ll} L(u_1^n) = f, \text{ in } \Omega_1, & (S_1+\partial_x)(u_1^n)(\ell,.,.) = (S_1+\partial_x)(u_2^{n-1})(\ell,.,.), \\ L(u_2^n) = f, \text{ in } \Omega_2, & (S_2+\partial_x)(u_2^n)(0,.,.) = (S_2+\partial_x)(u_1^{n-1})(0,.,.), \end{array} \tag{6}$$

where S_j, $j=1,2$, are linear operators along the interface that depend on y and z. As for the classical Schwarz method it suffices by linearity to consider the case $f=0$. Taking a Fourier transform of the new algorithm (6), we obtain

$$\begin{array}{r} (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_1^n = 0, \quad x<\ell,\ k\in\mathbb{R},\ m\in\mathbb{R}, \\ (\sigma_1(k,m)+\partial_x)(\hat{u}_1^n)(\ell,k,m) = (\sigma_1(k,m)+\partial_x)(\hat{u}_2^{n-1})(\ell,k,m), \end{array} \tag{7}$$

$$\begin{array}{r} (\eta + k^2 + m^2 - \partial_{xx})\hat{u}_2^n = 0, \quad x>0,\ k\in\mathbb{R},\ m\in\mathbb{R}, \\ (\sigma_2(k,m)+\partial_x)(\hat{u}_2^n)(0,k,m) = (\sigma_2(k,m)+\partial_x)(\hat{u}_1^{n-1})(0,k,m), \end{array} \tag{8}$$

where $\sigma_j(k,m)$ is the symbol of the operator $S_j(y,z)$. We proceed as in the case of the classical Schwarz method and using transmission conditions, we obtain

$$\hat{u}_1^{2n}(0,k,m) = \frac{\sigma_1(k,m)-\kappa}{\sigma_1(k,m)+\kappa}\cdot\frac{\sigma_2(k,m)+\kappa}{\sigma_2(k,m)-\kappa}e^{-2\ell\kappa}\hat{u}_1^0(0,k,m). \tag{9}$$

Defining the new convergence factor ρ_{opt} by

$$\rho_{opt} = \rho_{opt}(\eta,k,m,\ell,\sigma_1,\sigma_2) := \frac{\sigma_1(k,m)-\kappa}{\sigma_1(k,m)+\kappa}\cdot\frac{\sigma_2(k,m)+\kappa}{\sigma_2(k,m)-\kappa}e^{-2\ell\kappa}. \tag{10}$$

We compare the convergence factor $\rho_{opt}(\eta,k,m,\ell,\sigma_1,\sigma_2)$ with the one of the classical Schwarz method given in (5), and one can see that they differ only by the factor in front of the exponential term. Choosing for the symbols

$$\sigma_1(k,m) := \kappa \qquad \text{and} \quad \sigma_2(k,m) := -\kappa, \tag{11}$$

the new convergence factor vanishes identically, $\rho_{opt}\equiv 0$, and the algorithm converges in two iterations, independently of the initial guess, the overlap size ℓ and the problem parameter η. This is an optimal result since convergence in less than two iterations is impossible, due to the exchange information necessity between the subdomains. Furthermore, with this choice of σ_j the exponential factor in the convergence factor becomes irrelevant and one can have Schwarz methods without overlap. In practice we need to back transform the transmission conditions with σ_1 and σ_2 from the Fourier domain to the physical domain to obtain S_1 and S_2. The fact that σ_j contain a square-root, the optimal operators S_j are non-local operators. In the next section we will approximate σ_j by polynomials in ik and im, so S_j would consist of derivatives in y and z and thus be local operators.

4 Optimized Schwarz Methods

We approximate the symbols $\sigma_j(k,m)$ found in (11) as follows

$$\sigma_1^{app}(k,m) = p_1 + q_1(k^2+m^2) \quad \text{and} \quad \sigma_2^{app}(k,m) = -p_2 - q_2(k^2+m^2). \tag{12}$$

Hence the convergence factor (10) of the optimized Schwarz methods becomes

$$\rho = \rho(\eta, k, m, \ell, p_1, p_2, q_1, q_2) := \frac{\kappa - p_1 - q_1(k^2+m^2)}{\kappa + p_1 + q_1(k^2+m^2)} \cdot \frac{\kappa - p_2 - q_2(k^2+m^2)}{\kappa + p_2 + q_2(k^2+m^2)} e^{-2\ell\kappa}. \tag{13}$$

Theorem 1. *The optimized Schwarz method (6) with transmission conditions defined by the symbols (12) converges for $p_j > 0$, $q_j \geq 0$, $j = 1, 2$, faster than the classical Schwarz method (2), $|\rho| < |\rho_{cla}|$ for all k and m.*

Proof. The absolute value of the term in front of the exponential in the convergence factor (13) of the optimized Schwarz method is strictly smaller than 1 provided $p_j > 0$, and $q_j \geq 0$ which shows that $|\rho| < |\rho_{cla}|$ for all k and m.

Now, we introduce a low frequency approximations using a Taylor expansions about zero. Expanding the symbols $\sigma_j(k,m)$, $j = 1, 2$, we obtain

$$\begin{aligned} \sigma_1(k,m) &= \sqrt{\eta} + \tfrac{1}{2\sqrt{\eta}}(k^2+m^2) + \mathcal{O}_1(k^4, m^4), \\ \sigma_2(k,m) &= -\sqrt{\eta} - \tfrac{1}{2\sqrt{\eta}}(k^2+m^2) + \mathcal{O}_2(k^4, m^4), \end{aligned} \tag{14}$$

where $\mathcal{O}_1(k^4, m^4)$ and $\mathcal{O}_2(k^4, m^4)$ contain high order terms in m and k. The convergence factor ρ_{T0} of the zeroth order Taylor approximation is defined by

$$\rho_{T0}(\eta, k, m, \ell) = \left(\frac{\kappa - \sqrt{\eta}}{\kappa + \sqrt{\eta}}\right)^2 e^{-2\ell\kappa}, \tag{15}$$

and the convergence factor ρ_{T2} of the second order Taylor approximation would have the form

$$\rho_{T2}(\eta, k, m, \ell) = \left(\frac{\kappa - \sqrt{\eta} - \frac{1}{2\sqrt{\eta}}(k^2+m^2)}{\kappa + \sqrt{\eta} + \frac{1}{2\sqrt{\eta}}(k^2+m^2)}\right)^2 e^{-2\ell\kappa}. \tag{16}$$

Figure 1 on the left shows the convergence factors obtained with this choice of transmission conditions compared to the convergence factor ρ_{cla}. One can clearly see that OSM are uniformly better than the classical Schwarz method, in particular the low frequency behavior is greatly improved. Note that OSM converge even without overlap. In particular, we have the following theorem.

Theorem 2. *The optimized Schwarz methods with Taylor transmission conditions and overlap ℓ have an asymptotically superior performance than the classical Schwarz method with the same overlap. As ℓ goes to zero, we have*

$$\max_{|k|\leq\frac{\pi}{\ell},|m|\leq\frac{\pi}{\ell}} |\rho_{cla}(\eta,k,m,\ell)| = 1-2\sqrt{\eta}\ell+\mathcal{O}(\ell^2),$$

$$\max_{|k|\leq\frac{\pi}{\ell},|m|\leq\frac{\pi}{\ell}} |\rho_{T0}(\eta,k,m,\ell)| = 1-4\sqrt{2}\eta^{1/4}\sqrt{\ell}+\mathcal{O}(\ell),$$

$$\max_{|k|\leq\frac{\pi}{\ell},|m|\leq\frac{\pi}{\ell}} |\rho_{T2}(\eta,k,m,\ell)| = 1-8\eta^{1/4}\sqrt{\ell}+\mathcal{O}(\ell).$$

Without overlap, the optimized Schwarz methods with Taylor transmission conditions are asymptotically comparable to the classical Schwarz method with overlap ℓ. As ℓ goes to zero, we have

$$\max_{|k|\leq\frac{\pi}{\ell},|m|\leq\frac{\pi}{\ell}} |\rho_{T0}(\eta,k,m,0)| = 1-4\frac{\sqrt{\eta}}{\pi}\ell+\mathcal{O}(\ell^2),$$

$$\max_{|k|\leq\frac{\pi}{\ell},|m|\leq\frac{\pi}{\ell}} |\rho_{T2}(\eta,k,m,0)| = 1-8\frac{\sqrt{\eta}}{\pi}\ell+\mathcal{O}(\ell^2).$$

Proof. The proof is based on a Taylor expansion of the convergence factors, where we estimate the maximum frequency by π/ℓ.

Zeroth Order Optimized Transmission Conditions

Using the same zeroth order transmission conditions on both sides of the interface, $p_1 = p_2 = p$ and $q_1 = q_2 = 0$, the convergence factor in (13) becomes

$$\rho_{OO0}(\eta,k,m,\ell,p) := \left(\frac{\kappa-p}{\kappa+p}\right)^2 e^{-2\kappa\ell}. \tag{17}$$

To find the optimal parameter p^* of the associated Schwarz method, known as Optimized of Order 0 ($OO0$), we need to solve the following min-max problem

$$\min_{p\geq 0}(\max_{k,m}|\rho_{OO0}(\eta,k,m,\ell,p)|) = \min_{p\geq 0}\left(\max_{k,m}\left(\frac{\kappa-p}{\kappa+p}\right)^2 e^{-2\kappa\ell}\right). \tag{18}$$

We introduce the minimum and the maximum frequencies $f_{\min}$ and $f_{\max}$ of all the frequencies k and m. The asymptotic performance of the Optimized zeroth order Schwarz method is given by the next theorem, where we omit the proof due to the restriction on the present paper.

Theorem 3. *(Robin asymptotic)*
The asymptotic performance of the Schwarz method with optimized Robin transmission conditions and overlap ℓ, as ℓ goes to zero, is given by

$$\max_{\substack{k,m \\ f_{\min}\leq\sqrt{k^2+m^2}\leq\frac{\pi}{\ell}}} |\rho_{OO0}(\eta,k,m,\ell,p^*)| = 1-4.2^{1/6}(f_{\min}^2+\eta)^{1/6}\ell^{1/3}+\mathcal{O}(\ell^{2/3}). \tag{19}$$

The asymptotic performance of $OO0$ without overlap is asymptotically equivalent to the classical Schwarz method with overlap ℓ, as ℓ goes to zero, we have

$$\max_{\substack{k,m\\ f_{\min}\le\sqrt{k^2+m^2}\le\frac{\pi}{\ell}}} |\rho_{OO0}(\eta,k,m,0,p^*)| = 1-4\frac{(f_{\min}^2+\eta)^{1/4}}{\sqrt{\pi}}\sqrt{\ell}+\mathcal{O}(\ell). \quad (20)$$

Proof. The idea of the proof in the case of overlapping subdomains is based on the ansatz $p^* = C\ell^{\alpha}$, where $\alpha < 0$ and Taylor expansion of the convergence factor with $p = p^*$. A computation shows that $p^* = \frac{(4(f_{min}^2+\eta))^{1/3}}{2}\ell^{-1/3}$.

Second Order Optimized Transmission Conditions

Using the same second order transmission conditions on both sides of the interface, $p_1 = p_2 = p$ and $q_1 = q_2 = q$, the expression (13) of the convergence factor simplifies to

$$\rho_{OO2}(\eta,k,m,\ell,p,q) = \left(\frac{\kappa-p-q(k^2+m^2)}{\kappa+p+q(k^2+m^2)}\right)^2 e^{-2\kappa\ell}. \quad (21)$$

To determine the optimal parameters p^* and q^* for OSM of Order 2 ($OO2$), we need to solve the min-max problem

$$\min_{p,q\ge 0}(\max_{k,m}|\rho_{OO2}(\eta,k,m,\ell,p,q)|) = \min_{p,q\ge 0}\left(\max_{k,m}\left(\frac{\kappa-p-q(k^2+m^2)}{\kappa+p+q(k^2+m^2)}\right)^2 e^{-2\kappa\ell}\right). \quad (22)$$

We have the following.

Theorem 4. *(Second order)*
The asymptotic performance of the Schwarz method with optimized second order transmission conditions and overlap ℓ, as ℓ goes to zero, is given by

$$\max_{\substack{k,m\\ f_{\min}\le\sqrt{k^2+m^2}\le f_{\max}}} |\rho_{OO2}(\eta,k,m,\ell,p^*,q^*)| = 1-4.2^{3/5}(f_{\min}^2+\eta)^{1/10}\ell^{1/5}+\mathcal{O}(\ell^{2/5}). \quad (23)$$

The asymptotic performance of $OO2$ without overlap is equivalent to the classical Schwarz with overlap ℓ. As ℓ approaches zero, we obtain

$$\max_{\substack{k,m\\ f_{\min}\le\sqrt{k^2+m^2}\le f_{\max}}} |\rho_{OO2}(\eta,k,m,0,p^*,q^*)| = 1-4\frac{\sqrt{2}(f_{\min}^2+\eta)^{1/8}}{\pi^{1/4}}\ell^{1/4}+\mathcal{O}(\ell^{1/2}). \quad (24)$$

Proof. We do a Taylor expansion of the convergence factor with $p^* = C_1\ell^{\alpha}$ and $q^* = C_2\ell^{\beta}$, where $\alpha < 0$ and $\beta > 0$, we show that $p^* = 2^{-3/5}(f_{min}^2+\eta)^{2/5}\ell^{-1/5}$ and $q^* = 2^{-1/5}(f_{min}^2+\eta)^{-1/5}\ell^{3/5}$.

Figure 1 on the right shows a comparison of the convergence factors of the optimized Schwarz methods with the classical Schwarz method. We also compare the convergence factor of the classical Schwarz method with the convergence factor of the two-sided optimized Schwarz method, where we use different Robin transmission conditions between the two subdomains. As one can see the optimized Schwarz methods have a great performance compared to the classical Schwarz method.

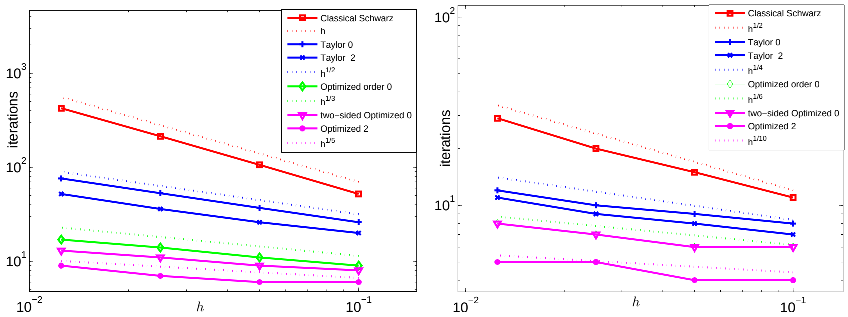

Fig. 2. Number of iterations required by the classical and the optimized Schwarz methods, with overlap $\ell = h$. On the left the methods are used as iterative solvers, and on the right as preconditioners for a Krylov method.

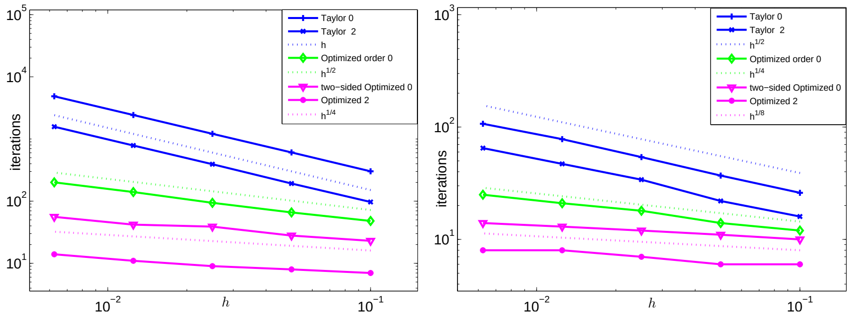

Fig. 3. Number of iterations required by the optimized Schwarz methods without overlap between subdomains. On the left the methods are used as iterative solvers, and on the right as preconditioners for a Krylov method.

5 Numerical Experiments

We perform numerical experiments for our model problem (1) on the unit cube, $\Omega = (0,1)^3$. We decompose the unit cube Ω into two subdomains $\Omega_1 = (0,b) \times (0,1)^2$ and $\Omega_2 = (a,1) \times (0,1)^2$, where $0 < a \leq b < 1$, so that the overlap is $\ell = b - a$. We use a finite difference discretization with the classical seven-point discretization and a uniform mesh parameter h. In practice, we usually use a small overlap between subdomains, in our experiments we chose the overlap ℓ to be exactly the mesh parameter h, i.e., $\ell = h$. Figure 2 on the left shows the number of iterations versus the mesh parameter h in the case of an overlap, for all the methods used as an iterative solvers, on the right the methods are used as preconditioners for a Krylov method. In figure 3 we show the number of iterations in the case of non-overlapping subdomains. On the left the methods are used as iterative solvers, whilst on the right the methods are used as preconditioners for a Krylov method. For both decompositions the numerical results show the asymptotic behavior predicted by the analysis.

6 Conclusion

In this paper we presented an extension of the optimal and optimized Schwarz methods to cover three-dimensional partial differential equations. We showed the impact of transmission conditions on the convergence factor of Classical Schwarz method. We also showed theoretically and numerically that the optimized Schwarz methods are fast and have a great improved performance compared to the classical Schwarz method.

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