Domain Decomposition Algorithms for Mortar Discretizations

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Summary. Mortar discretizations have been developed for coupling different approximations in different subdomains, that can arise from engineering applications in complicated structures with highly non-uniform materials. The complexity of the mortar discretizations requires fast algorithms for solving the resulting linear systems. Several domain decomposition algorithms, that have been successfully applied to conforming finite element discretizations, have been extended to the linear systems of mortar discretizations. They are overlapping Schwarz methods, FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) methods, and BDDC (Balancing Domain Decomposition by Constraints) methods. The new result is that complete analysis, providing the optimal condition number estimate, has been done for geometrically non-conforming subdomain partitions and for problems with discontinuous coefficients. These algorithms were further applied to the two-dimensional Stokes and three-dimensional elasticity. In addition, a BDDC algorithm with an inexact coarse problem was developed.

1 Introduction

Mortar discretizations were introduced in [2] to couple different approximations in different subdomains so as to obtain a good global approximate solution. They are useful for modeling multi-physics, adaptivity, and mesh generation for three dimensional complex structures. The coupling is done by enforcing certain constraints on solutions across the subdomain interface using Lagrange multipliers. We call these constraints the mortar matching conditions.

The complexity of the discretizations requires fast algorithms for solving the resulting linear systems. We focus on extension of several domain decomposition algorithms, that have been successfully applied to conforming finite element discretizations, to solving such linear systems. They are overlapping Schwarz methods, FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) methods, and BDDC (Balancing Domain Decomposition by Constraints) methods, see Section 3 of [19], [5, 4], and [16, 17].

The new result is that complete analysis, providing the optimal condition number bound, was done for geometrically non-conforming subdomain partitions and for problems with discontinuous coefficients. These algorithms are further extended to the Stokes problem and three-dimensional elasticity. In addition, using an inexact solver for the coarse problem the BDDC method was extended to a three–level algorithm.

Throughout this paper, h_i and H_i denote the mesh size and the subdomain diameter, and C is a generic positive constant independent of the mesh parameters and problem coefficients.

2 Mortar Discretization

We consider a model elliptic problem,

$$\begin{aligned} -\nabla \cdot (\rho(x)\nabla u(x)) = f(x), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a polyhedral domain in $\mathbb{R}^3$, $f(x)$ is a square integrable function in Ω, $\rho(x)$ is a positive and bounded function. We decompose Ω into a non-overlapping subdomain partition $\{\Omega_i\}_i$, that can be geometrically non-conforming. In a geometrically non-conforming partition, a subdomain can intersect its neighbors in a part of a face, a part of an edge, or a vertex. This allows a subdomain partition that is not necessarily a triangulation of Ω. We then introduce a triangulation $\mathcal{T}_i$ to each subdomain Ω_i and denote by X_i the conforming piecewise linear finite element space associated to the triangulation $\mathcal{T}_i$. These triangulations can be non matching across the subdomain interface $\Gamma = \bigcup_{i,j}(\partial\Omega_i \cap \partial\Omega_j)$. We can select a set of subdomain faces of which union covers Γ, see [18, Section 4.1]. We then denote those faces $\{F_n\}_n$ and call them nonmortar faces.

A subdomain Ω_i, with a nonmortar face F_n as its face, can intersect more than one neighbors $\{\Omega_j\}_j$ through F_n. This gives a partition $\{F_{n(i,j)}\}_j$ to F_n, where $F_{n(i,j)}$ is the common part of Ω_i and Ω_j. We call $F_{n(i,j)}$ mortar faces. We note that the mortar faces can be only part of subdomain faces while nonmortar faces are a full subdomain face. On each nonmortar $F_n \subset \Omega$, we introduce a Lagrange multiplier space $M(F_n)$ based on the finite element space X_i, see [2, 22, 6] for the detailed construction.

We define a product space

$$X = \prod_i X_i,$$

and introduce a mortar matching condition on $(v_1, \cdots, v_N) \in X$

$$\int_{F_n} (v_i - \phi)\psi \, ds = 0, \quad \forall \psi \in M(F_n), \tag{2}$$

where $\phi = v_j$ on $F_{n(i,j)} \subset F_n$. A mortar finite element space is defined by

$$\widehat{X} = \{v \in X \,:\, v \text{ satisfies } (2)\}\,,$$

and mortar discretization is to approximate the solution u of (1) in the mortar finite element space $\widehat{X}$. The approximation error is given by

$$\sum_{i=1}^{N} \|u - u^h\|^2_{H^1(\Omega_i)} \le C \sum_{i=1}^{N} h_i^2 |\log(h_i)| \|u\|^2_{H^2(\Omega_i)},$$

where u^h is the approximate solution, see [1]. The additional log factor does not appear when the subdomain partition is geometrically conforming.

3 An Overlapping Schwarz Method

To build an overlapping Schwarz preconditioner, we introduce two auxiliary partitions of Ω. They are an overlapping subregion partition $\{\widetilde{\Omega}_j\}_j$ and a coarse triangulation $\{T_k\}_k$ of Ω.

For each subregion $\widetilde{\Omega}_j$, we introduce a finite element space $\widetilde{X}_j$ as a subspace of $\widehat{X}$ in the following way. Among the nodes in the finite element space X, we define by genuine unknowns the nodes that are not contained in the interior of the nonmortar faces. The space $\widetilde{X}_j$ is then built from the basis functions of each genuine unknowns, that are supported in $\widetilde{\Omega}_j$. By assigning values of the basis functions at the nodes on the nonmortar faces using the mortar matching condition (2), we can obtain the resulting basis elements contained in $\widehat{X}$.

Similarly, we can construct a coarse finite element space $\widetilde{X}_0$ that belongs to $\widehat{X}$. Let X^H be the piecewise linear conforming finite element space associated to the coarse triangulation $\{T_k\}_k$. First we interpolate a function $v \in X^H$ to the produce space X using the nodal interpolant $I^h : X^H \to X$ such that

$$I^h(v) = (I_1^h(v), \cdots, I_N^h(v)),$$

where $I_i^h(v)$ denote the nodal interpolant of v to the space X_i. We then modify values of $I^h(v)$ at the nodes on the nonmortar faces using the mortar matching condition so that obtain the resulting interpolant $I^m(v)$ contained in $\widehat{X}$. The coarse finite element space $\widetilde{X}_0$ is given by

$$\widetilde{X}_0 = I^m(X^H) \subset \widehat{X}.$$

The two–level overlapping Schwarz algorithm consists of solving the local and coarse (when $j = 0$) problems; find $T_j u \in \widetilde{X}_j$ such that

$$a(T_j u, v_j) = a(u, v_j), \quad \forall v_j \in \widetilde{X}_j \; (j \ge 0).$$

For the overlapping Schwarz algorithm applied to the mortar discretization of the elliptic problem (1), we proved the condition number estimate, see [13].

Theorem 1. *We assume that the diameter of Ω_i is comparable to any coarse triangle T_k that intersects Ω_i and the diameter H_i of Ω_i satisfy $H_i \leq C\widetilde{H}_j$, where $\widetilde{H}_j$ is the diameter of subregion $\widetilde{\Omega}_j$ that intersects Ω_i. In addition, we assume that the mesh sizes of subdomains that intersect along a common face are comparable. We then obtain the condition number bound for the overlapping Schwarz algorithm,*

$$\kappa(\sum_{j=0}^{J} T_j) \leq C \max_{j,k} \left\{ \left(1 + \frac{\widetilde{H}_j}{\delta_j}\right) \left(1 + \log \frac{H_k}{h_k}\right) \right\},$$

where δ_j are the overlapping width of the subregion partition $\{\widetilde{\Omega}_j\}_j$, and H_k/h_k denote the number of nodes across subdomain Ω_k.

4 BDDC and FETI–DP Algorithms

In this section, we construct BDDC and FETI–DP algorithms for the mortar discretization. We first derive the primal form of the mortar discretization and then introduce a BDDC algorithm for solving the primal form. Secondly we introduce the dual form and build a FETI–DP algorithm that is closely related to the BDDC algorithm.

We separate unknowns in the finite element space X_i into interior and interface unknowns and after selecting appropriate primal unknowns among the interface unknowns we again decompose the interface unknowns into dual and primal unknowns,

$$X_i = X_I^{(i)} \times X_\Gamma^{(i)}, \quad X_\Gamma^{(i)} = W_\Delta^{(i)} \times W_\Pi^{(i)}, \tag{3}$$

where I, Γ, Δ, and Π denote the interior, interface, dual, and primal unknowns, respectively.

The primal unknowns are related to certain primal constraints selected from the mortar matching condition (2). They result in a coarse component of the BDDC preconditioner so that a proper selection of such constraints is important in obtaining a scalable BDDC algorithm. We consider $\{\psi_{ij,k}\}_k$, the basis functions in $M(F_n)$ that are supported in $\overline{F}_{n(i,j)}$, and introduce

$$\psi_{ij} = \sum_k \psi_{ij,k}.$$

We assume that at least one such basis function $\psi_{ij,k}$ exists for each $F_{n(i,j)} \subset F_n$. On each interface $F_{n(i,j)}$, we select the primal constraints for $(w_1, \cdots, w_N) \in X_\Gamma (= \prod_i X_\Gamma^{(i)})$ as

$$\int_{F_{n(i,j)}} (w_i - w_j)\psi_{ij}\, ds = 0. \tag{4}$$

For the case of a geometrically conforming partition, i.e., when $F_{n(i,j)}$ is a full face of two subdomains, the above constraints are the face average matching condition because $\psi_{ij} = 1$. We can change the variables to make the primal constraints explicit, see [14, Sec 6.2], [15, Sec 2.3], and [9, Sec. 2.2]. We then separate the unknowns in the space X_i as described in (3). We will also assume that all the matrices and vectors are written in terms of the new variables.

Throughout this paper, we use the notation V for the product space of local finite element spaces $V^{(i)}$. The same applies to the vector notations v and $v^{(i)}$. In addition, we use the notation $\widehat{V}$ for the subspace of V satisfying mortar matching condition (2) and the notation $\widetilde{V}$ for the subspace satisfying only the primal constraints (4). For example, we can represent the space

$$\widetilde{X}_\Gamma = \{w \in X_\Gamma \;:\; w \text{ satisfies the primal constraints (4)}\},$$

in the following way,

$$\widetilde{X}_\Gamma = W_\Delta \times \widehat{W}_\Pi.$$

We further decompose the dual unknowns into the part interior to the nonmortar faces and the other part to obtain

$$W_\Delta = W_{\Delta,n} \times W_{\Delta,m},$$

where n and m denote unknowns at nonmortar faces (open) and the other unknowns, respectively.

After enforcing the mortar matching condition (2) on functions in the space $\widetilde{X}_\Gamma$, we obtain the matrix representation,

$$B_n w_n + B_m w_m + B_\Pi w_\Pi = 0. \tag{5}$$

Here we enforced the mortar matching condition using a reduced Lagrange multiplier space, since the functions in the space $\widetilde{X}_\Gamma$ satisfy the primal constraints selected from the mortar matching condition (2). The reduced Lagrange multiplier space is obtained after eliminating one basis element among $\{\psi_{ij,k}\}_k$ for each $F_{ij} \subset F_l$ so that the matrix B_n in (5) is invertible. Therefore the unknowns w_n of the nonmortar part are determined by the other unknowns, w_m, and w_Π, which are called genuine unknowns. We define the space of genuine unknowns by

$$W_G = W_{\Delta,m} \times \widehat{W}_\Pi$$

and define the mortar map,

$$\widetilde{R}_\Gamma = \begin{pmatrix} -B_n^{-1} B_m & -B_n^{-1} B_\Pi \\ I & 0 \\ 0 & I \end{pmatrix}, \tag{6}$$

that maps the genuine unknowns in W_G into the unknowns in $\widetilde{X}_\Gamma$ which satisfy the mortar matching condition.

To derive the linear system of the mortar discretization, we introduce several matrices. The matrix $S_\Gamma^{(i)}$ is the local Schur complement matrix obtained from the local stiffness matrix $A^{(i)}$ by eliminating the subdomain interior unknowns,

$$S_\Gamma^{(i)} = A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}(A_{\Gamma I}^{(i)})^T = \begin{pmatrix} S_{\Delta\Delta}^{(i)} & (S_{\Pi\Delta}^{(i)})^t \\ S_{\Pi\Delta}^{(i)} & S_{\Pi\Pi}^{(i)} \end{pmatrix},$$

where Δ and Π stand for the blocks corresponding to dual and primal unknowns, respectively. We define extensions of the spaces by

$$W_G \xrightarrow{\widetilde{R}_\Gamma} \widetilde{X}_\Gamma \xrightarrow{\overline{R}_\Gamma} X_\Gamma,$$

where $\widetilde{R}_\Gamma$ is the mortar map in (6) and $\overline{R}_\Gamma$ is the product of restriction maps,

$$\overline{R}_\Gamma^{(i)} : \widetilde{X}_\Gamma \to X_\Gamma^{(i)}.$$

We next introduce the matrices S_Γ and $\widetilde{S}_\Gamma$, the block diagonal matrix and the partially assembled matrix at the primal unknowns, respectively, as

$$S_\Gamma = \mathrm{diag}_i(S_\Gamma^{(i)}), \quad \widetilde{S}_\Gamma = \overline{R}_\Gamma^t S_\Gamma \overline{R}_\Gamma.$$

The linear system of the mortar discretization is then written as:
find $u_G \in W_G$ such that

$$\widetilde{R}_\Gamma^t \widetilde{S}_\Gamma \widetilde{R}_\Gamma u_G = \widetilde{R}_\Gamma^t g_G, \tag{7}$$

where $g_G \in W_G$ is the part of genuine unknowns, i.e., the unknowns other than the nonmortar part, of $g \in X_\Gamma$, that is given by

$$g^{(i)} = f_\Gamma^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1} f_I^{(i)}.$$

Here $f^{(i)} = \begin{pmatrix} f_I^{(i)} \\ f_\Pi^{(i)} \end{pmatrix}$ is the local force vector. In the BDDC algorithm, we solve (7) using a preconditioner M^{-1} of the form,

$$M^{-1} = \widetilde{R}_{D,\Gamma}^t \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma},$$

where the weighted extension operator $\widetilde{R}_{D,\Gamma}$ is given by

$$\widetilde{R}_{D,\Gamma} = D\widetilde{R}_\Gamma = \begin{pmatrix} D_n & 0 & 0 \\ 0 & D_m & 0 \\ 0 & 0 & D_\Pi \end{pmatrix} \widetilde{R}_\Gamma.$$

Later, we will specify the weight D_n, D_m, and D_Π.

We now develop a FETI–DP algorithm closely related to the BDDC algorithm. In the FETI–DP algorithm, we solve the dual form of the mortar discretization that is derived from the constrained minimization problem,

$$\min_{w \in \widetilde{X}_\Gamma} \left\{ \frac{1}{2} w^t \widetilde{S}_\Gamma w - w^t \widetilde{g} \right\},$$

with w satisfying the mortar matching condition (5). The mixed form to the constrained minimization problem gives

$$\begin{aligned} \widetilde{S}_\Gamma w + B^t \lambda &= \widetilde{g}, \\ Bw &= 0, \end{aligned}$$

where $B = (B_n, B_m, B_\Pi)$. After eliminating w, we obtain the dual form,

$$B \widetilde{S}_\Gamma^{-1} B^t \lambda = B \widetilde{S}_\Gamma^{-1} \widetilde{g}. \tag{8}$$

We solve the equations of the dual form (8) iteratively using a preconditioner,

$$\widehat{F}_{DP}^{-1} = B_\Sigma \widetilde{S}_\Gamma B_\Sigma^t,$$

where

$$B_\Sigma^t = \Sigma B^t = \begin{pmatrix} \Sigma_n & 0 & 0 \\ 0 & \Sigma_m & 0 \\ 0 & 0 & \Sigma_\Pi \end{pmatrix} B^t.$$

As a result, we have obtained the two algorithms for solving the mortar discretization and we write them into

$$B_{DDC} = \widetilde{R}_{D,\Gamma}^t \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma} \widetilde{R}_\Gamma^t \widetilde{S}_\Gamma \widetilde{R}_\Gamma, \; F_{DP} = B_\Sigma \widetilde{S}_\Gamma B_\Sigma^t B \widetilde{S}_\Gamma^{-1} B^t.$$

The convergence of the two algorithms depends on the condition number of B_{DDC} and F_{DP}. We now show a close connection between them and then provide weights D and Σ leading to scalable preconditioners. Let

$$P_\Sigma = B_\Sigma^t B, \; E_D = \widetilde{R}_\Gamma \widetilde{R}_{D,\Gamma}^t.$$

Theorem 2. *Assume that P_Σ and E_D satisfy*
1. $E_D + P_\Sigma = I$,
2. $E_D^2 = E_D, \; P_\Sigma^2 = P_\Sigma$,
3. $E_D P_\Sigma = P_\Sigma E_D = 0$.

Then the operators F_{DP} and B_{DDC} have the same spectra except the eigenvalues 0 and 1.

The same result was first shown by [17] and later by [15] for the conforming finite element discretizations. We are able to extend the result to the mortar discretizations.

The Neumann-Dirichlet preconditioner for the FETI-DP algorithms suggested by [10] was shown to be the most efficient for the problems with discontinuous coefficients, see [3]. The weight of the Neumann-Dirichlet preconditioner is given by

$$\Sigma_n = (B_n^t B_n)^{-1},\ \Sigma_m = 0,\ \Sigma_\Pi = 0, \tag{9}$$

and the condition number of the FETI-DP algorithm was shown to be

$$\kappa(F_{DP}) \le C(1 + \log(H/h))^2,$$

when the subdomain with smaller ρ_i is selected as the nonmortar side.

When the weight of the BDDC preconditioner is selected to be

$$D_n = 0,\ D_m = I,\ D_\Pi = I, \tag{10}$$

the E_D and P_Σ satisfy the assumptions in Theorem 2. Therefore, the BDDC algorithm equipped with the weight in (10) has the condition number bound,

$$\kappa(B_{DDC}) \le C(1 + \log(H/h))^2,$$

and the BDDC algorithm is as efficient as the FETI-DP algorithm.

5 Applications of the BDDC and FETI-DP Algorithms

The BDDC and FETI-DP algorithms introduced in the previous section can be generalized to the mortar discretizations of the Stokes problem and three dimensional compressible elasticity problems. For these cases, the selection of primal constraints is important in obtaining a scalable preconditioner.

We assume that the subdomain partition is geometrically conforming. We denote the common face (edge) of two subdomains Ω_i and Ω_j by F_{ij} in three (two) dimensions. An appropriate Lagrange multiplier space $M(F_{ij})$ is then provided for the nonmortar part of F_{ij}. We note that the space $M(F_{ij})$ contains the constant functions.

For the Stokes problem, we select the average matching condition across the interface as the primal constraints, namely,

$$\int_{F_{ij}} \mathbf{v}_i \, ds = \int_{F_{ij}} \mathbf{v}_j \, ds,$$

where F_{ij} is the common face (edge) of $\partial\Omega_i$ and $\partial\Omega_j$ in three (two) dimensions. For the elasticity problems, we select

$$\int_{F_{ij}} \mathbf{v}_i \cdot I_{M_{ij}}(\mathbf{r}_k) \, ds = \int_{F_{ij}} \mathbf{v}_j \cdot I_{M_{ij}}(\mathbf{r}_k) \, ds, \quad k = 1, \cdots, 6,$$

where $\mathbf{r}_k$ are the six rigid body motions and $I_{M_{ij}}(\mathbf{r}_k)$ is the nodal interpolant of $\mathbf{r}_k$ to the Lagrange multiplier space $M(F_{ij})$ provided for the nonmortar face F_{ij}.

With the selection of the primal constraints, we showed the condition number bound of the two algorithms

$$F_{DP} \leq C\left(1+\log\frac{H}{h}\right)^2, \quad B_{DDC} \leq C\left(1+\log\frac{H}{h}\right)^2,$$

when the weight are given by (9) and (10); see [11] and [7, 8]. The BDDC and FETI-DP algorithms of the elasticity can be extended to the geometrically nonconforming subdomain partitions as well. For such a case, the Lagrange multiplier space $M(F_{ij})$ is the span of basis elements ψ_l of $M(F_n)$ that are supported in $F_{n(i,j)}$. Here $F_n(\subset \partial\Omega_i)$ is the nonmortar face that is partitioned by its mortar neighbors $\{\Omega_j\}_j$.

We note that the BDDC preconditioner consists of solving local problems and the coarse problem,

$$\begin{aligned} M^{-1} &= \widetilde{R}_D^t \widetilde{S}_\Gamma^{-1} \widetilde{R}_D, \\ &= \widetilde{R}_D^t \begin{pmatrix} I & 0 \\ -S_{\Pi\Delta}S_{\Delta\Delta}^{-1} & I \end{pmatrix} \begin{pmatrix} S_{\Delta\Delta}^{-1} & 0 \\ 0 & F_{\Pi\Pi}^{-1} \end{pmatrix} \begin{pmatrix} I & -S_{\Delta\Delta}^{-1}S_{\Delta\Pi} \\ 0 & I \end{pmatrix} \widetilde{R}_D. \end{aligned}$$

As the increase of the number of subdomains, the cost for solving the coarse component becomes a bottleneck of the computation. By solving the coarse problem inexactly, we can speed up the total computational time.

BDDC algorithms with an inexact coarse problem were developed by [21, 20] for conforming finite element discretizations of elliptic problems in both two and three dimensions. The idea is to group subdomains into a subregion and to obtain a subregion partition. Using the additional level, we construct a BDDC preconditioner of the coarse component $F_{\Pi\Pi}$ in M^{-1}. The resulting preconditioner, called a three-level BDDC preconditioner, is given by

$$\overline{M}^{-1} = \widetilde{R}_D^t \overline{S}_\Gamma^{-1} \widetilde{R}_D,$$

where $\overline{S}_\Gamma^{-1}$ denotes the matrix that is the part $F_{\Pi\Pi}^{-1}$ of $\widetilde{S}_\Gamma^{-1}$ is replaced by a BDDC preconditioner using the additional subregion level. The condition number bound of the three–level BDDC algorithm was shown to be

$$\kappa(\overline{M}^{-1}\widetilde{R}_\Gamma^t \widetilde{S} \widetilde{R}_\Gamma) \leq C\left(1+\log\frac{\widehat{H}}{H}\right)^2 \left(1+\log\frac{H}{h}\right)^2,$$

where $\widehat{H}$, H, and h denote the subregion diameters, subdomain diameters, and mesh sizes, respectively.

We obtain a subregion partition $\{\Omega^{(j)}\}_{j=1}^{N_c}$, where each subregion $\Omega^{(j)}$ is the union of N_j subdomains $\Omega_i^{(j)}$. An example of a subregion partition, that is

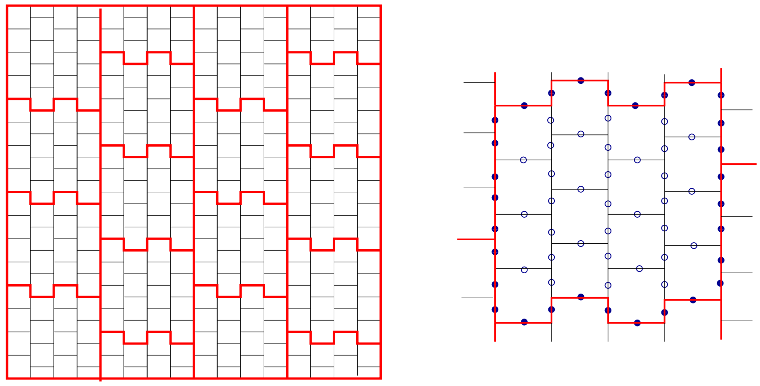

Fig. 1. A subregion partition (left) and unknowns at a subregion (right) when $\widehat{H}/H = 4$; small rectangles are subdomains in the left.

obtained from a geometrically non-conforming subdomain partition, is shown in Fig. 1.

In the subregion partition, we define faces as the intersection of two subregions and vertices (or edges) as the intersection of more than two subregions. Finite element spaces for the subregions are given by the primal unknowns of the two–level algorithm so that the subregion partition is equipped with a conforming finite element space, for which the unknowns match across the subregion interface. On this new level, the mortar discretization is no longer relevant. We can then develop the theory and algorithm for the subregion partition as in the two–level BDDC algorithm done for the conforming finite element discretization. Analysis and numerical computations of the three-level BDDC algorithm for mortar discretizations will be found in [12].

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