A Functional Analytic Framework for BDDC and FETI-DP

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1 Introduction

In this paper we present a concise common framework for the BDDC algorithm (cf. [4, 8, 9]) and the FETI-DP algorithm (cf. [6, 5, 10]), using the mathematical language of function spaces, their dual spaces and quotient spaces, and operators. This abstract framework will be illustrated in terms of the following model problem.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded polyhedral domain subdivided into J nonoverlapping polyhedral subdomains $\Omega_1, \ldots, \Omega_J$, $\mathcal{T}$ be a triangulation of Ω aligned with the boundaries of the subdomains, and $V(\Omega) \subset H_0^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$. For simplicity, we assume that the subdomains are geometrically conforming, i.e., the intersection of the closures of two distinct subdomains is either empty, a common vertex, a common edge or a common face. The interface of the subdomains is $\Gamma = \bigcup_{j=1}^J (\partial\Omega_j \setminus \partial\Omega)$.

The model problem is to find $u \in V(\Omega)$ such that

$$a(u,v) = \sum_{j=1}^{J} a_j(u_j, v_j) = \int_\Omega f v \, dx \qquad \forall\, v \in V(\Omega), \tag{1}$$

where $u_j = u\big|_{\Omega_j}$, $v_j = v\big|_{\Omega_j}$,

$$a_j(u_j, v_j) = \alpha_j \int_{\Omega_j} \nabla u_j \cdot \nabla v_j \, dx,$$

α_j is a positive constant, and $f \in L_2(\Omega)$.

The rest of the paper is organized as follows. The common framework for BDDC and FETI-DP will be presented in Section 2, followed by a discussion of the additive Schwarz formulations of these algorithms in Section 3. Condition number estimates for BDDC and FETI-DP (applied to the model problem) are then sketched in Section 4. Throughout the paper we use $\langle \cdot, \cdot \rangle$ to denote the canonical bilinear form between a vector space V and its dual space V', and the superscript t to denote the transpose of an operator with respect to the canonical bilinear forms.

2 A Common Framework for BDDC and FETI-DP

After parallel subdomain solves, the model problem (1) is reduced to the *interface problem* of computing $u_\Gamma \in V(\Gamma)$ such that

$$a(u_\Gamma, v_\Gamma) = \int_\Omega f v_\Gamma \, dx \qquad \forall\, v_\Gamma \in V(\Gamma), \tag{2}$$

where $V(\Gamma) \subset V(\Omega)$ is the space of discrete harmonic functions whose members satisfy

$$a(v_\Gamma, w) = 0 \qquad \forall\, w \in V(\Omega) \text{ that vanish on } \Gamma.$$

The interface problem (2) is solved by the BDDC method through a preconditioned conjugate gradient algorithm. In the FETI-DP approach, it is first transformed to a dual-primal problem and then solved by a preconditioned conjugate gradient algorithm.

The first ingredient in the common framework for BDDC and FETI-DP is of course the space $V(\Gamma)$. The restriction of $V(\Gamma)$ to Ω_j gives the space $\mathcal{H}_j$ of discrete harmonic functions on Ω_j. The second ingredient is a space $\mathcal{H}_c \subset \mathcal{H}_1 \times \cdots \times \mathcal{H}_J$ of constrained piecewise discrete harmonic functions. The subdomain components of a function in $\mathcal{H}_c$ share certain average values (constraints) along the interface Γ. In particular, we have $V(\Gamma) \subset \mathcal{H}_c$. The constraints (shared averages) are chosen so that (i) the bilinear form $a(\cdot,\cdot)$ remains positive definite on $\mathcal{H}_c$, (ii) the bilinear form $a_j(\cdot,\cdot)$ is positive definite on the subspace of $\mathcal{H}_j$ whose members have vanishing constraints, and (iii) the preconditioned systems in BDDC and FETI-DP have good condition numbers.

Example For the two-dimensional model problem, the constraints that define $\mathcal{H}_c$ are the values of the subdomain components at the corners of the subdomains that are interior to Ω, i.e., $\mathcal{H}_c$ is the space of piecewise discrete harmonic functions that are continuous at these interior corners (cross points). For the three-dimensional model problem, the constraints are the averages along the edges of the subdomains that are interior to Ω, i.e., $\mathcal{H}_c$ is the space of piecewise discrete harmonic functions whose average along any interior edge is continuous across the subdomains sharing the edge.

The third ingredient of the framework is the Schur complement operator $\mathbb{S} : \mathcal{H}_c \longrightarrow \mathcal{H}_c'$ defined by

$$\langle \mathbb{S} v, w \rangle = a(v, w) \qquad \forall\, v, w \in \mathcal{H}_c.$$

Let $I_\Gamma : V(\Gamma) \longrightarrow \mathcal{H}_c$ be the natural injection. Then the interface problem (2) can be written as $Su_\Gamma = \phi_\Gamma$, where

$$S = I_\Gamma^t \mathbb{S} I_\Gamma \tag{3}$$

and $\langle \phi_\Gamma, v \rangle = \int_\Omega f v \, dx \qquad \forall\, v \in V(\Gamma)$.

In the FETI-DP approach, the interface problem (2) is transformed to the equivalent dual-primal problem of finding $(u^c, \lambda) \in \mathcal{H}_c \times [\mathcal{H}_c/V(\Gamma)]'$ such that

$$\begin{aligned} \sum_{j=1}^{J} a_j(u_j^c, v_j) + \langle \lambda, Q_\Gamma v \rangle &= \int_\Omega f v \, dx \qquad && \forall\, v \in \mathcal{H}_c \\ \langle \mu, Q_\Gamma u^c \rangle &= \quad 0 && \forall\, \mu \in [\mathcal{H}_c/V(\Gamma)]' \end{aligned} \tag{4}$$

where $Q_\Gamma : \mathcal{H}_c \longrightarrow \mathcal{H}_c/V(\Gamma)$ is the canonical projection, and $[\mathcal{H}_c/V(\Gamma)]'$ plays the role of the space of Lagrange multipliers that enforce the continuity of the constraints along Γ for functions in $\mathcal{H}_c$. Eliminating u^c from (4), we find $S^\dagger \lambda = Q_\Gamma \mathbb{S}^{-1} \phi_c$, where the operator $S^\dagger : [\mathcal{H}_c/V(\Gamma)]' \longrightarrow [\mathcal{H}_c/V(\Gamma)]$ is defined by

$$S^\dagger = Q_\Gamma \mathbb{S}^{-1} Q_\Gamma^t \tag{5}$$

and $\langle \phi_c, v \rangle = \int_\Omega f v \, dx \quad \forall v \in \mathcal{H}_c$.

The final ingredient of the framework is a operator P_Γ that projects $\mathcal{H}_c$ onto $V(\Gamma)$. We can then define the preconditioners $B_{BDDC} : V(\Gamma)' \longrightarrow V(\Gamma)$ and $B_{FETI\text{–}DP} : [\mathcal{H}_c/V(\Gamma)] \longrightarrow [\mathcal{H}_c/V(\Gamma)]'$ by

$$B_{BDDC} = P_\Gamma \mathbb{S}^{-1} P_\Gamma^t, \qquad B_{FETI\text{–}DP} = L_\Gamma^t \mathbb{S} L_\Gamma, \tag{6}$$

where the lifting operator $L_\Gamma : \mathcal{H}_c/V(\Gamma) \longrightarrow \mathcal{H}_c$ is given by

$$L_\Gamma(v + V(\Gamma)) = v - I_\Gamma P_\Gamma v \qquad \forall\, v \in \mathcal{H}_c. \tag{7}$$

Example For our model problems the projection operator P_Γ is defined by weighted averaging:

$$(P_\Gamma v)(p) = \Big(\frac{1}{\sum_{j \in \sigma_p} \alpha_j^\gamma} \Big) \sum_{\ell \in \sigma_p} \alpha_\ell^\gamma v_\ell(p) \qquad \forall\, p \in \mathcal{N}_\Gamma, \tag{8}$$

where $\mathcal{N}_\Gamma$ = the set of nodes on Γ, σ_p = the index set for the subdomains that share p as a common boundary node, and γ is any number $\geq 1/2$. The key property of this weighted averaging is that

$$\alpha_k \alpha_\ell^\gamma / (\sum_{j \in \sigma_p} \alpha_j^\gamma) \leq \alpha_\ell \qquad \forall\, k, \ell \in \sigma_p. \tag{9}$$

In summary, the system operators for the BDDC and FETI-DP methods and their preconditioners are defined in terms of the four ingredients $V(\Gamma)$, $\mathcal{H}_c$, $\mathbb{S}$ and P_Γ through (3) and (5)–(7).

It is easy to see that

$$P_\Gamma I_\Gamma = Id_{V(\Gamma)}, \quad Q_\Gamma L_\Gamma = Id_{\mathcal{H}_c/V(\Gamma)} \quad \text{and} \quad I_\Gamma P_\Gamma + L_\Gamma Q_\Gamma = Id_{\mathcal{H}_c}. \quad (10)$$

The following result (cf. [9, 7, 3]) on the spectra of $B_{BDDC}S$ and $B_{FETI\text{–}DP}S^\dagger$ follows from the three relations in (10).

Theorem 1. *It holds that* $\lambda_{\min}(B_{BDDC}S) \geq 1$, $\lambda_{\min}(B_{FETI\text{–}DP}S^\dagger) \geq 1$, *and* $\sigma(B_{BDDC}S)\backslash\{1\} = \sigma(B_{FETI\text{–}DP}S^\dagger)\backslash\{1\}$. *Furthermore, the multiplicity of any common eigenvalue different from 1 is identical for* $B_{BDDC}S$ *and* $B_{FETI\text{–}DP}S^\dagger$.

3 Additive Schwarz Formulations

The additive Schwarz formulations of B_{BDDC} and $B_{FETI\text{–}DP}$ involve the spaces $\mathring{\mathcal{H}}_j = \{v \in \mathcal{H}_j : E_j v \in \mathcal{H}_c\}$, where E_j is the trivial extension operator defined by

$$E_j v = \begin{cases} v & \text{on } \Omega_j \\ 0 & \text{on } \Omega \setminus \Omega_j \end{cases}, \quad (11)$$

and the Schur complement operators $S_j : \mathring{\mathcal{H}}_j \longrightarrow \mathring{\mathcal{H}}_j'$ defined by

$$\langle S_j v, w\rangle = a_j(v, w) \qquad \forall\, v, w \in \mathring{\mathcal{H}}_j.$$

Example $\mathring{\mathcal{H}}_j$ is precisely the space of discrete harmonic functions on Ω_j whose interface constraints are identically zero. For the 2D model problem these functions vanish at the corners of Ω_j. For the 3D model problem, they have zero averages along the edges of Ω_j. Note that $a_j(\cdot,\cdot)$ is positive definite on $\mathring{\mathcal{H}}_j$.

We can now introduce the coarse space

$$\mathcal{H}_0 = \{v \in \mathcal{H}_c : a_j(v_j, w_j) = 0 \quad \forall\, w_j \in \mathring{\mathcal{H}}_j,\ 1 \leq j \leq J\},$$

and define the Schur complement operator $S_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_0'$ by

$$\langle S_0 v, w\rangle = a(v, w) \qquad \forall\, v, w \in \mathcal{H}_0.$$

Lemma 1. *The inverse of* $\mathbb{S}$ *can be written as*

$$\mathbb{S}^{-1} = \sum_{j=0}^{J} E_j S_j^{-1} E_j^t, \quad (12)$$

where E_j *for* $1 \leq j \leq J$ *is defined in* (11) *and* $E_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}_c$ *is the natural injection.*

Proof. Let $v \in \mathcal{H}_c$ be arbitrary. Then we have a unique decomposition $v = \sum_{k=0}^{J} E_k v_k$, where $v_0 \in \mathcal{H}_0$ and $v_k \in \mathring{\mathcal{H}}_k$ for $1 \le k \le J$, and

$$\Big[\sum_{j=0}^{J} E_j S_j^{-1} E_j^t\Big]\mathbb{S}v = \Big[\sum_{j=0}^{J} E_j S_j^{-1} E_j^t\Big]\mathbb{S}\Big[\sum_{k=0}^{J} E_k v_k\Big]$$
$$= \sum_{j=0}^{J} E_j S_j^{-1} E_j^t \mathbb{S} E_j v_j = \sum_{j=0}^{J} E_j v_j = v,$$

where we have used the facts that $E_j^t \mathbb{S} E_k = 0$ if $j \neq k$ and $S_j = E_j^t \mathbb{S} E_j$.

It follows from (6) and (12) that

$$B_{BDDC} = \sum_{j=0}^{J} (P_\Gamma E_j) S_j^{-1} (P_\Gamma E_j)^t. \tag{13}$$

Let $\mathring{\mathcal{H}} = \sum_{j=1}^{J} E_j \mathring{\mathcal{H}}_j$ be the subspace of $\mathcal{H}_c$ whose members have zero interface constraints. Note that the lifting operator defined by (7) actually maps $\mathcal{H}_c/V(\Gamma)$ to $\mathring{\mathcal{H}}$, since the interface constraints of a function $v \in \mathcal{H}_c$ are preserved by the weighted averaging operator P_Γ. Therefore we can factorize L_Γ as

$$L_\Gamma = \mathring{I} \circ \mathring{L}_\Gamma,$$

where $\mathring{L}_\Gamma : \mathcal{H}_c/V(\Gamma) \longrightarrow \mathring{\mathcal{H}}$ is defined by the same formula in (7) and the operator $\mathring{I} : \mathring{\mathcal{H}} \longrightarrow \mathcal{H}_c$ is the natural injection. We can then write

$$B_{FETI\text{–}DP} = \mathring{L}_\Gamma^t (\mathring{I}^t \mathbb{S} \mathring{I}) \mathring{L}_\Gamma. \tag{14}$$

The following lemma can be established by arguments similar to those in the proof of Lemma 1.

Lemma 2. *We have*

$$\mathring{I}^t \mathbb{S} \mathring{I} = \sum_{j=1}^{J} R_j^t S_j R_j, \tag{15}$$

where $R_j : \mathring{\mathcal{H}} \longrightarrow \mathring{\mathcal{H}}_j$ *is the restriction operator.*

It follows from (14) and (15) that

$$B_{FETI\text{–}DP} = \sum_{j=1}^{J} (R_j \mathring{L}_\Gamma)^t S_j (R_j \mathring{L}_\Gamma). \tag{16}$$

The formulations (13) and (16) allow both algorithms to be analyzed by the additive Schwarz theory (cf. [2, 11] and the references therein).

4 Condition Number Estimates

In view of Theorem 1, the preconditioned systems in the BDDC and FETI-DP methods have similar behaviors. Here we will sketch the condition number estimates for the BDDC method applied to our model problem. Since we already know that $\lambda_{\min}(B_{BDDC}S) \geq 1$, it only remains to find an upper bound for $\lambda_{\max}(B_{BDDC}S)$ using the following formula from the theory of additive Schwarz preconditioners (cf. [2]):

$$\lambda_{\max}(B_{BDDC}S) = \max_{v \in V(\Gamma)\setminus\{0\}} \frac{\langle Sv, v\rangle}{\displaystyle\min_{\substack{v=\sum_{j=0}^{J} P_\Gamma E_j v_j \\ v_0 \in \mathcal{H}_0,\, v_j \in \mathring{\mathcal{H}}_j\ (1\leq j\leq J)}} \sum_{j=0}^{J} \langle S_j v_j, v_j\rangle} \tag{17}$$

Let w be a discrete harmonic function on a subdomain Ω_j and the geometric object $\mathcal{G}$ be either a corner c ($\dim \mathcal{G} = 0$), an open edge e ($\dim \mathcal{G} = 1$) or an open face f ($\dim \mathcal{G} = 2$) of Ω_j. We will denote by $w_{\mathcal{G}}$ the discrete harmonic function that agrees with w at the nodes on $\mathcal{G}$ and vanishes at all other nodes. The following estimate (cf. [2, 11] and the references therein) is crucial for the condition number estimate of the model problem:

$$|w_{\mathcal{G}}|^2_{H^1(\Omega_j)} \leq C\Big(1 + \ln \frac{H_j}{h_j}\Big)^{3-d+\dim\mathcal{G}} |w|^2_{H^1(\Omega_j)}, \tag{18}$$

where $d = 2$ or 3, H_j is the diameter of Ω_j, and h_j is the mesh size of the quasi-uniform triangulation which is the restriction of $\mathcal{T}$ to Ω_j. We assume that w vanishes at one of the corners of Ω_j when $d = 2$ and that w has zero average along one of the edges of Ω_j. (Henceforth we use C to denote a generic positive constant that can take different values at different occurrences.)

Furthermore, if $w \in V(\Gamma)$, then it follows from the equivalence of $|w|_{H^1(\Omega_j)}$ and $|w|_{H^{1/2}(\partial\Omega_j)}$ (cf. [2, 11]) that

$$|w_{\mathcal{G}}|_{H^1(\Omega_k)} \leq C|w_{\mathcal{G}}|_{H^1(\Omega_\ell)} \tag{19}$$

if Ω_k and Ω_ℓ share the common geometric object $\mathcal{G}$.

Let $v \in V(\Gamma)$ be arbitrary and $v = \sum_{j=0}^{J} P_\Gamma E_j v_j$ be any decomposition of v, where $v_0 \in \mathcal{H}_c$ and $v_j \in \mathring{\mathcal{H}}_j$ for $1 \leq j \leq J$. We want to show

$$\langle Sv, v\rangle \leq C\Big(1 + \ln \frac{H}{h}\Big)^2 \sum_{j=0}^{J} \langle S_j v_j, v_j\rangle, \tag{20}$$

where $H/h = \max_{1\leq j\leq J}(H_j/h_j)$.

Observe first that

$$\langle Sv, v\rangle = \langle S \sum_{j=0}^{J} P_\Gamma E_j v, \sum_{k=0}^{J} P_\Gamma E_k v\rangle$$

$$\le 2\Big[\langle SP_\Gamma E_0 v_0, P_\Gamma E_0 v_0\rangle + \langle S\sum_{j=1}^{J} P_\Gamma E_j v_j, \sum_{j=k}^{J} P_\Gamma E_k v\rangle\Big] \tag{21}$$
$$\le C\sum_{j=0}^{J}\langle SP_\Gamma E_j v, P_\Gamma E_j v\rangle,$$

where we have used the fact that each v_j $(1 \le j \le J)$ only interacts with functions from a few subdomains. Therefore, it remains only to relate $\langle SP_\Gamma E_j v_j, P_\Gamma E_j v_j\rangle$ to $\langle S_j v_j, v_j\rangle$.

Let $w = v_0 - P_\Gamma E_0 v_0$. Then w vanishes at the corners of the subdomains when $d = 2$ and has zero averages along the edges of the subdomains when $d = 3$. We can write

$$w = \sum_{1\le j\le J}\Big(\sum_{\mathrm{c}\in\mathcal{C}_j} w_{\mathrm{c}} + \sum_{\mathrm{e}\in\mathcal{E}_j} w_{\mathrm{e}} + \sum_{\mathrm{f}\in\mathcal{F}_j} w_{\mathrm{f}}\Big)$$

where $\mathcal{C}_j$ (resp. $\mathcal{E}_j$ and $\mathcal{F}_j$) is the set of the corners (resp. edges and faces) of Ω_j ($\mathcal{C}_j = \emptyset = \mathcal{F}_j$ for $d = 2$), and apply (8), (9), (18) and (19) to obtain the estimate

$$\langle Sw, w\rangle \le C\sum_{j=1}^{J}\Big(1+\ln\frac{H_j}{h_j}\Big)^2\alpha_j|v_0|^2_{H^1(\Omega_j)} \le C\Big(1+\ln\frac{H}{h}\Big)^2\langle S_0 v_0, v_0\rangle,$$

which together with the triangle inequality implies that

$$\langle SP_\Gamma E_0 v_0, P_\Gamma E_0 v_0\rangle \le C\Big(1+\ln\frac{H}{h}\Big)^2\langle S_0 v_0, v_0\rangle. \tag{22}$$

Similarly, we have the estimate

$$\langle SP_\Gamma E_j v_j, P_\Gamma E_j v_j\rangle \le C\Big(1+\ln\frac{H_j}{h_j}\Big)^2\langle S_j v_j, v_j\rangle \quad \text{for} \quad 1\le j\le J. \tag{23}$$

The estimate (20) follows from (21)–(23).

We see from (20) that

$$\langle Sv, v\rangle \le C\Big(1+\ln\frac{H}{h}\Big)^2 \min_{\substack{v=\sum_{j=0}^{J} P_\Gamma E_j v_j \\ v_0\in\mathcal{H}_0,\, v_j\in\mathring{\mathcal{H}}_j\ (1\le j\le J)}} \sum_{j=0}^{J}\langle S_j v_j, v_j\rangle. \tag{24}$$

Combining (17) and (24) we have the estimate

$$\lambda_{\max}(B_{BDDC}S) \le C\Big(1+\ln\frac{H}{h}\Big)^2$$

and hence the following theorem on the condition number of $B_{BDDC}S$, which has also been obtained in [8] and [9] by a different approach.

Theorem 2. *For the model problem we have*

$$\kappa(B_{BDDC}S) = \frac{\lambda_{\max}(B_{BDDC}S)}{\lambda_{\min}(B_{BDDC}S)} \leq C\Big(1 + \ln\frac{H}{h}\Big)^2,$$

where the positive constant C is independent of h_j, H_j, α_j and J.

Finally we remark that for the model problem the estimate

$$\kappa(B_{FETI\text{–}DP}S^{\dagger}) \leq C\Big(1 + \ln\frac{H}{h}\Big)^2$$

follows from Theorem 1 and Theorem 2. A direct analysis of $B_{FETI\text{–}DP}S^{\dagger}$ by the additive Schwarz theory can also be found in [1].

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