An Additive Schwarz Method for the Constrained Minimization of Functionals in Reflexive Banach Spaces

Lori Badea

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania. `lori.badea@imar.ro`

Summary. In this paper, we show that the additive Schwarz method proposed in [3] to solve one-obstacle problems converges in a much more general framework. We prove that this method can be applied to the minimization of functionals over a general enough convex set in a reflexive Banach space. In the Sobolev spaces, the proposed method is an additive Schwarz method for the solution of the variational inequalities coming from the minimization of non-quadratic functionals. Also, we show that the one-, two-level variants of the method in the finite element space converge, and we explicitly write the constants in the error estimations depending on the overlapping and mesh parameters.

1 Introduction

The literature on the domain decomposition methods is very large. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting with [5], or those cited in the books [10, 11] and [13]. The multilevel or multigrid methods can be viewed as domain decomposition methods and we can cite, for instance, the results obtained in [7, 9, 11].

In [3], an additive Schwarz method has been proposed for symmetric variational inequalities. Although this method does not assume a decomposition of the convex set according to the domain decomposition, the convergence proof is given only for the one-obstacle problems. In Section 2 of this paper, we prove that the method converges in a much more general framework, i.e. we can apply it to the minimization of functionals over a general enough convex set in a reflexive Banach space. In Section 3, we show that, in the Sobolev spaces, the proposed method is an additive Schwarz method and it converges for variational inequalities coming from the minimization of non-quadratic functionals. Also, in Section 4, we show that the one-, two-level variants of the method in the finite element space converge, and we explicitly write the constants in the error estimations depending on the overlapping and mesh parameters. The convergence rates we find are similar with those obtained in the literature for symmetric inequalities or equations, i.e. they are almost independent on the overlapping and mesh parameters in the case of the two-level method.

2 General Convergence Result

Let us consider a reflexive Banach space V, some closed subspaces of V, $V_1, \cdots, V_m$, and $K \subset V$ a non empty closed convex subset. We make the following

Assumption 1 *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ there exist $v_i \in V_i$, $i = 1, \dots, m$, which satisfy*

$$v - w = \sum_{i=1}^{m} v_i, \quad w + v_i \in K \text{ and } \sum_{i=1}^{m} ||v_i|| \leq C_0 ||v - w||.$$

We consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, which is assumed to be coercive on K, in the sense that $\frac{F(v)}{||v||} \to \infty$, as $||v|| \to \infty$, $v \in K$, if K is not bounded. Also, we assume that there exist two real numbers $p,\ q > 1$ such that for any real number $M > 0$ there exist $\alpha_M,\ \beta_M > 0$ for which

$$\begin{aligned} &\alpha_M ||v - u||^p \leq \langle F'(v) - F'(u), v - u\rangle \text{ and} \\ &||F'(v) - F'(u)||_{V'} \leq \beta_M ||v - u||^{q-1} \end{aligned} \tag{1}$$

for any $u, v \in V$ with $||u||, ||v|| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (1) holds, then for any $u, v \in V$, $||u||, ||v|| \leq M$, we have $\alpha_M ||v - u||^p \leq \langle F'(v) - F'(u), v - u\rangle \leq \beta_M ||v - u||^q$. Following the way in [6], we can prove that for any $u, v \in V$, $||u||, ||v|| \leq M$, we have

$$\langle F'(u), v - u\rangle + \frac{\alpha_M}{p}||v - u||^p \leq F(v) - F(u) \leq \langle F'(u), v - u\rangle + \frac{\beta_M}{q}||v - u||^q. \tag{2}$$

We point out that since F is Gâteaux differentiable and satisfies (1), then F is a convex functional (see Proposition 5.5 in [4], p. 25).

We consider the minimization problem

$$u \in K : F(u) \leq F(v), \text{ for any } v \in K, \tag{3}$$

and since the functional F is convex and differentiable, it is equivalent with the variational inequality

$$u \in K : \langle F'(u), v - u\rangle \geq 0, \text{ for any } v \in K. \tag{4}$$

We can use, for instance, Theorem 8.5 in [8], p. 251, to prove that problem (3) has a unique solution if F has the above properties. In view of (2), for a given $M > 0$ such that the solution $u \in K$ of (3) satisfies $||u|| \leq M$, we have

$$\frac{\alpha_M}{p}||v - u||^p \leq F(v) - F(u) \text{ for any } v \in K,\ ||v|| \leq M. \tag{5}$$

To solve the minimization problem (3), we propose the following additive subspace correction algorithm corresponding to the subspaces $V_1, \dots, V_m$ and the convex set K.

ALGORITHM 1 *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we solve the inequalities*

$$\begin{aligned} & w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle \geq 0, \\ & \text{for any } v_i \in V_i,\ u^n + v_i \in K, \end{aligned} \tag{6}$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \rho \sum_{i=1}^{m} w_i^{n+1}$, where ρ is chosen such that $u^{n+1} \in K$ for any $n \geq 0$.

A possible choice of ρ to get $u^{n+1} \in K$, is $\rho \leq \frac{1}{m}$. Indeed, if we write $0 < r = \rho m \leq 1$, then $u^{n+1} = (1-r)u^n + r \sum_{i=1}^{m} \frac{1}{m}(u^n + w_i^{n+1}) \in K$. Evidently, problem (6) has an unique solution and it is equivalent with

$$\begin{aligned} & w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : F(u^n + w_i^{n+1}) \leq F(u^n + v_i), \\ & \text{for any } v_i \in V_i,\ u^n + v_i \in K. \end{aligned} \tag{7}$$

Let us now give the convergence result of Algorithm 1.

Theorem 1. *We consider that V is a reflexive Banach, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V satisfying Assumption 1, and F is a Gâteaux differentiable functional on K which is supposed to be coercive if K is not bounded, and satisfies (1). On these conditions, if u is the solution of problem (3) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 1, then there exists $M > 0$ such that the following error estimations hold:*

(i) if $p = q$ we have

$$F(u^n) - F(u) \leq \left(\frac{C_1}{C_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{8}$$

$$||u^n - u||^p \leq \frac{p}{\alpha_M} \left(\frac{C_1}{C_1 + 1} \right)^n \left[F(u^0) - F(u) \right], \tag{9}$$

where C_1 is given in (14), and

(ii) if $p > q$ we have

$$F(u^n) - F(u) \leq \frac{F(u^0) - F(u)}{\left[1 + nC_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \tag{10}$$

$$||u - u^n||^p \leq \frac{p}{\alpha_M} \frac{F(u^0) - F(u)}{\left[1 + nC_2 \left(F(u^0) - F(u) \right)^{\frac{p-q}{q-1}} \right]^{\frac{q-1}{p-q}}}, \tag{11}$$

where C_2 is given in (18).

Proof. We first prove that the approximation sequence $(u^n)_{n \geq 0}$ of u obtained from Algorithm 1 is bounded for $\rho = \frac{r}{m}$, $0 \leq r \leq 1$. In view of the convexity of F and equation (7), we get

$$F(u^{n+1}) = F(u^n + \frac{r}{m}\sum_{i=1}^m w_i^{n+1}) = F((1-r)u^n + \sum_{i=1}^m \frac{r}{m}(u^n + w_i^{n+1}))$$
$$\leq (1-r)F(u^n) + \frac{r}{m}\sum_{i=1}^m F(u^n + w_i^{n+1}) \leq F(u^n).$$

Consequently, using (3), we have $F(u) \leq F(u^{n+1}) \leq F(u^n) \leq \cdots \leq F(u^0)$, and, from the coercivity of F if K is not bounded, we get that there exists $M > 0$, such that $||u|| \leq M$ and $||u^n|| \leq M$, $n \geq 0$.

In view of (2) and (6), we get $\frac{\alpha_M}{p}||w_i^{n+1}||^p \leq F(u^n) - F(u^n + w_i^{n+1})$. Using this equation in the place of (7), with a proof similar with the above one, we get

$$\rho\frac{\alpha_M}{p}\sum_{i=1}^m ||w_i^{n+1}||^p \leq F(u^n) - F(u^{n+1}) \tag{12}$$

Now, writing $\bar{u}^{n+1} = u^n + \sum_{i=1}^m w_i^{n+1}$ in view of the convexity of F, we have

$$F(u^{n+1}) = F(u^n + \frac{r}{m}\sum_{i=1}^m w_i^{n+1}) = F((1-\frac{r}{m})u^n + \frac{r}{m}(u^n + \sum_{i=1}^m w_i^{n+1})$$
$$\leq (1-\frac{r}{m})F(u^n) + \frac{r}{m}F(u^n + \sum_{i=1}^m w_i^{n+1}) \leq (1-\frac{r}{m})F(u^n) + \frac{r}{m}F(\bar{u}^{n+1}).$$

With $v := u$ and $w := u^n$, we get a decomposition $v_i^n \in V_i$ of $u - u^n$ satisfying the conditions of Assumption 1. Using this decomposition, the above equation, (2) and inequalities (6),

$$F(u^{n+1}) - F(u) + \rho\tfrac{\alpha_M}{p}||\bar{u}^{n+1} - u||^p$$
$$\leq (1-\rho)(F(u^n) - F(u)) + \rho\left(F(\bar{u}^{n+1}) - F(u) + \tfrac{\alpha_M}{p}||\bar{u}^{n+1} - u||^p\right)$$
$$\leq (1-\rho)(F(u^n) - F(u)) + \rho\langle F'(\bar{u}^{n+1}), \bar{u}^{n+1} - u\rangle$$
$$= (1-\rho)(F(u^n) - F(u)) + \rho\sum_{i=1}^m \langle F'(\bar{u}^{n+1}), w_i^{n+1} - v_i^n\rangle$$
$$\leq (1-\rho)(F(u^n) - F(u)) + \rho\sum_{i=1}^m \langle F'(u^n + w_i^{n+1}) - F'(\bar{u}^{n+1}), v_i^n - w_i^{n+1}\rangle$$
$$\leq (1-\rho)(F(u^n) - F(u)) + \rho\beta_M\left(\sum_{i=1}^m ||w_i^{n+1}||\right)^{q-1}\sum_{i=1}^m ||v_i^n - w_i^{n+1}||$$
$$\leq (1-\rho)(F(u^n) - F(u)) + \rho\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^m ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\sum_{i=1}^m (||v_i^n|| + ||w_i^{n+1}||)$$
$$\leq (1-\rho)(F(u^n) - F(u))$$
$$+ \rho\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^m ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\left(C_0||u - u^n|| + \sum_{i=1}^m ||w_i^{n+1}||\right)$$
$$\leq (1-\rho)(F(u^n) - F(u))$$
$$+ \rho\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^m ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\left(C_0||u - \bar{u}^{n+1}|| + (1 + C_0)\sum_{i=1}^m ||w_i^{n+1}||\right)$$

$$\leq (1-\rho)(F(u^n)-F(u)) + \rho\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}} ||u-\bar{u}^{n+1}||$$
$$+ \rho\beta_M(1+C_0)m^{\frac{(p-1)q}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q}{p}} .$$

But, for any $\varepsilon > 0$ $r > 1$ and $x, y \geq 0$, we have $x^{\frac{1}{r}} y \leq \varepsilon x + \frac{1}{\varepsilon^{\frac{1}{r-1}}} y^{\frac{r}{r-1}}$. Consequently, we get

$$F(u^{n+1}) - F(u) + \rho\frac{\alpha_M}{p}||\bar{u}^{n+1}-u||^p$$
$$\leq (1-\rho)(F(u^n)-F(u)) + \rho\beta_M(1+C_0)m^{\frac{(p-1)q}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q}{p}}$$
$$+ \rho\beta_M C_0 \frac{m^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p-1}} + \rho\beta_M C_0 \varepsilon m^{\frac{(p-1)(q-1)}{p}} ||u-\bar{u}^{n+1}||^p .$$

With $\varepsilon = \frac{\alpha_M}{p} \frac{1}{\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}}}$, the above equations become

$$F(u^{n+1}) - F(u) \leq \frac{1-\rho}{\rho}(F(u^n)-F(u^{n+1})) + \beta_M(1+C_0)m^{\frac{(p-1)q}{p}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q}{p}}$$
$$+ \left(\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}}\right)^{\frac{p}{p-1}} \left(\frac{p}{\alpha_M}\right)^{\frac{1}{p-1}} \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p-1}} .$$

In view of this equation and (12), we have

$$\begin{aligned} F(u^{n+1}) - F(u) \leq & \frac{1-\rho}{\rho}\left(F(u^n)-F(u^{n+1})\right) \\ & + \frac{1}{\rho^{\frac{q}{p}}} \frac{\beta_M(1+C_0)m^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \left(F(u^n)-F(u^{n+1})\right)^{\frac{q}{p}} \\ & + \frac{1}{\rho^{\frac{q-1}{p-1}}} \frac{\left(\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}}\right)^{\frac{p}{p-1}}}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p-1}}} \left(F(u^n)-F(u^{n+1})\right)^{\frac{q-1}{p-1}} . \end{aligned} \tag{13}$$

We notice that because of (2) we must have $p \geq q$. Also, using (5), we see that error estimations in (9) and (11) can be obtained from (8) and (10), respectively. Now, if $p = q$, from the above equation, we easily get equation (8), where

$$C_1 = \frac{1}{\rho}\left(1-\rho+m^{p-1}\frac{\beta_M(1+C_0)}{\frac{\alpha_M}{p}} + m^{p-1}\left(\frac{\beta_M C_0}{\frac{\alpha_M}{p}}\right)^{\frac{p}{p-1}}\right). \tag{14}$$

Finally, if $p > q$, from (13), we have

$$F(u^{n+1}) - F(u) \leq C_3\left(F(u^n)-F(u^{n+1})\right)^{\frac{q-1}{p-1}} \tag{15}$$

where

$$C_3 = \frac{1-\rho}{\rho}\left(F(u^0)-F(u)\right)^{\frac{p-q}{p-1}} + \frac{m^{\frac{(p-1)q}{p}}}{\rho^{\frac{q}{p}}}\frac{\beta_M(1+C_0)}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p}}}\left(F(u^0)-F(u)\right)^{\frac{p-q}{p(p-1)}}$$
$$+\frac{m^{q-1}}{\rho^{\frac{q-1}{p-1}}}\frac{(\beta_M C_0)^{\frac{p}{p-1}}}{\left(\frac{\alpha_M}{p}\right)^{\frac{q}{p-1}}}. \tag{16}$$

From (15), we get $F(u^{n+1}) - F(u) + \frac{1}{C_3^{\frac{p-1}{q-1}}}\left(F(u^{n+1})-F(u)\right)^{\frac{p-1}{q-1}} \le F(u^n) - F(u)$, and we know (see Lemma 3.2 in [12]) that for any $r > 1$ and $c > 0$, if $x \in (0, x_0]$ and $y > 0$ satisfy $y + cy^r \le x$, then $y \le \left(\frac{c(r-1)}{crx_0^{r-1}+1} + x^{1-r}\right)^{\frac{1}{1-r}}$. Consequently, we have $F(u^{n+1}) - F(u) \le \left[C_2 + (F(u^n)-F(u))^{\frac{q-p}{q-1}}\right]^{\frac{q-1}{q-p}}$, from which,

$$F(u^{n+1}) - F(u) \le \left[(n+1)C_2 + \left(F(u^0)-F(u)\right)^{\frac{q-p}{q-1}}\right]^{\frac{q-1}{q-p}}, \tag{17}$$

where

$$C_2 = \frac{p-q}{(p-1)\left(F(u^0)-F(u)\right)^{\frac{p-q}{q-1}} + (q-1)C_3^{\frac{p-1}{q-1}}}. \tag{18}$$

Equation (17) is another form of equation (10).

3 Additive Schwarz Method as a Subspace Correction Method

The proofs of the results in this section are similar with those in the case of the multiplicative Schwarz method which are given in [1] for the infinite dimensional case, and in [2] for the one- and two-level methods. Detailed proofs for the additive method will be given in a forthcoming paper.

Let Ω be an open bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We take $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$, and a convex closed set $K \subset V$ satisfying

Property 1. If $v, w \in K$ and $\theta \in C^1(\bar{\Omega})$, with $0 \le \theta \le 1$, then $\theta v + (1-\theta)w \in K$.

We consider an overlapping decomposition of the domain Ω, $\Omega = \cup_{i=1}^m \Omega_i$, in which Ω_i are open subdomains with Lipschitz continuous boundary. We associate to this domain decomposition the subspaces $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \ldots, m$. In this case, Algorithm 1 represents an additive Schwarz method. We can show that Assumption 1 holds for any convex set K having Property 1. Consequently, the additive Schwarz method geometrically converges if the convex set has this property, but the constant C_0 in Assumption 1 depends on the domain decomposition parameters. Therefore, since the constants C_1 and C_2 in the error estimations in Theorem 1 depend on C_0, then these estimations will depend on the domain decomposition parameters, too.

When we use the linear finite element spaces we introduce similar spaces to the above ones, V_h and V_h^i, $i = 1, \ldots, m$, which are considered as subspaces of $W_0^{1,s}$. For the one- and two-level additive Schwarz methods, we can show that Assumption 1 also holds for any closed convex set K_h satisfying

Property 2. If $v, w \in K_h$, and if $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.

We have denoted by $\mathcal{T}_h$ the mesh partition of the domain, and by L_h the $\mathbb{P}_1$-Lagrangian interpolation operator which uses the function values at the mesh nodes. We can prove that Assumption 1 holds for any convex set K_h having Property 2. Moreover, in this case, we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters.

In the case of the one-level method, this constant can be written as

$$C_0 = Cm\,(1 + 1/\delta)\,, \tag{19}$$

where δ is the overlapping parameter and C is independent of the mesh parameter and the domain decomposition. In the case of the two-level method, we introduce a new subspace V_H^0 associated with the coarse mesh $\mathcal{T}_H$. The constant C_0 can be written as

$$C_0 = C(m+1)\,(1 + H/\delta)\,C_{d,s}(H,h), \tag{20}$$

where

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or } 1 \le d < s \le \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty. \end{cases} \tag{21}$$

We notice that, if the overlapping size δ and the mesh sizes H and h are chosen such that H/h and H/δ are constant, then the convergence rate of the two-level additive Schwarz method is independent of the mesh and domain decomposition parameters.

Acknowledgement. The author acknowledges the partial financial support of IMAR under the contracts CEEX05-D11-23/2005 and CEEX06-11-12/2006.

References

[1] L. Badea. Convergence rate of a multiplicative Schwarz method for strongly nonlinear inequalities. In *Analysis and Optimization of Differential Systems (also available from http://www.imar.ro/lbadea)*, pages 31–42. Kluwer Academic Publishers, Boston, 2003.

[2] L. Badea. Convergence rate of a Schwarz multilevel method for the constrained minimization of nonquadratic functionals. *SIAM J. Numer. Anal.*, 44(2):449–477, 2006.

[3] L. Badea and J. Wang. An additive Schwarz method for variational inequalities. *Math. Comp.*, 69(232):1341–1354, 1999.

[4] I. Ekeland and R. Temam. *Analyse Convexe et Problèmes Variationnels.* Dunod, Paris, 1974.

[5] R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périeux (Eds.). *First Int. Symp. on Domain Decomposition Methods.* SIAM, Philadelphia, 1988.

[6] R. Glowinski, J. L. Lions, and R. Trémolières. *Analyse Numérique des Inéquations Variationnelles.* Dunod, Paris, 1976.

[7] R. Kornhuber. *Adaptive Monotone Multigrid Methods for Nonlinear Variational Problems.* Teubner-Verlag, Stuttgart, 1997.

[8] J. L. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires.* Dunod, Paris, 1969.

[9] J. Mandel. A multilevel iterative method for symmetric, positive definite linear complementary problems. *Appl. Math. Optimization*, 11:77–95, 1984.

[10] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford Science Publications, 1999.

[11] B. F. Smith, P. E. Bjørstad, and W. Gropp. *Domain Decomposition.* Cambridge University Press, 1996.

[12] X.-C. Tai and J. Xu. Global and uniform convergence of subspace correction methods for some convex optimization problems. *Math. Comp.*, 71:105–124, 2002.

[13] A. Toselli and O. Widlund. *Domain Decomposition Methods – Algorithms and Theory.* Springer-Verlag, Berlin, 2005.