Extending the theory for iterative substructuring algorithms to less regular subdomains

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Limitations of the standard theory

We will consider finite element approximations of a selfadjoint elliptic problem on a region Ω (scalar elliptic or linear elasticity.) The domain Ω is subdivided into nonoverlapping subdomains Ω_i. In between the interface Γ.

We will consider tools for proofs of results on iterative substructuring methods, such as FETI-DP and BDDC.

We will also consider two-level overlapping Schwarz methods with a coarse space component borrowed from iterative substructuring methods, in particular BDDC. In addition, the preconditioner will also have local components based on overlapping subregions. Related work in the past by Dryja, Sarkis, and W. (on special multigrid methods); cf. Numer. Math. 1996. That paper introduced quasi-monotonicity. More recent work by Sarkis et al.

Assumptions in previous work

In the theory for methods involving such coarse components, we typically assume that:
The partition into subdomains Ω_i is such that each subdomain is the union of shape-regular coarse tetrahedral elements of a global conforming mesh $\mathcal{T}_{\mathcal{H}}$ and the number of such tetrahedra forming an individual subdomain is uniformly bounded.

In the theory for two-level Schwarz methods, we often assume that a conventional coarse space is used, defined on a coarse triangulation, and that the coefficients do not vary a lot or that they are at least quasi-monotone.

Why are these assumptions unsatisfactory?

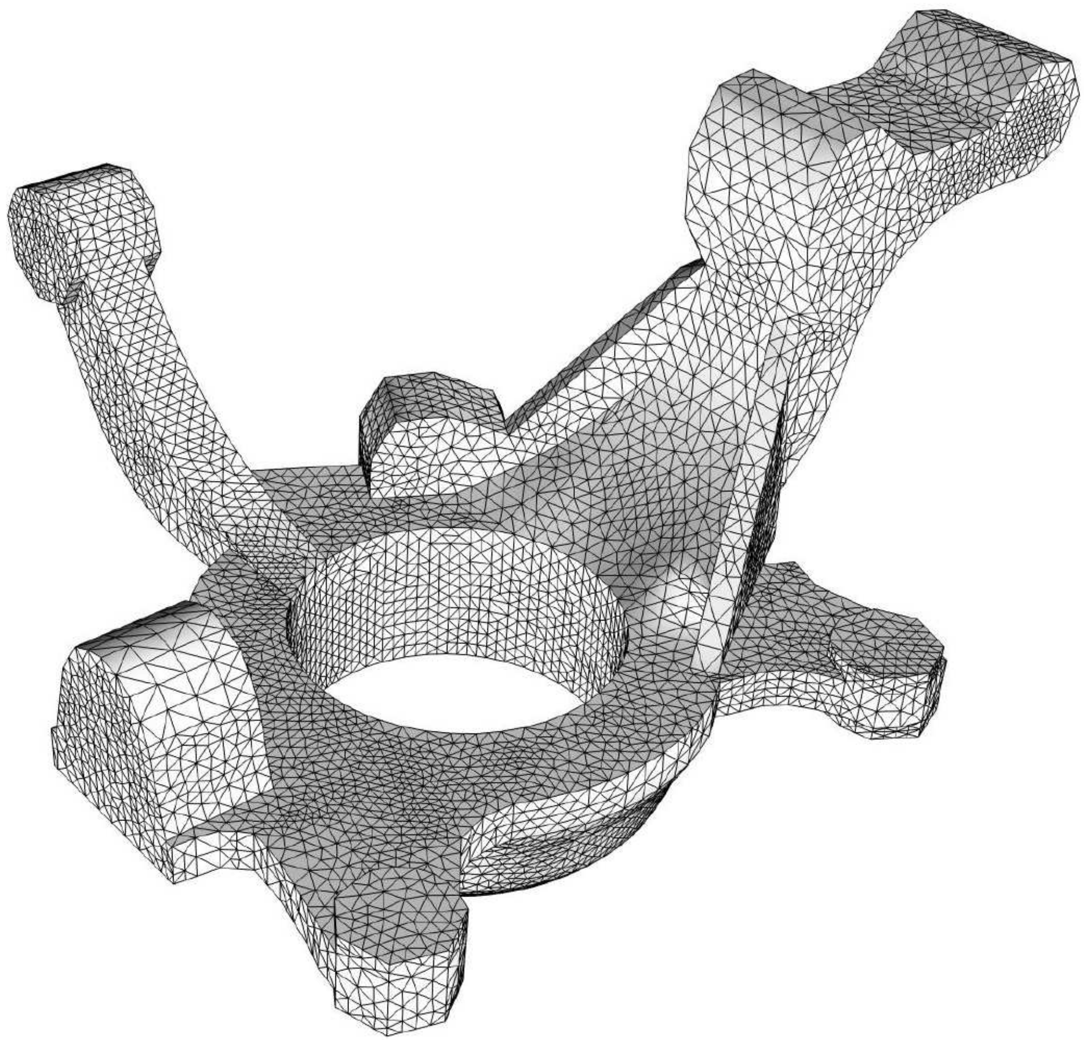

Figure 1: Finite element meshing of a mechanical object.

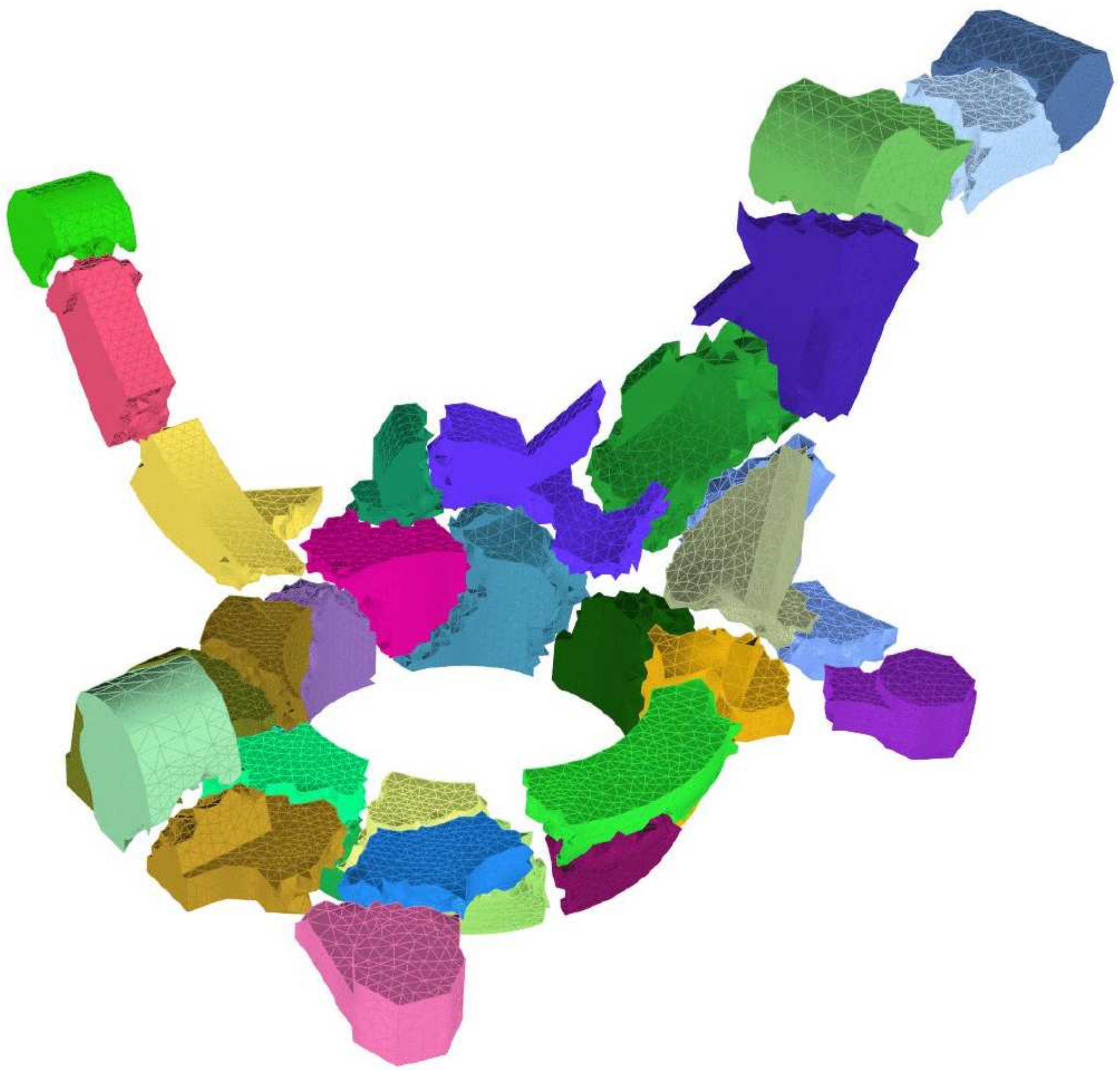

Figure 2: Partition into thirty subdomains. Courtesy Charbel Farhat.

What is needed for more general results?

In all theory for multi-level domain decomposition methods, we need a Poincaré inequality.

Theorem [Poincaré's Inequality and a Relative Isoperimetric Inequality] *Let $\Omega \subset R^n$ be open, bounded and connected. Then,*

$$\inf_{k \in R} \left(\int_\Omega |f - k|^{n/(n-1)} \, dx \right)^{(n-1)/n} \leq \gamma(\Omega, n) \int_\Omega |\nabla f| \, dx,$$

if and only if,

$$[\min(\|A\|, \|B\|)]^{1-1/n} \leq \gamma(\Omega, n) \|\partial A \cap \partial B\|. \tag{1}$$

Here $A \subset \Omega$, $B = \Omega \setminus A$.

This result can be found in a book Lin and Yang, "Geometric Measure Theory – An Introduction".

Using Hölder's inequality several times, we find, for $n = 3$, that

$$\inf_{k \in R} \|u - k\|_{L^2(\Omega)} \leq \gamma(\Omega, n) \mathsf{Vol}(\Omega)^{1/3} \|\nabla u\|_{L^2(\Omega)}.$$

This is the conventional form of Poincaré's inequality. (Thanks to Fanghua Lin and Hyea Hyun Kim.)

The parameter in this inequality enters into all bounds of our result and it is closely related to the second eigenvalue of the Laplacian with Neumann boundary conditions.

We (re)learn from this result that we have to expect slow convergence if the subdomains are not shape regular. We can also have problems if elements at the boundary are not shape regular. (Consider a slim bar.)

But we also see that under reasonable assumptions on our subregions, we can expect a satisfactory parameter in the Poincaré inequality.

Another important tool is a simple trace theorem:

$$\beta \|u\|^2_{L^2(\partial\Omega)} \leq C(\beta^2 |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)}).$$

The parameter β measures the thickness of Ω. This result is borrowed from Nečas' 1967 book and it is proven under the assumption that the region is Lipschitz; C is proportional to the Lipschitz constant.

One can easily construct subdomains which are not Lipschitz, but there are also trace theorems for more general regions under reasonable geometric assumptions.

I have not yet put these matters in a final form. In the literature, we find Fritz John regions, carrots, cigars, etc.

Overlapping Schwarz methods

Consider a scalar elliptic equation defined by a bilinear form

$$\sum \int_{\Omega_j} \rho_j \, \nabla u \cdot \nabla v \, dx.$$

The coefficients ρ_j are arbitrary positive constants and the Ω_j are quite general subdomains.

A natural coarse space is the range of the following interpolation operator

$$I_B^h u(x) = \sum_{V^k \in \Gamma} u(V^k)\theta_{V_k}(x) + \sum_{E^i \subset W} \bar{u}_{E^i}\theta_{E^i}(x) + \sum_{F^k \subset \Gamma} \bar{u}_{F^k}\theta_{F^k}(x).$$

Here $\bar{u}_{E^i}$ and $\bar{u}_{F^k}$ are averages over edges and faces of the subdomains.

$\theta_{V_k}(x)$ the standard nodal basis functions of the vertices of the subdomains, $\theta_{E^i}(x) = 1$ at the nodes of the edge E^i and vanishes at all other interface nodes, and $\theta_{F^k}(x)$ is a similar function defined for the face F^k. These functions are extended as discrete harmonic functions in the interior of the subdomains. Note that this interpolation operator, I_B, preserves constants. A slightly richer coarse space will preserve all linear functions; useful for elasticity.

Faces, edges, and vertices of quite general subdomains can be defined in terms of certain equivalence classes. We will now consider the energy of the face terms and estimate their energy in terms of the energy of the function interpolated. We can estimate the averages $\bar{u}_{F^k}$ by Cauchy-Schwarz and the trace theorem.

Estimates of the energy of $\theta_{F^k}(x)$ well known for special regions, e.g., tetrahedra; bounds are $C(1 + \log(H/h)H$. We will consider, in detail, two dimensions only, and construct functions ϑ_E forming a partition of unity.

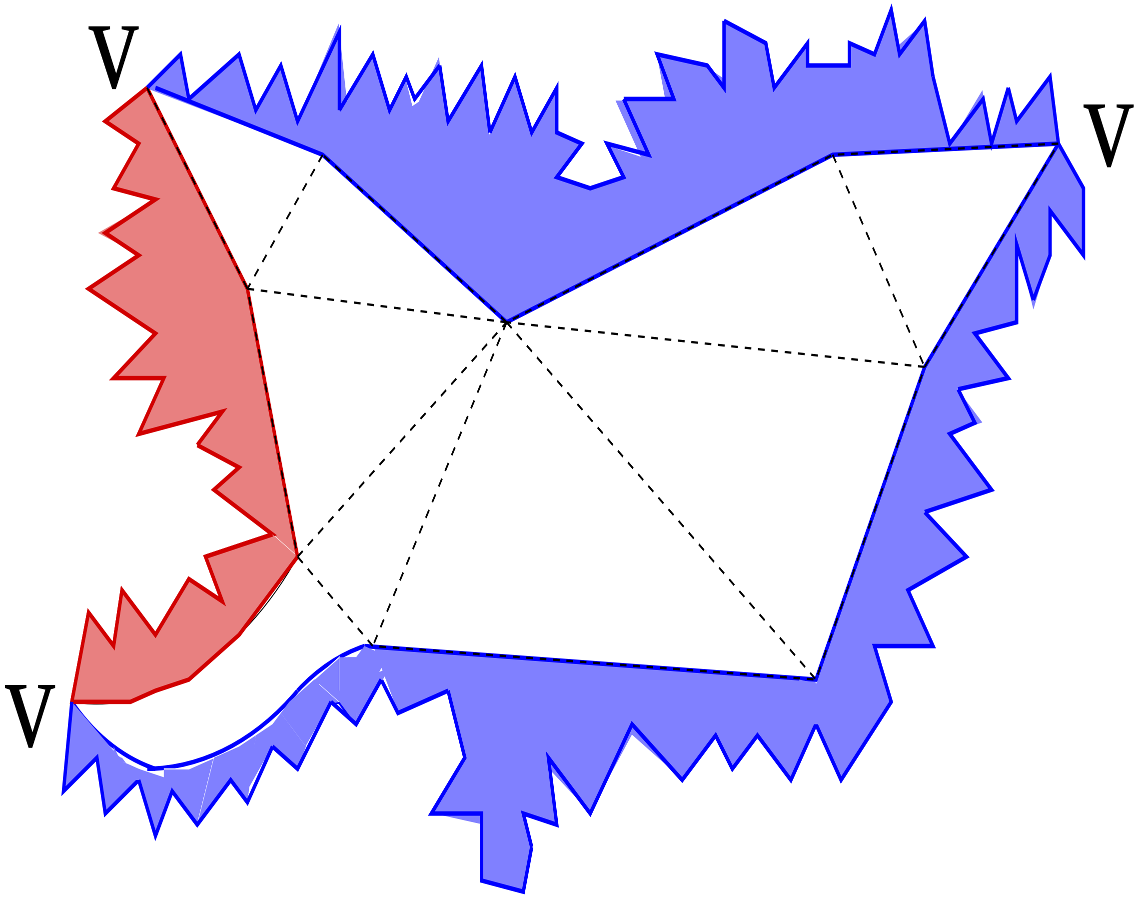

Figure 3: Construction of ϑ_E in 2D.

The overlapping subregions are unions of elements and can be chosen quite generally. We assume that they have satisfactory Poincaré parameters and each has a diameter comparable to the subregions which they intersect. The interface Γ can intersect these subregions arbitrarily.

The proof of our result uses a traditional argument on stable decompositions, which is a main part of the abstract Schwarz theory. The coarse component contributes a logarithmic factor that orginates with the bound for θ_F functions and a bound on the edge averages $\bar{u}_E$. The bounds for the local components are done using a partition of unity related to the overlapping subregions and a Friedrichs inequality on patches of diameter δ. The patches are chosen so that the coefficient of the elliptic problem is constant in each of them; see also Chap. 3 of the T. & W. book. A second factor $(1 + H/\delta)$ comes from estimates of the local components; Brenner has shown that this factor cannot be improved.

Result on the two-level overlapping Schwarz method

Theorem. *Under the given assumptions, the condition number κ of the preconditioned operator satisfies*

$$\kappa \leq C(1 + H/\delta)(1 + \log(H/h)).$$

Here C is independent of the mesh size, the number of subdomains, the coefficients ρ_i, etc. H/δ measures the relative overlap between neighboring overlapping subregions. H/h measures the maximum number of elements across any subregion. The logarithmic factor can be removed, in some cases, if the coefficients are comparable and the coarse space contains the linear functions.

Extension of theory for iterative substructuring methods

The technical tools necessary for the traditional analysis of the rate of convergence of iterative substructuring methods are collected in Section 4.6 of the T. & W. book. Among the tools necessary for the analysis of BDDC and FETI-DP in three dimensions is a bound on the energy of

$$I^h(\vartheta_{F^k} u)$$

and bounds on the corresponding edge functions. These bounds feature a second logarithmic factor. The old results on special subdomains can be extended to much more general subdomains; no new ideas are required.

It is known that the estimate of the condition numbers of BDDC and FETI–DP can be reduced to bound of an averaging operator E_D across the interface. On each subdomain face, e.g., we have a weighted average of the traces of functions defined in the relevant pair of subdomains. The weights depend on the coefficients of the elliptic problem. We have to cut the traces using ϑ_{F^k}, etc. We then estimate the energy of resulting components in terms of the energy of the functions, given on the subdomains, from which the averages are computed. Two logarithmic factors result.

These bounds have previously been developed quite rigorously for the case of simple polyhedral subdomains, for scalar elliptic problems, compressible elasticity, flow in porous media, Stokes and almost incompressible elasticity. For each of these cases, we have to select the coarse component and certain scale factors of the preconditioner quite carefully; that is not today's story. What is new is that we can obtain bounds, in many cases, which are of good quality for more general subdomains.