An all-floating formulation of the BETI method

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Outline

- Linear elastostatics and boundary element method
- Dirichlet domain decomposition method
- Boundary Element Tearing and Interconnecting method
- Inversion of Steklov Poincaré operators
- Floating subdomains
- All–floating BETI formulation
- Numerical examples

Linear elastostatics

Mixed boundary value problem:

$$\begin{aligned} -\mathrm{div}\,(\sigma(u)) &= 0 && \text{for } x \in \Omega \subset \mathbb{R}^3, \\ u_i(x) &= g_{D,i}(x) && \text{for } x \in \Gamma_{D,i}, i = 1, \ldots, 3, \\ t_i(x) := (T_x u)_i(x) = (\sigma(u)n(x))_i &= g_{N,i}(x) && \text{for } x \in \Gamma_{N,i}, i = 1, \ldots, 3. \end{aligned}$$

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The stress tensor $\sigma(u)$ is related to the strain tensor $e(u)$ by Hooke's law

$$\sigma(u) = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\text{tr}\; e(u) I + \frac{E}{(1+\nu)} e(u)\right).$$

Young modulus $E > 0$, Poisson ratio $\nu \in (0, \frac{1}{2})$,
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$$e(u) = \frac{1}{2}(\nabla u^\top + \nabla u).$$

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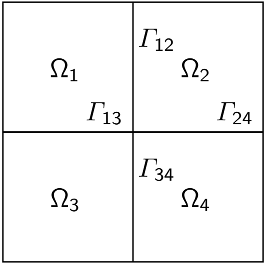
Bounded Lipschitz domain Ω given by a domain decomposition into p non–overlapping subdomains.

Assumptions:

piecewise constant: $E_i > 0$ and $\nu_i \in (0, 1/2)$.



Boundary integral formulation

Representation formula:

$$u(x) = \int_{\Gamma} [U^*(x,y)]^\top t(y) ds_y - \int_{\Gamma} [T_y U^*(x,y)]^\top u(y) ds_y \qquad \text{for } x \in \Omega.$$

Fundamental solution of linear elastostatics:

$$U^*_{kl}(x-y) = \frac{1+\nu}{8\pi E(1-\nu)} \left[(3-4\nu) \frac{\delta_{kl}}{|x-y|} + \frac{(x_k - y_k)(x_l - y_l)}{|x-y|^3} \right].$$

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Calderon projector for the Cauchy data $u(x)$ and $t(x)$ on the boundary Γ:

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} \qquad \text{on } \Gamma$$

Boundary integral operators:

$$(Vt)(x) = \int_\Gamma [U^*(x,y)]^\top t(y) ds_y, \; (Du)(x) = -T_x \int_\Gamma [T_y U^*(x,y)]^\top u(y) ds_y,$$

$$(Ku)(x) = \int\!\!\!\!\!\!- _\Gamma [T_y U^*(x,y)]^\top u(y) ds_y, \; (K't)(x) = \int\!\!\!\!\!\!- _\Gamma [T_x U^*(x,y)]^\top t(y) ds_y.$$

Dirichlet domain decomposition method

Solve the global system of linear equations iteratively:

$$\widetilde{S}_h \underline{u} = \sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \underline{u} = \sum_{i=1}^{p} A_i^\top \underline{f_i},$$

with some connectivity matrices $A_i \in \mathbb{R}^{M_i \times M}$ mapping from the global nodes to the local nodes such that $\underline{v}_i = A_i \underline{v}$.

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Local realization of the matrices of the Steklov Poincaré operators:

$$\widetilde{S}_{i,h} = D_{i,h} + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) V_{i,h}^{-1} (\frac{1}{2} M_{i,h} + K_{i,h}),$$

with the boundary element matrices realized by the Fast Multipole Method (integration by parts ⇒ single and double layer potentials of the Laplacian)

$$\begin{aligned} V_{i,h}[\ell,k] &= \langle V_i \varphi_k^i, \varphi_\ell^i \rangle_{L_2(\Gamma_i)}, & K_{i,h}[\ell,n] &= \langle K_i \psi_n^i, \varphi_\ell^i \rangle_{L_2(\Gamma_i)}, \\ D_{i,h}[m,n] &= \langle D_i \psi_n^i, \psi_m^i \rangle_{L_2(\Gamma_i)}, & M_{i,h}[\ell,n] &= \langle \psi_n^i, \varphi_\ell^i \rangle_{L_2(\Gamma_i)}. \end{aligned}$$

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Preconditioning:

$$C_{\widetilde{S}}^{-1} = \sum_{i=1}^{p} A_i^\top V_{i,lin,h} A_i$$

BETI method

Starting from the equivalent minimization problem

$$F(\underline{u}) = \min_{\underline{v}\in\mathbb{R}^M} \sum_{i=1}^{p} \left[\frac{1}{2}(\widetilde{S}_{i,h}A_i\underline{v}, A_i\underline{v}) - (\underline{f}_i, A_i\underline{v})\right]$$

one can derive the Boundary Element Tearing and Interconnecting method [Langer, Steinbach 2003] (FETI [Farhat, Roux 1991; Klawonn,Widlund 2001; ...]):

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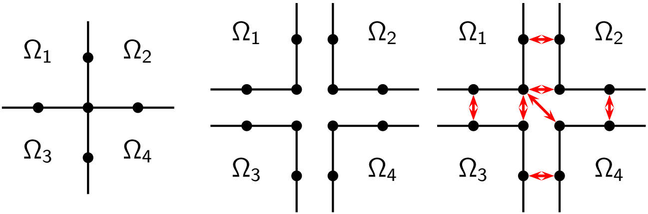
- introducing local vectors $\underline{u}_i = A_i\underline{u}$.
- describing the connections across the interfaces by introducing the constraints

$$\sum_{i=1}^{p} B_i\underline{u}_i = \underline{0} \qquad \text{where } B_i \in \mathbb{R}^{\widetilde{M}\times M_i}.$$



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$$\begin{pmatrix} \widetilde{S}_{1,h} & & & -B_1^\top \\ & \ddots & & \vdots \\ & & \widetilde{S}_{p,h} & -B_p^\top \\ B_1 & \dots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{0} \end{pmatrix}.$$

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Dirichlet b.c.,$\widetilde{S}_{i,h}$ are invertible: $\underline{u}_i = \widetilde{S}_{i,h}^{-1}(\underline{f}_i + B_i^\top \underline{\lambda})$.

Then the bottom line of the linear system leads to the Schur complement system:

$$\sum_{i=1}^{p} B_i \widetilde{S}_{i,h}^{-1} B_i^\top \underline{\lambda} = -\sum_{i=1}^{p} B_i \widetilde{S}_{i,h}^{-1} \underline{f}_i.$$

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Scaled hypersingular BETI preconditioner (appropriate scaling matrices C_E):

$$C^{-1} = (BC_E^{-1}B^\top)^{-1} BC_E^{-1} D_h C_E^{-1} B^\top (BC_E^{-1}B^\top)^{-1}.$$

Condition number for the scaled hypersingular BETI preconditioner:

$$\kappa(C^{-1}B\widetilde{S}^{-1}B^\top) \le c\,(1 + \log H/h)^2\,.$$

Inversion of the Steklov Poincaré operator

$$u \in H_*^{1/2}(\Gamma) : Su(x) = g(x) \text{ for } x \in \Gamma \text{ with } \int_\Gamma v_k(x) \cdot g(x) ds_x = 0 \quad \forall v_k \in \mathcal{R}.$$

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Modified variational formulation in $H^{1/2}(\Gamma)$:

$$\langle \widetilde{S} u, v\rangle_\Gamma := \langle Su, v\rangle_\Gamma + \sum_{k=1}^{6} \alpha_k \langle u, \widetilde{w}_k\rangle_\Gamma \langle v, \widetilde{w}_k\rangle_\Gamma = \langle g, v\rangle_\Gamma$$

with orthogonal basis $\tilde{v}_k$ of the rigid body motions (Gram–Schmidt):

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Theorem: Spectral equivalence inequalities:

$$\sigma_1 \langle V_L^{-1} v, v \rangle_\Gamma \le \langle \widetilde{S} v, v \rangle_\Gamma \le \sigma_2 \langle V_L^{-1} v, v \rangle_\Gamma \text{ für alle } v \in H^{1/2}(\Gamma)$$

where

$$\begin{aligned} \sigma_1 &= \min\{ c_1^{V_L} \tilde{c}_1^D, \alpha_k \langle \tilde{v}_k, \widetilde{w}_k \rangle_\Gamma \}, \\ \sigma_2 &= \max\left\{ \left(\frac{1}{4} + c_K \right) \frac{E}{1 - 2\nu} \frac{1 - \nu}{1 + \nu}, \alpha_k \langle \widetilde{v}_k, \widetilde{w}_k \rangle_\Gamma \right\}. \end{aligned}$$

Preconditioning by boundary integral operators of opposite order:
[Steinbach, Wendland 95,98; McLean, Steinbach 99]

Floating subdomains

Without Dirichlet bc: local Steklov Poincaré operators $S_{i,h}$ are singular.
Solve local subproblems by the modified operator: $(\gamma_{k,i} \in \mathbb{R})$

$$\underline{u}_i = \widetilde{S}_{i,h}^{-1}(\underline{f}_i - B_i^\top \underline{\lambda}) + \sum_{k=1}^{6} \gamma_{k,i} \underline{v}_{k,i} \qquad \text{für } i = 1, \ldots, q.$$

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From BETI constraints: $\left(G = (B_1 \mathcal{R}_1, \ldots, B_q \mathcal{R}_q) \in \mathbb{R}^{M \times 6 \cdot q}\right)$

$$\left[\sum_{i=1}^{q} B_i \widetilde{S}_{i,h}^{-1} B_i^\top + \sum_{i=q+1}^{p} B_i S_{i,h}^{-1} B_i^\top \right] \underline{\lambda} - G \underline{\gamma} = \underline{d},$$

rewritten as

$$F \underline{\lambda} - G \underline{\gamma} = \underline{d} \qquad \text{unter} \qquad G^\top \underline{\lambda} = ((\underline{f}_i, \mathcal{R}_i))_{i=1:q}\,.$$

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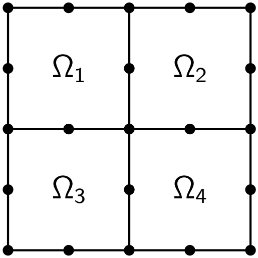
By an orthogonal projection $P = I - QG(G^\top QG)^{-1}G^\top$:
Lagrangian multipliers $\underline{\lambda}$ and constants $\underline{\gamma}$ from:

$$P^\top F\underline{\lambda} = P^\top \underline{d} \qquad \text{and then} \qquad \underline{\gamma} = (G^\top QG)^{-1} G^\top Q(F\underline{\lambda} - \underline{d}).$$

Idea of the all–floating BETI formulation



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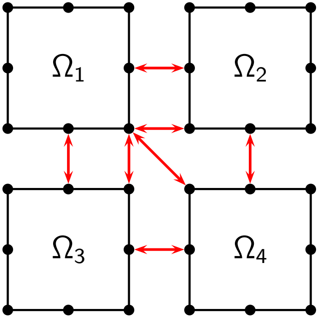

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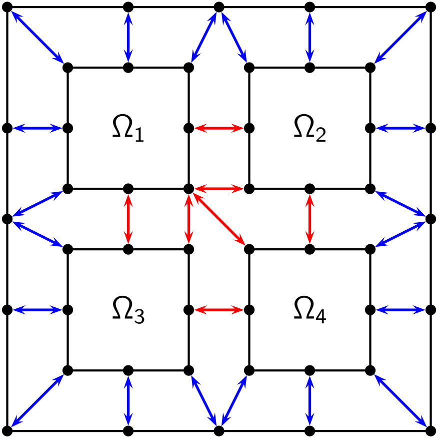

Derivation of the all–floating formulation

Starting from the system of linear equations

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respectively the equivalent minimization problem

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Using the BETI ideas and $\widetilde{v}_i = v_i + \widetilde{g}_{|\Gamma_i} \longrightarrow$ local minimization problems

$$\widetilde{F}(\underline{\widetilde{u}}_i) = \min_{\underline{\widetilde{v}}_i \in \mathbb{R}^{M_i}} \frac{1}{2}(\widetilde{S}_{i,h} \underline{\widetilde{v}}_i, \underline{\widetilde{v}}_i).$$

under the constraints

$$\sum_{i=1}^{p} B_i \underline{\widetilde{v}}_i = \underline{0} \text{ on } \Gamma_C, \qquad \underline{\widetilde{v}}_i = \underline{g} \text{ on } \Gamma_D.$$

Derivation of an all floating formulation

Introducing Lagrange multipliers $\underline{\lambda} \in \mathbb{R}^{M_L}$ we get the system

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$$\sum_{i=1}^{p} \widetilde{B}_i \widetilde{S}_{i,h}^{-1} \widetilde{B}_i^\top \underline{\lambda} - G\underline{\gamma} = \underline{b}.$$

- larger number of unkowns
- larger blocks $\widetilde{S}_{i,h}$
- better condition number for the preconditioner of $\widetilde{S}_{i,h}$
- unified treatment of all subdomains $\longrightarrow$ linear elastostatics (Neumann b.c.)

BETI as saddle point problems

BETI skew–symmetric system of linear equations (Bramble and Pasciak, 1988)

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BETI twofold saddle point problem ($\widetilde{K}_{i,h} = \frac{1}{2} M_{i,h} + K_{i,h}$, Zulehner 2005)

$$\begin{pmatrix} V_{1,h} & & & -\widetilde{K}_{1,h} & & & \\ & \ddots & & & \ddots & & \\ & & V_{p,h} & & & -\widetilde{K}_{p,h} & \\ \widetilde{K}_{1,h}^\top & & & D_{1,h} & & & -B_1^\top \\ & \ddots & & & \ddots & & \vdots \\ & & \widetilde{K}_{p,h}^\top & & & D_{p,h} & -B_p^\top \\ & & & B_1 & \dots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \vdots \\ \underline{t}_p \\ \underline{u}_1 \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \vdots \\ \underline{0} \\ \underline{f}_1 \\ \vdots \\ \underline{f}_p \\ \underline{0} \end{pmatrix}.$$

Complexity of BETI methods

Theorem

Requirement: algebraic multigrid preconditioner for V_i is optimal.
Solving by Bramble Pasciak transformation and conjugate gradient method:

Twofold saddle point problem of the standard BETI method:

- *number of iterations* $\mathcal{O}((1+\log(H/h))^2)$
- $\mathcal{O}((H/h)^2(1+\log(H/h))^4)$ *arithmetical operations*

Complexity of BETI methods

Theorem

Requirement: algebraic multigrid preconditioner for V_i is optimal.
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Twofold saddle point problem of the all–floating BETI method:

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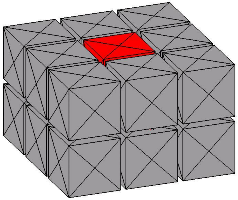
The use of fast boundary element methods does not perturb the convergence rates of the approximation.

Example: steel and concrete



18 subdomains

	BETI		all–floating	
L	t_2	It.	t_2	It.
0	31	19(21(10))	39	22(17(10))
1	217	28(33(14))	170	24(27(14))
2	2129	35(44(14))	1437	27(33(14))
3	14149	42(51(14))	9005	32(36(14))
4	116404	47(54(14))	77111	38(38(15))

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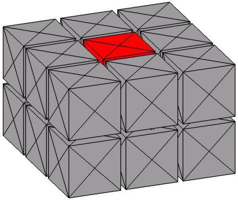

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		Dirichlet DD		BETI			all–floating		
L	N_i	t_2	It.	#duals	t_2	It.	#duals	t_2	It.
0	24	7	53(10)	267	7	78	537	8	65
1	96	25	110(14)	927	19	100	1629	19	82
2	384	181	130(14)	3435	112	114	5649	115	85
3	1536	986	148(14)	13203	562	129	21033	476	95
4	6144	6902	154(14)	51747	4352	153	81177	3119	105
5	24576	59264	166(16)	204867	31645	172	318969	23008	120

Scalability: cube, mixed bvp, Laplace equation

	8 finer subdomains 64 subdomains					
L	N_i	#duals	t_1	t_2	It.	D-error
0	96	221	4	3	29	$1.35e-2$
	24	613	5	6	32	$1.10e-2$
1	384	753	8	7	36	$3.88e-3$
	96	1865	7	5	37	$3.45e-3$
2	1536	2777	21	34	41	$9.91e-4$
	384	6481	11	10	47	$9.09e-4$
3	6144	10665	82	194	46	$2.28e-4$
	1536	24161	23	48	52	$2.22e-4$
4	24576	41801	287	1811	53	$6.13e-5$
	6144	93313	90	307	60	$4.67e-5$
5	98304	165513	1358	10485	62	$1.61e-5$
	24576	366785	312	2658	70	$1.40e-5$

total number of degrees of freedom on finest level: 1345177 and 2726209

Conclusions and future work

All–floating BETI formulation:

- unified treatment of all subdomains
- simpler implementation
- improved asymptotic complexity

- real life applications
- nearly incompressible materials
- coupling with finite elements
- coupled field problems
- automatic generation of domain decompositions
- Helmholtz
- Maxwell
- ...