Hp-spectral FEM's in fast domain decomposition algorithms

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An outline of the lecture

- Introduction: the state of art in developing fast solvers.
- Finite-difference/fem preconditioners for hierarchical and spectral p elements.
- Factorized preconditioners for spectral elements and their similarity to the preconditioners-solvers for hierarchical elements.
- Examples of the factorized fast solvers for spectral elements :
  - ✓ `2-d multigrid solver,`
  - ✓ `3-d fast solver based on the wavelet multilevel decompositions,`
  - ✓ `multilevel solver for faces.`
- Almost optimal in the arithmetic cost domain decomposition preconditioner-solver for hp spectral element methods.
- Conclusions.

Preconditioners for hierarchical elements

$\mathcal{M}_{1,p} = (\mathcal{L}_i(s)\,,\ i = 0, 1, \ldots, p)$ – set of polynomials on (-1,1):

$$\mathcal{L}_0(s) = \tfrac{1}{2}(1 + s)\,, \qquad \mathcal{L}_1(s) = \tfrac{1}{2}(1 - s)\,,$$

$$\mathcal{L}_i(s) := \beta_i \int_{-1}^{s} P_{i-1}(t)\,dt = \gamma_i[P_i(s) - P_{i-2}(s)]\,, \quad i \geq 2\,,$$

P_i are Legendre's polynomials and

$$\beta_i = \frac{1}{2}\sqrt{(2j-3)(2j-1)(2j+1)}\,, \gamma_i = 0.5\sqrt{(2i-3)(2i+1)/(2i-1)}\,.$$

Therefore, $\mathcal{L}_i$ are specially normalized integrated Legendre's polynomials.

By hierarchical ref. el. $\mathcal{E}_{\rm hi}$ is understood ref.el. on the cube $\tau_0 = (-1,1)^d$ with the basis in the space $\mathcal{Q}_{p,x}$

$$\mathcal{M}_{d,p} = \big(L_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathcal{L}_{\alpha_1}(x_1)\mathcal{L}_{\alpha_2}(x_2)...\mathcal{L}_{\alpha_d}(x_d)\ ,\ \boldsymbol{\alpha} \in \omega\big)\ ,$$

$$\omega := (\boldsymbol{\alpha} = (\alpha_1, \alpha_2, .., \alpha_d) : 0 \le \alpha_1, \alpha_2, .., \alpha_d \le p)\ ,$$

and with the stiffness matrix $\mathbf{A}$, induced by $\mathcal{M}_{d,p}$ and Dirichlet integral

$$a_{\tau_0}(u,v) = \int_{\tau_0} \nabla u \cdot \nabla v \, d\mathbf{x}\, .$$

$\mathbf{A}_I$ – *internal* stiff. matrix, generated by $\overset{\circ}{\mathcal{M}}_{d,p} = (L_{\boldsymbol{\alpha}},\ 2 \le \alpha_k \le p)$.

If to reorder set $\mathring{\mathcal{M}}_{d,p}$, matrices $\mathbf{A}_I$, $\mathbf{M}_I$ in $d = 3$ become block diagonal

$$\mathbf{A}_I = \text{diag}\left[\mathbf{A}_{eee}, \mathbf{A}_{eeo}, \ldots, \mathbf{A}_{ooe}, \mathbf{A}_{ooo}\right] \,,$$

$$\mathbf{M}_I = \text{diag}\left[\mathbf{M}_{eee}, \mathbf{M}_{eeo}, \ldots, \mathbf{M}_{ooe}, \mathbf{M}_{ooo}\right] \,.$$

At $p = 2N + 1$ all 8 blocks are $N^3 \times N^3$ matrices and, *e.g.*,

$$\mathbf{A}_{a_1 a_2 a_3} = \left(a_{\tau_0}(L\boldsymbol{\alpha}, L\boldsymbol{\alpha}')\right)_{\alpha_k, \alpha'_k = 1}^{N} \,,$$

with α_k, α'_k even/odd respectively to even/odd a_k.

These blocks are sums

$$\mathbf{A}_{abc} = \mathbb{K}_{1,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{0,c} + \mathbb{K}_{0,a} \otimes \mathbb{K}_{1,b} \otimes \mathbb{K}_{0,c} + \mathbb{K}_{0,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{1,c} \ ,$$

$$\mathbf{M}_{abc} = \mathbb{K}_{0,a} \otimes \mathbb{K}_{0,b} \otimes \mathbb{K}_{0,c} \, , \qquad a, b, c = e, o$$

of Kronecker products of triplets of $N \times N$ matrices, which may be preconditioned by simple matrices

$$\boldsymbol{\mathcal{D}} = \mathrm{diag}\,[4i^2]_{i=1}^{N} \ , \qquad \boldsymbol{\Delta} = \frac{1}{2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & & \cdots & \cdots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} .$$

Lemma 1. For 1-d preconditioners $\boldsymbol{\mathcal{D}}, \boldsymbol{\Delta}$ and 3-d preconditioners

$$\boldsymbol{\Lambda}_{\mathrm{e}} = \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{D}} + \boldsymbol{\Delta} \times \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta} + \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\Delta}\,, \qquad \boldsymbol{\mathcal{M}} = \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\Delta}\,,$$

there hold the inequalities

$$\boldsymbol{\Delta} \prec \mathbb{K}_{0,a} \prec \boldsymbol{\Delta}\,, \qquad \boldsymbol{\mathcal{D}} \prec \mathbb{K}_{1,a} \prec \boldsymbol{\mathcal{D}}\,,$$

$$\boldsymbol{\Lambda}_{\mathrm{e}} \prec \mathbf{A}_{abc} \prec \boldsymbol{\Lambda}_{\mathrm{e}}\,, \qquad \boldsymbol{\mathcal{M}} \prec \mathbf{M}_{abc} \prec \boldsymbol{\mathcal{M}}\,.$$

Proof. Ivanov/Korneev [1995] and Korneev/Jensen [1997],
Korneev/Langer/Xanthis [2003]. □

Finite-difference interpretation

In 2-d

$$\boldsymbol{\Lambda}_{\mathrm{e}} = \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{D}} \otimes \boldsymbol{\Delta}$$

and is the F-D approximation of the differential operator

$$Lu \equiv -2\left(x_1^2 \frac{\partial^2 u}{\partial x_2^2} + x_2^2 \frac{\partial^2 u}{\partial x_1^2}\right) , \quad x \in \pi_1 := (0,1)^2 , \quad u|_{\partial\pi_1} = 0 ,$$

on the square mesh of size $\hbar = 1/(N+1)$. In 3-d, $\hbar^{-2}\boldsymbol{\Lambda}_{\mathrm{e}}$ is the F-D approximation on the same mesh of the 4-th order operator

$$Lu \equiv x_3^2 u_{,1,1,2,2} + x_2^2 u_{,1,1,3,3} + x_1^2 u_{,2,2,3,3} = f(x) , \quad x \in \pi_1 := (0,1)^3 , \quad u|_{\partial\pi_1} = 0 ,$$

where, e.g., $u_{,1,1,2,2} = \partial^4 u / \partial x_1^2 \partial x_2^2$.

FEM preconditioner

Suppose, $d = 3$, $\overset{\circ}{\mathcal{V}}(\pi_1)$ is the space of continuous on $\overline{\pi}_1$ and piece wise trilinear on each cell of the cubic mesh functions, vanishing on $\partial\pi_1$, and $\mathbf{\Lambda}_{\rm e,fem}$ is the corresponding to this space matrix of bilinear form

$$b_{\pi_1}(u,v) = \sum_{k=1}^{3} \int_{\pi_1} \varphi_k u_{,k+1,k+2} v_{,k+1,k+2} dx \,, \qquad \varphi_k = x_k^2 \,.$$

Lemma 2. The matrix $\frac{1}{\hbar}\mathbf{\Lambda}_{\rm e,fem}$ is spectrally equivalent to $\mathbf{A}_{abc}$ and $\mathbf{\Lambda}_{\rm e}$ uniformly in p.

Proof. See, *e.g*, Korneev [2002]. □

In 2-d, one can use the FE space $\overset{\circ}{\mathcal{V}}_{\triangle}(\pi_1)$ of continuous and piece wise linear functions on the triangulation, obtained by subdivision of each square nest of the mesh in two triangles. Preconditioner $\mathbf{\Lambda}_{\mathrm{e,fem}}$ is matrix of the bilinear form

$$b_{\pi_1}(u,v) = \sum_{k=1}^{2} \int_{\pi_1} \varphi_k u_{,3-k} v_{,3-k} dx \,,$$

on the space $\overset{\circ}{\mathcal{V}}_{\triangle}(\pi_1)$. We have $\mathbf{\Lambda}_{\mathrm{e,fem}} \asymp \hbar^2 \mathbf{A}_{abc}, \hbar^2 \mathbf{\Lambda}_{\mathrm{e}}$.

Preconditioners for the spectral elements

GLL (Gauss-Lobatto-Legendre) nodes η_i satisfy equation

$$(1-\eta_i^2)P_p'(\eta_i)=0\,, \qquad i=0,1,..,p\,,$$

whereas for GLC (Gauss-Lobatto-Chebyshev) nodes we have

$$\eta_i=\cos\left(\frac{\pi}{p}(p-i)\right)\,, \qquad i=0,1,..,p\,.$$

Orthogonal tensor product grid $\mathbf{x}=\boldsymbol{\eta}_{\boldsymbol{\alpha}}=(\eta_{\alpha_1},\eta_{\alpha_2},..,\eta_{\alpha_d})$, $\boldsymbol{\alpha}\in\omega$, with GLC or GLC nodes is termed Gaussian, whereas both types of the Lagrange reference elements are termed (for brevity) spectral. In their coordinate polynomials $L_{\boldsymbol{\alpha}}(\mathbf{x})=\mathcal{L}_{\alpha_1}(x_1)\mathcal{L}_{\alpha_2}(x_2)...\mathcal{L}_{\alpha_d}(x_d)$, 1-d polynomials satisfy $\mathcal{L}_i(\eta_j)=\delta_{i,j}$, $0\le j\le p$, where $\delta_{i,j}$ is the Kronecker's delta.

For steps $\hbar_i := \eta_i - \eta_{i-1}, i \le N$, of the Gaussian mesh, we have $\hbar_i \asymp i/p^2$. Mesh of a more general class satisfy

$$c_1 \frac{i^\gamma}{\aleph} \le \hbar_i \le c_2 \frac{i^\gamma}{\aleph}\,, \qquad \aleph = \sum_{i=1}^N i^\gamma, \qquad \gamma \ge 0,$$

on [-1,0] and is continued on [0,1] by symmetry.

✓ At $\gamma = 0 \;\Rightarrow\; \aleph = N$ – quasiuniform mesh,

✓ at $\gamma = 1 \;\Rightarrow\; \aleph = N(N+1)/2$ – mesh, called pseudospectral, for which at $c_1 = c_2 = 1$, we have

$$\hbar_i = i/\aleph = \frac{i}{(N^2+N)} = \beta(p)\, i/p^2\,, \qquad \beta \in [4,8]\,.$$

$\mathbf{A}_{\rm Sp}$, $\mathbf{A}_{\rm Psp}$ – notations for ref. el. stiffness matrices for Gaussian and pseudospectral nodes, respectively,

$\boldsymbol{\mathcal{A}}_{\rm Sp}$, $\boldsymbol{\mathcal{A}}_{\rm Psp}$ – notations for preconditioners, which are FE matrices, induced by the space $\mathcal{H}(\tau_0) \cap C(\overline{\tau}_0)$ of continuous functions belonging to $\mathcal{Q}_{1,x}$ on each square nest of the corresponding mesh.

Simpler preconditioner

$$\mathbb{A}_\hbar = \mathbf{\Delta}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \mathbf{\Delta}_\hbar \otimes \mathbb{D}_\hbar + \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbf{\Delta}_\hbar \,,$$

where

$$\mathbb{D}_\hbar = \operatorname{diag}[\widetilde{\hbar}_i = \frac{1}{2}(\hbar_i + \hbar_{i+1})]_{i=0}^{p} \,, \quad \widetilde{\hbar}_i = 0 \text{ for } i = 0, p+1 \,,$$

and $\mathbf{\Delta}_\hbar$ is FE matrix:

$$(\mathbf{\Delta}_\hbar\, \mathbf{u})|_i = -\frac{1}{\hbar_i} u_{i-1} + (\frac{1}{\hbar_i} + \frac{1}{\hbar_{i+1}}) u_i - \frac{1}{\hbar_{i+1}} u_{i+1} \,, \quad i = 1, 2, .., p-1 \,,$$

$$(\mathbf{\Delta}_\hbar\, \mathbf{u})|_{i=0} = -\frac{1}{\hbar_1}(u_1 - u_0) \,, \quad (\mathbf{\Delta}_\hbar\, \mathbf{u})|_{i=p} = \frac{1}{\hbar_p}(u_p - u_{p-1}) \,.$$

Lemma 3. Let $\mathbb{A}_\hbar$ be obtained on Gaussian or pseudospectral ($\gamma = 1$) mesh. Stiffness matrix $\mathbf{A}_{\rm Sp}$ of the spectral reference element and matrices $\boldsymbol{\mathcal{A}}_{\rm Psp}, \mathbb{A}_\hbar$ are spectrally equivalent uniformly in p, i.e.,

$$\boldsymbol{\mathcal{A}}_{\rm Psp}, \boldsymbol{\mathcal{A}}_{\rm Sp}, \mathbb{A}_\hbar \prec \mathbf{A}_{\rm Sp} \prec \mathbb{A}_\hbar, \boldsymbol{\mathcal{A}}_{\rm Sp}, \boldsymbol{\mathcal{A}}_{\rm Psp} \,.$$

Let $\mathbf{M}_{\rm Sp}$ be mass matrix of spectral element, $\boldsymbol{\mathcal{M}}_{\rm Sp}$, $\boldsymbol{\mathcal{M}}_{\rm P/Sp}$ be its FE preconditioners, generated by space $\mathcal{H}(\tau_0)$ on Gaussian or pseudospectral mesh , and $\mathbb{M}_\hbar := \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar \otimes \mathbb{D}_\hbar$. Then uniformly in p

$$\boldsymbol{\mathcal{M}}_{\rm Psp}, \boldsymbol{\mathcal{M}}_{\rm Sp}, \mathbb{M}_\hbar \prec \mathbf{M}_{\rm Sp} \prec \mathbb{M}_\hbar, \boldsymbol{\mathcal{M}}_{\rm Sp}, \boldsymbol{\mathcal{M}}_{\rm Psp} \,.$$

Proof. Most important contribution by Bernardi/Maday [1992],who studied 1-d case. Step to more dimensions in Canuto [1994] Casarin [1997].

Note :

– in the multi-d preconditioner $\mathbf{\Lambda}_{\mathrm{e}}$ for hierarchical ref. el. $\mathcal{E}_{\mathrm{hi}}$, matrices $\boldsymbol{\Delta}$ and $\boldsymbol{\mathcal{D}}$ are preconditioners for the **mass and stiffness** matrices in 1-d, respectively,

– whereas, in the multi-d preconditioner $\mathbb{A}_{\hbar}$ for spectral ref. el. $\mathcal{E}_{\mathrm{sp}}$, matrices $\boldsymbol{\Delta}_{\hbar}$ and $\mathbb{D}_{\hbar}$ are preconditioners for the **stiffness and mass** matrices in 1-d.

Factored preconditioners for spectral elements

Let us introduce $(p-1)\times(p-1)$ matrices

$$\begin{aligned}\mathbf{\Delta}_{\mathrm{Sp}} &= \text{tridiag}\,[-1,2,-1]\,,\\ \boldsymbol{\mathcal{D}}_{\mathrm{Sp}} &= \text{tridiag}\,[1,4,..,N^2,(N-1)^2,(N-2)^2,..,4,1]\,,\end{aligned}$$

$(p-1)^3\times(p-1)^3$ matrices

$$\begin{aligned}\widetilde{\mathbf{\Lambda}}_{I,\mathrm{Sp}} = {} & \boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes(\mathbf{\Delta}_{\mathrm{Sp}}+\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}^{-1})+\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes(\mathbf{\Delta}_{\mathrm{Sp}}+\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}^{-1})\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}+\\ & (\mathbf{\Delta}_{\mathrm{Sp}}+\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}^{-1})\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\,,\end{aligned}$$

$$\mathbf{\Lambda}_{I,\mathrm{Sp}} = \boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\mathbf{\Delta}_{\mathrm{Sp}}+\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\mathbf{\Delta}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}+\mathbf{\Delta}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\otimes\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}\,,$$

diagonal transformation $(\mathbf{p}-\mathbf{1})^3\times(\mathbf{p}-\mathbf{1})^3$ matrix

$$\mathbf{C} = p^{-4}\,\mathbb{D}_{\hbar}^{-1/2}\otimes\mathbb{D}_{\hbar}^{-1/2}\otimes\mathbb{D}_{\hbar}^{-1/2} \qquad (\mathbf{C} = p^{-2}\,\mathbb{D}_{\hbar}^{-1/2}\otimes\mathbb{D}_{\hbar}^{-1/2}\ \text{ for } 2\!-\!\text{d})\,,$$

and matrices $\widetilde{\boldsymbol{\Delta}}_\hbar = \mathbb{D}_\hbar^{1/2} \boldsymbol{\Delta}_\hbar \mathbb{D}_\hbar^{1/2}$ and

$$\widetilde{\mathbb{A}}_\hbar := \mathbf{C}^{-1} \mathbb{A}_\hbar \mathbf{C}^{-1} = p^8 \, \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} \mathbb{A}_\hbar \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} \otimes \mathbb{D}_\hbar^{1/2} =$$

$$p^8 \left(\mathbb{D}_\hbar^2 \otimes \mathbb{D}_\hbar^2 \otimes \widetilde{\boldsymbol{\Delta}}_\hbar + \mathbb{D}_\hbar^2 \otimes \widetilde{\boldsymbol{\Delta}}_\hbar \otimes \mathbb{D}_\hbar^2 + \mathbb{D}_\hbar^2 \otimes \mathbb{D}_\hbar^2 \otimes \widetilde{\boldsymbol{\Delta}}_\hbar \right) .$$

Theorem1. If matrices $\widetilde{\mathbb{A}}_{I,\hbar}$, $\boldsymbol{\Lambda}_{I,\mathrm{Sp}}$, $\widetilde{\boldsymbol{\Lambda}}_{I,\mathrm{Sp}}$ are obtained on Gaussian or pseudospectral mesh $\hbar_i \asymp i/p^2$ for $1 \le i \le N$, then they are spectrally equivalent uniformly in p.

Proof. Korneev/Rytov [2005].

Corollary 1. Let $\boldsymbol{\Lambda}_{I,C} := \mathbf{C}\boldsymbol{\Lambda}_{I,\mathrm{Sp}}\mathbf{C}$ and $\widetilde{\boldsymbol{\Lambda}}_{I,C} := \mathbf{C}\widetilde{\boldsymbol{\Lambda}}_{I,\mathrm{Sp}}\mathbf{C}$. Under conditions of Theorem 1

$$\boldsymbol{\Lambda}_{I,C}, \widetilde{\boldsymbol{\Lambda}}_{I,C} \prec \mathbf{A}_{I,\mathrm{Sp}} \prec \boldsymbol{\Lambda}_{I,C}, \widetilde{\boldsymbol{\Lambda}}_{I,C} \, .$$

Finite – difference interpretation

Matrix $\mathbf{\Lambda}_{I,\mathrm{Sp}}$ is 7-point F-D approximation of diff. operator

$$L_{\mathrm{Sp}}u = -\left[\phi^2(x_2)\phi^2(x_3)u_{,1,1} + \phi^2(x_1)\phi^2(x_3)u_{,2,2} + \phi^2(x_1)\phi^2(x_2)u_{,3,3}\right] ,$$

at $u|_{\partial\tau_0} = 0$ and $\phi(x) = \min(x+1, x-1)$. Indeed, for $\hbar = 2/p$, $\phi_i = \phi(-1+i\hbar)$ and $\mathbf{u} = (u_{\mathbf{i}})_{i_1,i_2,i_3=1}^{p-1}$,

$$\mathbf{\Lambda}_{I,\mathrm{sp}}\mathbf{u}|_{\mathbf{i}} = -\frac{1}{\hbar^2}\sum_{k=1,2,3}\phi^2_{i_{k+1}}\phi^2_{i_{k+2}}[u_{\mathbf{i}-\mathbf{e}_k} - 2u_{\mathbf{i}} + u_{\mathbf{i}+\mathbf{e}_k}] , \quad 1 \le i_1, i_2, i_3 \le (p-1) ,$$

where $\mathbf{i} = (i_1, i_2, i_3)$, indices $k, k+1, k+2$ are understood modulo 3, $\mathbf{e}_k = (\delta_{k,l})_{l=1}^3$ is the unite vector. For $d = 2$,

$$L_{\mathrm{Sp}}u = -\left[\phi^2(x_2)u_{,1,1} + \phi^2(x_1)u_{,2,2}\right] , \quad u|_{\partial\tau_0} = 0 ,$$

$$\mathbf{\Lambda}_{I,\mathrm{sp}}\mathbf{u}|_{\mathbf{i}} = -\sum\nolimits_{k=1,2}\phi^2_{i_{3-k}}[u_{\mathbf{i}-\mathbf{e}_k} - 2u_{\mathbf{i}} + u_{\mathbf{i}+\mathbf{e}_k}] , \quad \mathbf{i} = (i_1, i_2) .$$

Finite element preconditioners

Let $d = 2$. We divide square nests of size $\hbar$ in pairs of triangles and, on such triangulation, introduce the space $\overset{\circ}{\mathcal{V}}_{\triangle}(\tau_0) \in C(\overline{\tau}_0)$ of piece wise linear functions, vanishing on $\partial\tau_0$. The FE preconditioner $\mathbf{B}_{I,\mathrm{sp}}$ is the matrix of the bilinear form

$$b_{\tau_0}(u,v) = \sum_{k=1}^{3} \int_{\tau_0} \phi_{3-k}^2 u_{,k} v_{,k}\, dx$$

on this space. In 2 and 3-d, $\mathbf{B}_{I,\mathrm{sp}}$ can be defined by the FE spaces of bilinear and trilinear functions, respectively. We have

$$\mathbf{B}_{I,\mathrm{sp}} \asymp \hbar^{4-d} \mathbf{\Lambda}_{I,\mathrm{Sp}}\,.$$

Comparison

- At $d = 2$ in each quarter of τ_0, operator L_{Sp} coincides with L up to the constant multiplier (and rotation and transportation of the axes).
- The same is true for F-D operators $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_{I,\mathrm{sp}}$.
- At $d = 3$, differential and F-D operators are different even in the order: L is of 4-th order, whereas L_{Sp} is of 2-nd.
- However, multipliers $\mathbf{\Delta}$, $\boldsymbol{\mathcal{D}}$ and respectively $\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}$, $\mathbf{\Delta}_{\mathrm{Sp}}$ in representations of $\mathbf{\Lambda}_e$, $\mathbf{\Lambda}_{I,\mathrm{sp}}$ by sums of Kroneckers products are similar.
- An additional difficulty for deriving fast solvers for 3-d hierarchical elements directly on the basis of $\mathbf{\Lambda}_e$ is that it is a F-D analogue of 4-th order differential operator. More over, this operator contains only mixed derivatives. The use of the spectral elements and the preconditioner $\mathbf{\Lambda}_{I,C}$ simplifies the problem by reducing it to designing a fast solver for $\mathbf{\Lambda}_{I,\mathrm{sp}}$, which is the F-D approximation of the 2-nd order differential operator containing only derivatives $\partial^2/\partial x_k^2$, $k = 1, 2, 3$.

Conclusions

★All fast solvers for systems with the hierarchical reference element stiffness matrices (or spectrally equivalent , *e.g.*, $\mathbf{\Lambda}_e$) are easily adjusted into fast solvers for systems with the spectral reference element stiffness matrices or spectrally equivalent to them matrices like $\mathbf{\Lambda}_{I,\mathrm{sp}}$

★The arithmetic costs of the latter and the former solvers are the same in the order.

★At least, these conclusions are true for the all known fast solvers see, *e.g.*, [K1],[K2],[KA],[B],[BSS], for systems with matrices $\mathbf{\Lambda}_e$.

Example 1

Algebraic multilevel solver for 2d spectral elements

We set $p = 2N,\ N = 2^{\ell_0 - 1}$ and introduce

- sequence of ℓ_0 embedded meshes of the sizes $\hbar_l = 2^{-l},\ l = 1, 2, .., \ell_0$, with the nodes $x = \hbar_l(i, j) - (1, 1)$,
- sequence of spaces $\mathcal{V}_l(\tau_0)$ with $\mathcal{V}_{\ell_0}(\tau_0) = \overset{\circ}{\mathcal{V}}_{\triangle}(\tau_0)$ and
- FE matrices $\mathbf{B}_l$ with $\mathbf{B}_{\ell_0} = \mathbf{B}_{I,\mathrm{Sp}}$.

Each space $\mathcal{V}_l(\tau_0)$ and the matrix $\mathbf{B}_l$ are the space $\overset{\circ}{\mathcal{V}}_{\triangle}(\tau_0)$ and the matrix $\mathbf{B}_{I,\mathrm{Sp}}$ for the mesh of the level l.

Also the following notations are used:

- X_l – the subset of internal nodes,
- V_l and W_l – vector-spaces, related to subsets of nodes X_l and $X_{W,l} := X_l \setminus X_{l-1}$, so that

$$V_l = V_{l-1} \oplus W_l = W_l \oplus W_{l-1} \oplus \ldots \oplus W_2 \oplus V_1 \, .$$

- $\mathbf{P}_{l-1} : V_{l-1} \rightarrow V_l$ – usual interpolation matrix from the mesh "$l-1$" on the next finer mesh "l".
- $\mathbf{R}_l : V_l \rightarrow W_l$ – restriction matrix to the set of nodes $X_{W,l}$.
- $\mathbf{B}_{V_l}$, $\mathbf{B}_{W_l}$ – blocks on the diagonal of $\mathbf{B}_l$ related to the subspaces V_l and W_l.

One multilevel iteration

If $\boldsymbol{\mathcal{B}}_{W_l}$ is a preconditioner for $\mathbf{B}_{W_l}$, one multigrid iteration for $\mathbf{B}_l\mathbf{u} = \mathbf{F}$, producing $\mathbf{u}^{k+1,l} := \mathrm{Mgm}(l, \mathbf{B}_l, \mathbf{F}, \mathbf{u}^{k,l})$ for a given $\mathbf{u}^{k,l}$ is:

If $\mathbf{l > 1}$, **then do**

Pre-smoothing in the subspace W_l:

$\mathbf{v} := \mathbf{u}^{k,l}$;

do ν **times** $\quad \mathbf{v} := \mathbf{v} - \sigma_l^{-1}\mathbf{R}_l^\top \boldsymbol{\mathcal{B}}_{W_l}^{-1}\mathbf{R}_l(\mathbf{B}_l\mathbf{v} - \mathbf{F})$;

Correction of the solution on the lower level in the space V_{l-1}:

$\mathbf{d}_{l-1} := \mathbf{P}_{l-1}^*(\mathbf{F} - \mathbf{B}_l\mathbf{v})$; $\mathbf{w} = 0$;

do μ_{l-1} **iterations** $\mathbf{w} = \mathbf{Mgm}(l-1, \mathbf{B}_{l-1}, \mathbf{d}_{l-1}, \mathbf{w})$;

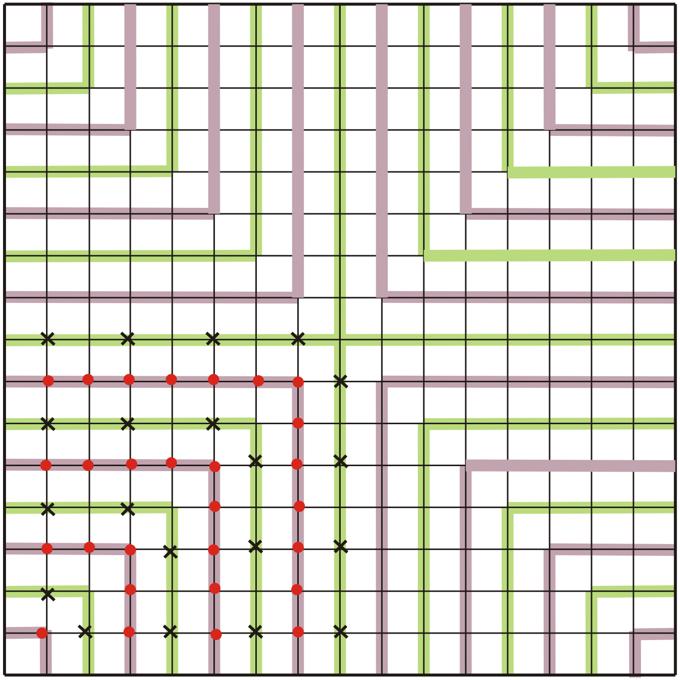
$\mathbf{v} := \mathbf{v} + \mathbf{P}_{l-1}\mathbf{w}$;

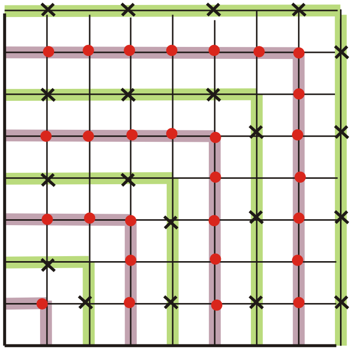
Post-smoothing in the subspace W_l:

do ν **times** $\quad \mathbf{v} := \mathbf{v} - \sigma_l^{-1}\mathbf{R}_l^\top \boldsymbol{\mathcal{B}}_{W_l}^{-1}\mathbf{R}_l(\mathbf{B}_l\mathbf{v} - \mathbf{F})$;

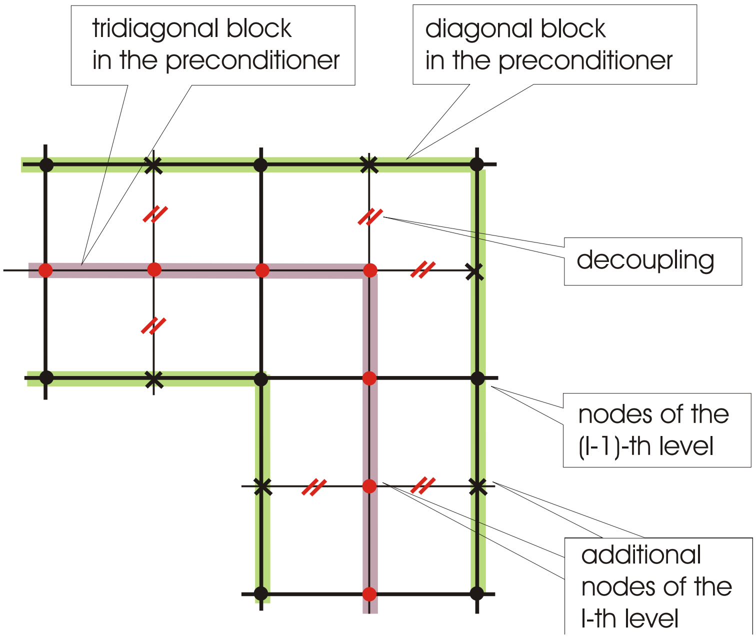
$\mathbf{u}^{k+1,l} = \mathbf{v}$

else, then solve $\mathbf{B_l u = F}$ **by the exact method**





Lines $\Im_j$ along which smoothing is performed in the multigrid solvers for spectral (left) and hierarchical reference elements (right)



Line preconditioning.

Two factors influence efficiency : 1) efficiency of preconditioners $\boldsymbol{\mathcal{B}}_{W_l}$, i.e., the values of $c_k > 0$ in the inequalities

$$c_1 \boldsymbol{\mathcal{B}}_{W_l} \le \mathbf{B}_{W_l} \le c_2 \boldsymbol{\mathcal{B}}_{W_l} \ ,$$

and the cost of solving systems with the matrices $\boldsymbol{\mathcal{B}}_{W_l}$. 2) the value of c_0 in the strengthened Cauchy inequality

$$(b_{\tau_0}(u,v))^2 \le c_0\, b_{\tau_0}(u,u) b_{\tau_0}(v,v), \quad c_0 < 1, \quad \forall u \in \mathcal{V}_{l-1}, \ \forall v \in \mathcal{W}_l \ ,$$

where $\mathcal{W}_l(\tau_0) := \mathcal{V}_l(\tau_0) \ominus \mathcal{V}_{l-1}(\tau_0)$.

Lemma 4. $c_1 \ge 1 - 2/\sqrt{11}, \ c_2 \le 1 + 2/\sqrt{11}, \ c_0 \le 97/176 < 2/3 \,.$

Proof. Repeats the proof of Beuchler [2002] for hierarchical reference element.

Convergence of the multigrid iterations

Theorem 2 (Korneev/Rytov [2005]). Let $\mathbf{B}_l\mathbf{u} = \mathbf{F}$ be solved by the multigrid method in which $\sigma = 2/(c_1 + c_2)$, $\mu \geq 3$ and ν be greater than some $\nu_o(c_0, c_1, c_2)$. Then the convergence factor

$$\rho_{l,\text{mult}} := \sup_{\mathbf{u}^k \in U_l} \|\mathbf{u}^{k+1} - \mathbf{u}\|_{\mathbf{B}_l} / \|\mathbf{u}^k - \mathbf{u}\|_{\mathbf{B}_l}$$

is bounded by the constant $\rho < 1$ independent of p, l and $\mathbf{u}^k$.

Proof. Follows from results of Schieweck [1985] and Pflaum [2000] and Lemma 4.

Multigrid iteration as a preconditioner

Let $\mathbf{M}_\mu$ be the linear error transmission operator for one multigrid iteration for system $\mathbf{B}_{I,\mathrm{sp}}\mathbf{u} = \mathbf{F}$. Then $\varkappa$ multigrid iterations implicitly define the preconditioner $\mathbf{Mg}_{\mathrm{Sp}}$ for $\widehat{\boldsymbol{\Lambda}}_{I,C}$ and $\mathbf{A}_{I,\mathrm{sp}}$, the inverse to which is $\mathbf{Mg}_{\mathrm{sp}}^{-1} = \hbar^{-2}\mathbf{C}(\mathbf{I} - \mathbf{M}_\mu^\varkappa)\mathbf{B}_{I,\mathrm{sp}}^{-1}\mathbf{C}$, $\mathbf{C} = p^{-2}\mathbb{D}_\hbar^{-1/2} \otimes \mathbb{D}_\hbar^{-1/2}$.

Theorem 3 (Korneev/Rytov [2005]). Let $\mu = 3$, $\nu \geq 3$ and $\varkappa \geq 1$. Then

$$\underline{c}\, \mathbf{Mg}_{\mathrm{sp}}^{-1} \leq \mathbf{A}_{I,\mathrm{sp}}^{-1} \leq \overline{c}\, \mathbf{Mg}_{\mathrm{sp}}^{-1}\ ,$$

with constants $\underline{c}, \overline{c} > 0$ independent of p (and $\varkappa$). The procedure of the matrix-vector multiplication by $\mathbf{Mg}_{\mathrm{sp}}^{-1}$ requires $\mathcal{O}(p^2)$ arithmetic operations.

Example 2

Multiresolution wavelet solver for 3d spectral elements

Since, e.g., $\boldsymbol{\Lambda}_{I,\mathrm{Sp}}$ is a sum of Kronecker products of matrices $\boldsymbol{\Delta}_{\mathrm{Sp}}$, $\boldsymbol{\mathcal{D}}_{\mathrm{Sp}}$ related to 1-d integrals, fast solver for $\boldsymbol{\Lambda}_{I,\mathrm{Sp}}$ is constructed by deriving multilevel preconditioners for these matrices.

For simplicity, we set again $p = 2N$, $N = 2^{\ell_0 - 1}$, and for $l = 1, 2, ..., l_0$ introduce

- uniform mesh of size $\hbar_l = 2^{1-l}$ on the interval $(-1, 1)$

$$x_i^l = -1 + i\hbar_l,\ \ i = 0, 1, 2, .., 2N_l, \quad x_0 = -1,\ \ x_{2N_l} = 1, \quad N_l = 2^{l-1}$$

- space $\mathcal{V}_l(-1, 1)$ of continuous piece wise linear functions, vanishing at $x = -1, 1$,
- nodal=hat basis function $\sigma_i^l \in \mathcal{V}_l(-1, 1)$, such that $\sigma_i^l(x_j^l) = \delta_{i,j}$ and

$$\mathcal{V}_l(-1, 1) = \mathrm{span}\left(\sigma_i^l\right)_{i=1}^{p_l - 1}, \qquad p_l = 2^l,$$

- Gram matrices in the nodal basis

$$\boldsymbol{\Delta}_l = \hbar_l \left(\int_{-1}^{1} (\sigma_i^l)', (\sigma_j^l)' \right)_{i,j=1}^{p_l - 1}, \qquad \boldsymbol{\mathcal{M}}_l = \hbar_l^{-1} \left(\int_{-1}^{1} \phi^2 \, \sigma_i^l, \sigma_j^l \right)_{i,j=1}^{p_l - 1},$$

- single scale wavelet basis $(\psi_k^l)_{k=1}^{p_{l-1}}$ in the space $\mathcal{W}_l := \mathcal{V}_l \ominus \mathcal{V}_{l-1}$, so that $\mathcal{W}_l = \mathrm{span}\,[\,\psi_k^l\,]_{k=1}^{p_{l-1}}$,
- multiscale wavelet basis $(\psi_k^l\,)_{k,l=1}^{p_{l-1},\,l_0}$, composed of single scale bases according to the representation

$$\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus .. \oplus \mathcal{W}_{l_1}, \quad \text{where } \mathcal{V} = \mathcal{V}_{l_0}, \;\; \mathcal{W}_1 = \mathcal{V}_1,$$

- Gram matrices in the multiscale wavelet basis

$$\boldsymbol{\Delta}_{\text{wlet}} = \left((\hbar_k \hbar_l)^{1/2} \int_{-1}^{1} (\psi_i^k)', (\psi_j^l)' \, dx \right)_{i,j=1;\,k,l=1}^{p_{l-1};\,l_0},$$

$$\boldsymbol{\mathcal{M}}_{\text{wlet}} = \left((\hbar_k \hbar_l)^{-1/2} \int_{-1}^{1} \phi^2 \, \psi_i^k, \psi_j^l \, dx \right)_{i,j=1;\,k,l=1}^{p_{l-1};\,l_0},$$

- diagonal matrices with the main diagonals from $\mathbf{\Delta}_{\text{wlet}}$ and $\boldsymbol{\mathcal{M}}_{\text{wlet}}$

$$\mathbb{D}_1 = \operatorname{diag}\left[\hbar_l \int_{-1}^{1} ((\psi_i^l)')^2\, dx\right]_{i,l=1}^{p_{l-1},l_0}, \quad \mathbb{D}_0 = \operatorname{diag}\left[\hbar_l^{-1} \int_{-1}^{1} \phi^2 (\psi_i^l)^2\, dx\right]_{i,l=1}^{p_{l-1},l_0}.$$

The transformation matrix from the multiscale wavelet basis to the basis $(\sigma_k^{l_0})_{k=1}^{p-1}$ is denoted by $\mathbf{Q}$. If $\mathbf{v}$ and $\mathbf{v}_{\text{wavelet}}$ are the vectors of the coefficients of a function from $\mathcal{V}(0,1)$ in the one scale nodal and the multiscale wavelet bases, respectively, then $\mathbf{v} = \mathbf{Q}\,\mathbf{v}_{\text{wavelet}}$.

Theorem 4. There exist wavelet bases $(\psi_k^l)_{k,l=1}^{p_{l-1},l_0}$ such that matrices $\mathbf{\Delta}_{\text{wlet}}$ and $\boldsymbol{\mathcal{M}}_{\text{wlet}}$ are simultaneously spectrally equivalent to their diagonals $\mathbb{D}_1$ and $\mathbb{D}_0$, respectively, (uniformly in p) and multiplications $\mathbf{Q}\,\mathbf{v}_{\text{wlet}}$ and $\mathbf{Q}^T\,\mathbf{v}$ require $\mathcal{O}(p)$ arithmetic operations.

Proof. Basically it is the same as the proof of a similar result by Beuchler/Schneider/Schwab [2004] in the case of hierarchical element.

Theorem 5. Let

$$\boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w}^{-1} = \begin{cases} (\mathbf{Q}^T \otimes \mathbf{Q}^T)[\mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_1 \otimes \mathbb{D}_0]^{-1}(\mathbf{Q} \otimes \mathbf{Q}), & d = 2, \\ (\mathbf{Q}^T \otimes \mathbf{Q}^T \otimes \mathbf{Q}^T)[\mathbb{D}_0 \otimes \mathbb{D}_1 \otimes \mathbb{D}_1 + \mathbb{D}_1 \otimes \mathbb{D}_0 \otimes \mathbb{D}_1 + & \\ \qquad \mathbb{D}_1 \otimes \mathbb{D}_1 \otimes \mathbb{D}_0]^{-1}(\mathbf{Q} \otimes \mathbf{Q} \otimes \mathbf{Q}), & d = 3\,, \end{cases}$$

then $\boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w} \asymp \mathbf{A}_I$ and therefore

$$\mathrm{cond}\,[\boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w}^{-1}\mathbf{A}_I] \prec 1\,.$$

The computational cost of the operation $\boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w}^{-1}\mathbf{v}$ for any $\mathbf{v}$ is $\mathrm{ops}\,[\boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w}^{-1}\mathbf{v}] = \mathcal{O}(p^d)$.

Example 3

Multiresolution wavelet solver for faces

Good master preconditioner-solver for one face subproblem may be matrix spectrally equivalent to the matrix of the norm

$$ {}_{00}| \, v \, |^2_{1/2,F_0} = | \, v \, |^2_{1/2,F_0} + \int_{F_0} \frac{|v(x)|^2}{\operatorname{dist}\,[x, \partial F_0]} \, dx \, , \quad \forall \; v \in \mathring{\mathcal{Q}}_{p,x} \, , $$

for a typical face $F_0 = (-1,1) \times (-1,1)$ of the reference element. By diagonal entries $d_{0,i}, d_{1,i}$ of $\mathbb{D}_0, \mathbb{D}_1$, respectively, one can define diagonal $(2N-1)^2 \times (2N-1)^2$ matrix $\mathbb{D}_{1/2}$ with diagonal entries

$$ d^{(1/2)}_{i,j} = d_{0,i} d_{0,j} \sqrt{d_{1,i}/d_{0,i} + d_{1,j}/d_{0,j}} \, . $$

Theorem 6 (Korneev/Rytov [2005]). Let

$$\mathbb{S}_0^{-1} = (\mathbf{Q}^\top \otimes \mathbf{Q}^\top)\, \mathbb{D}_{1/2}^{-1} (\mathbf{Q} \otimes \mathbf{Q})\,, \qquad \boldsymbol{\mathcal{S}}_0 = \mathbf{C}\, \mathbb{S}_0\, \mathbf{C}\,.$$

Then for all $v \in \overset{\circ}{\mathcal{Q}}_{p,x}$ and the corresponding vectors $\mathbf{v}$, the norms ${}_{00}|\, v\, |_{1/2,\tau_0}$ and $||\mathbf{v}||_{\boldsymbol{\mathcal{S}}_0}$, respectively, are equivalent uniformly in p, i.e.,

$${}_{00}|\, v\, |_{1/2,\tau_0} \asymp ||\mathbf{v}||_{\boldsymbol{\mathcal{S}}_0}\,.$$

Proof. Basis tool is Peetre's K-interpolation method.

$\mathbb{S}_0$ is a multiscale wavelet precoditioner for which $\mathrm{ops}\,[\mathbb{S}_0^{-1}\mathbf{v}] = \mathcal{O}(p^2)$, $\forall \mathbf{v}$, and, therefore, $\mathrm{ops}\,[\boldsymbol{\mathcal{S}}_0^{-1}\mathbf{v}] = \mathcal{O}(p^2)$ as well. Similar preconditioner-solver for faces of hierarchical elements was approved in Korneev/Langer/Xanthis [2003].

DOMAIN DECOMPOSITION ALGORITHM

The problem to be solved

$$a_\Omega(u,v) := \int_\Omega \varrho(x) \nabla u \cdot \nabla v \, dx = (f, v)_\Omega \,, \qquad \forall\, v \in \overset{\circ}{H}{}^1(\Omega)\,,$$

in the domain $\overline{\Omega} = \cup_{r=1}^{\mathcal{R}} \overline{\tau}_r$, which is an assemblage of compatible and in general curvilinear finite elements occupying domains τ_r. It is assumed that finite elements satisfy the generalized conditions of shape regularity. The positive coefficient $\varrho(x)$ is assumed to be pice wise constant, *i.e.*, $\varrho(x) = \varrho_r$ for $x \in \tau_r$.

The finite element stiffness matrix may be represented in the block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IW} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FW} \\ \mathbf{K}_{WI} & \mathbf{K}_{WF} & \mathbf{K}_{WW} \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IE} & \mathbf{K}_{IV} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FE} & \mathbf{K}_{FV} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_E & \mathbf{K}_{EV} \\ \mathbf{K}_{VI} & \mathbf{K}_{VF} & \mathbf{K}_{VE} & \mathbf{K}_V \end{pmatrix}, \qquad \text{where}$$

I – stands for internal d.o.f., F – faces, E – edges, V – vertices, B – interface boundary, W – wire basket.

We consider the DD Dirichlet-Dirichlet preconditioner-solver $\boldsymbol{\mathcal{K}}$

$$\begin{aligned} \boldsymbol{\mathcal{K}}^{-1} &= \overline{\boldsymbol{\mathcal{K}}}_I^{\,+} + \mathbf{P}_{V_B\to V}\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{P}_{V_B\to V}^{\top}\,, \\ \boldsymbol{\mathcal{S}}_B^{-1} &= \overline{\boldsymbol{\mathcal{S}}}_F^{\,+} + \mathbf{P}_{V_W\to V_B}(\boldsymbol{\mathcal{S}}_W^{B})^{-1}\mathbf{P}_{V_W\to V_B}^{\top}\,. \end{aligned} \tag{0.1}$$

i) The block diagonal preconditioner-solver for the internal Dirichlet problems on finite elements has the form

$$\overline{\boldsymbol{\mathcal{K}}}_I^{\,+} := \begin{pmatrix} \boldsymbol{\mathcal{K}}_I^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\quad \boldsymbol{\mathcal{K}}_I = \mathrm{diag}\left[h_1\varrho_1\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, h_2\varrho_2\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}, \ldots, h_{\mathcal{R}}\varrho_{\mathcal{R}}\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}}\right]$

$\boldsymbol{\mathcal{B}}_{I,\mathrm{sp}} = \boldsymbol{\mathcal{A}}_{I,\mathrm{sp}\leftarrow w}$ – multiresolution preconditioner-solver of Theorem 5.

ii) Block diagonal preconditioner-solver for internal problems on faces

$$\overline{\boldsymbol{\mathcal{S}}}_F^{\,+} = \begin{pmatrix} \boldsymbol{\mathcal{S}}_F^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\mathcal{S}}_F = \operatorname{diag}\left[\kappa_1 \boldsymbol{\mathcal{S}}_0, \kappa_2 \boldsymbol{\mathcal{S}}_0, \ldots, \kappa_Q \boldsymbol{\mathcal{S}}_0\right],$$

– Q is the number of faces $F_k \subset \Omega$,
– κ_k are multipliers

$$\kappa_k = \left(h_{r_1(k)} \varrho_{r_1(k)} + h_{r_2(k)} \varrho_{r_2(k)}\right),$$

with $r_1(k)$, $r_2(k)$ being numbers of two elements $\overline{\tau}_{r_1(k)}$ and $\overline{\tau}_{r_2(k)}$, sharing the face F_k,
– h_r is the characteristic size of an element,
– $\boldsymbol{\mathcal{S}}_0$ is the preconditioner-solver for one face, defined in Theorem 4.

iii) Preconditioner-solver $\boldsymbol{\mathcal{S}}_W^B$ for wire basket subproblem of relatively small dimension $\mathcal{O}(\mathcal{R}p) \times \mathcal{O}(\mathcal{R}p)$. We borrow it from Casarin [1997] and Pavarino/Widlund [1996], assuming that its arithmetical cost does not disturb optimality of DD solver, *i.e.*, $\text{ops}[(\boldsymbol{\mathcal{S}}_W^B)^{-1}\mathbf{v}] = \mathcal{O}(\mathcal{R}p^3)$.

The prolongation operations include :

iv) prolongation $\mathbf{P}_{V_B \to V}$ from interelement boundary on the whole computational domain $\overline{\Omega}$, completed by means of inexact solver with the preconditioner $\boldsymbol{\mathcal{B}}_{I,\text{sp}}$,
v) simple prolongation $\mathbf{P}_{V_W \to V_B}$ from wire basket on interelement boundary, not requiring solution of any systems, which is the same as in Pavarino/Widlund [1996] and Casarin [1997].

Theorem 7. Suppose, the generalized conditions of shape regularity are fulfilled and the coefficient $\rho > 0$ is piece wise constant. Then the bound for the relative condition number of DD preconditioner-solver $\boldsymbol{\mathcal{K}}$ is

$$\mathrm{cond}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \le c(1+\log p)^2\,.$$

Suppose additionally that the wire basket solver satisfy the above assumption **iii**). Then the number of arithmetic operations needed for solving the system $\boldsymbol{\mathcal{K}}^{-1}\mathbf{v} = \mathbf{f}$ has the majorant

$$\mathrm{ops}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{f}] \le \mathcal{O}(p^3(1+\log p)\mathcal{R})\,, \qquad \forall \mathbf{f}\,.$$

CONCLUSIONS

Factored precondtioners, presented in this lecture for the spectral reference element stiffness and mass matrices, allow to design almost optimal in computationl work preconditioners-solvers for three most important subproblems, arising in DD algorithms for elliptic equations in 3d domains. Indeed, two of these preconditioners-solvers are optimal.

In the presented DD preconditioner-solver, only one sparse subsystem of the relatively small dimension $\mathcal{O}(\mathcal{R}) \times \mathcal{O}(\mathcal{R})$, which is a part of the wire basket subproblem, was not supplied with the solver optimal with the respect to its dimension $\mathcal{O}(\mathcal{R})$.

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