17th International Conference on Domain Decomposition Methods

On a two-level domain decomposition preconditioner for 3D flows in anisotropic highly heterogeneous porous media

(work in progress)

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Statement of the problem

Continuity equation + **Darcy´s law**

$$\nabla \cdot v = f$$

$$v = -K \cdot \nabla p$$

↓

Pressure equation

$$-\nabla \cdot \left(K \cdot \nabla p\right) = f$$

Permeability tensor

$$K = \begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{pmatrix} > 0$$

Boundary conditions

Dirichlet

$$p = p_0$$

Neumann

$$v \cdot n = v_0$$

Applications

Saturated flow in anisotropic heterogeneous porous media

$$\nabla \cdot v = f, \quad v = -K \cdot \nabla p$$

Two-phase flow in heterogeneous porous media

$$\nabla \cdot v = 0, \quad v = -\lambda(S_w)\, K \cdot \nabla p,$$

$$\frac{\partial S_w}{\partial t} + v \cdot \nabla f_w(S_w) = 0$$

Fine grid – isotropic permeability tensor

Coarse grid – ***full*** tensor (effective permeability)

Finite volume discretization

Continuity equation

$$\nabla v = f \quad \Longrightarrow \quad \iiint_V \nabla v = \iiint_V f$$

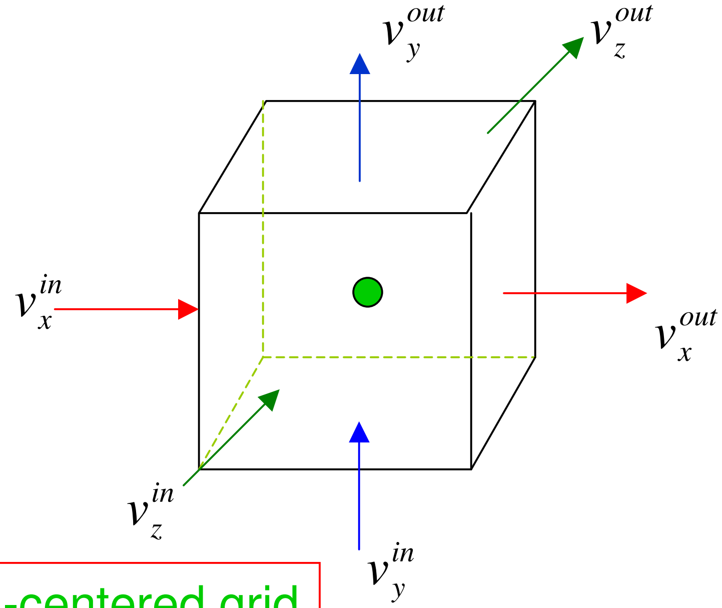
Velocity vector in 3D

$$v = \left(v_x, v_y, v_z\right)$$

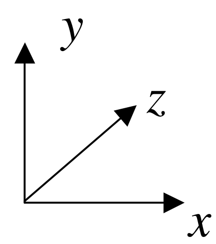
FV scheme

$$\frac{v_x^{out} - v_x^{in}}{h_x} + \frac{v_y^{out} - v_y^{in}}{h_y} + \frac{v_z^{out} - v_z^{in}}{h_z} = f$$

Finite volume

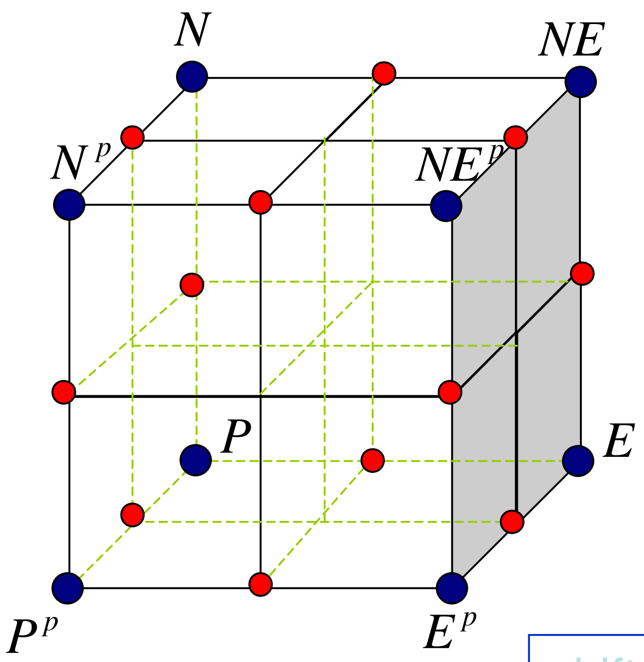


Cell-centered grid



Finite volume discretization

Multipoint Flux Approximation



shifted control volume

- Pressure is given at 8 points — 8 eqns
- Pressure is continuous at 12 points — 12 eqns
- Velocities are continuous along 12 interfaces — 12 eqns

32 equations

32 unknowns

Polynomials:

$$p = a^i x + b^i y + c^i z + d^i, \quad i = \overline{1,8}$$

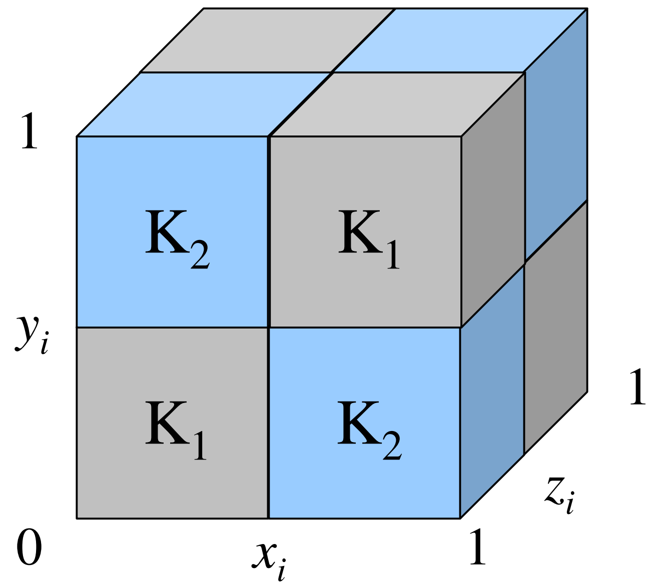
Ware A.F., Parrott A.K., and Rogers C.

FV discretization (validation)

Permeability tensor

$$K_1 = \begin{pmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.25 \\ 0.25 & 0.25 & 1 \end{pmatrix}, \quad K_2 = \alpha K_1$$

α - jump discontinuity



Exact solution

$$p = (x - x_i)^2 (y - y_i)^2 (z - z_i)^2 \cos(\pi(x + y + z))$$

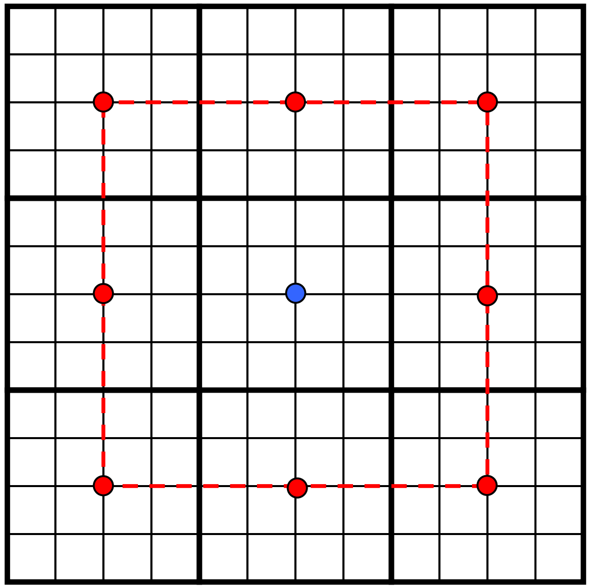
FV discretization (validation)

Grid	$\alpha = 10^{-2}$		$\alpha = 10^{-5}$	
	$\|\|p - p_h\|\|_{L2}$	$\|\|p - p_h\|\|_C$	$\|\|p - p_h\|\|_{L2}$	$\|\|p - p_h\|\|_C$
4 x 4 x 4	0.1709	0.2174	0.1711	0.2174
8 x 8 x 8	0.0395	0.0284	0.0395	0.0284
16 x 16 x 16	0.0087	0.0075	0.0087	0.0075
32 x 32 x 32	0.0020	0.0018	0.0020	0.0018

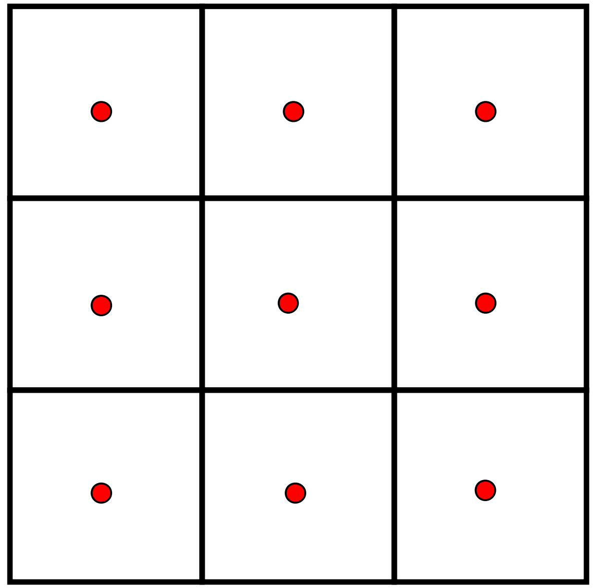
Convergence rate $O(h)^2$ doesn‘t depend on jump discontinuity

Two-grid method

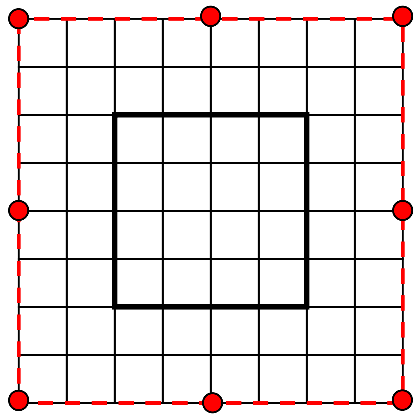
Fine grid



Coarse grid



Extended subdomain



$$Ax = b$$

One sweep of TGM

$$x^n \to x^{n+1}$$

- Smooth with DD (2-3 iterations) $\qquad \tilde{x}^n$
- Calculate the residual $\qquad r_h^n = b - A_h \tilde{x}^n$
- Restrict the residual in each subdomain $\qquad r_H = \frac{1}{m}\sum_{i=1}^{m} r_h^i$
- Discretize and solve on coarse grid $\qquad A_H c_H = r_H$
- Prolong coarse grid correction by solving local problems in shifted subdomains $\qquad c_h$
- Correct the solution $\qquad \tilde{\tilde{x}}^{n+1} = x^n + c^n$
- Post smooth with DD $\qquad x^{n+1}$

Neuss N., Jäger W., Wittum G.

DD smoothing

$$x^n \to \tilde{x}^n$$

Additive Schwarz

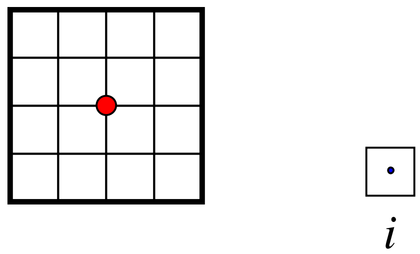
Multiplicative Schwarz

With overlapping

Without overlapping

Restriction

$$r_H = \frac{1}{m}\sum_{i=1}^{m} r_h^i$$



- r_H - residual on a coarse grid
- r_h^i - residual on a fine grid

$m = m_x m_y m_z$ -number of fine grid blocks in a coarse one

Coarse grid operator

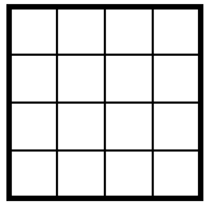
Coarse scale Darcy´s law

$$\langle v \rangle = -K^{eff} \cdot \langle \nabla p \rangle$$

$\langle f \rangle$ **- volume average**

Local flow problems

$$\langle v \rangle^{i} = -K^{eff} \cdot \langle \nabla p \rangle^{i}, \quad i = \overline{1,3}.$$



Boundary conditions and RHS for local flow problem

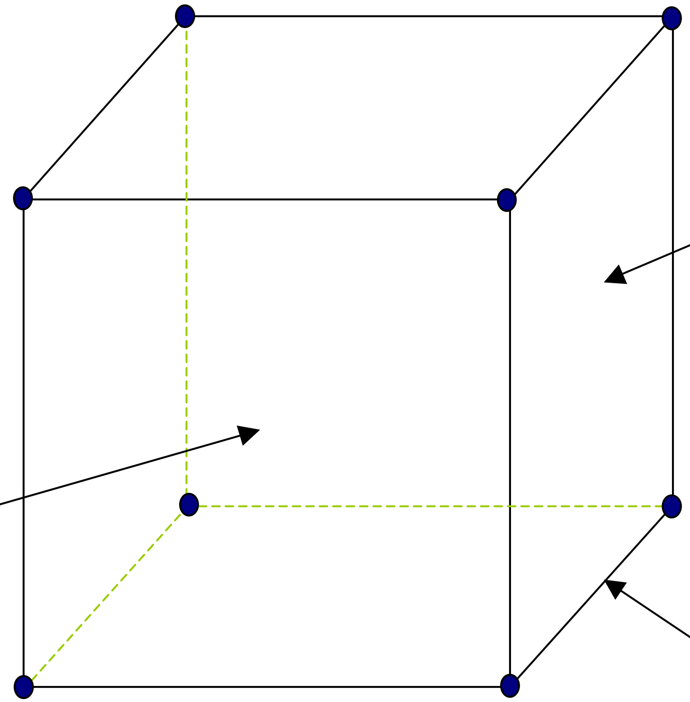
1-0 Dirichlet + Neumann (v = 0) b.c.

1-0 Dirichlet + piecewise linear b.c.

RHS = 0

Prolongation

● -coarse grid solution



$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{yy}\frac{\partial p}{\partial y}\right)=0$$

2D problem for the planes

BCs for 3D problem

3D problem inside the cell

$$-\nabla\cdot\left(K\cdot\nabla p\right)=f$$

1D by TDMA at the edges

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial p}{\partial x}\right)=0$$

BCs for 2D problem

Numerical results

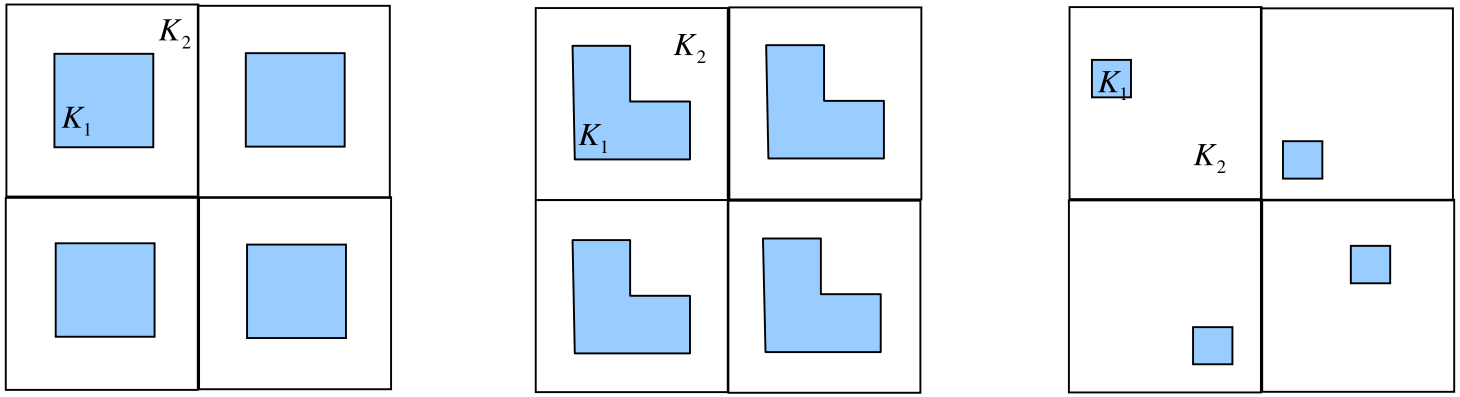
Periodic

Non-periodic

Cubic inclusion

L-shaped inclusion

Random inclusion

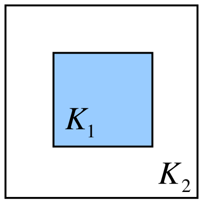


Permeability tensor

$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

Convergence of TGM depends on overlapping and number of subdomains

One- and two-level DD



$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

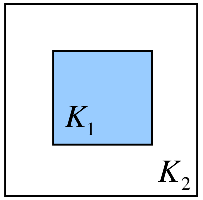
64 inclusions,
acc=1e-4
ovrlp=2: 1.5h

Coarse grid	4x4x4			
Fine grid	4x4x4	8x8x8	16x16x16	32x32x32
DD iter.	--	95	162	247
TGM iter.	--	4	5	7

Coarse grid 8x8x8

Fine grid	4x4x4	8x8x8	16x16x16
DD iter.	158	266	
TGM iter.	3	4	5

DD smoothing (overlapping)



$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

2 presmooth.
2 postsmooth.

Coarse grid 8x8x8,

fine grid 8x8x8

ovrlp = 1

TGM iter = 13

Coarse grid 8x8x8,

fine grid 16x16x16

ovrlp = 2

TGM iter = 7

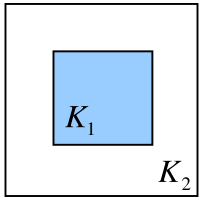
Acc = 1E-5

Coarse grid 8x8x8,

fine grid 16x16x16

ovrlp = 1	ovrlp = 2	ovrlp = 3
TGM iter = 23	TGM iter = 7	TGM iter = 6

DD pre- and post-smoothing

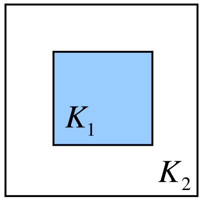


$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

8x8x8 coarse blocks, 8x8x8 fine blocks
Accuracy for TGM = 1E-5

DD smoother:
2-pre, 2-post: 13 TGM iter
0-pre, 2-post: 47 TGM iter
0-pre, 4-post: 24 TGM iter

DD smoothing



$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

2 presmooth.
2 postsmooth.

Coarse grid 8x8x8,

fine grid 8x8x8

ovrlp = 1

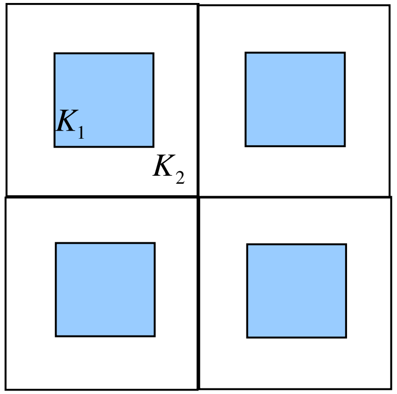
Acc = 1E-5

Additive Schwarz	Multiplicative Schwarz
TGM iter = 13	TGM iter = 7

TGM for different geometries

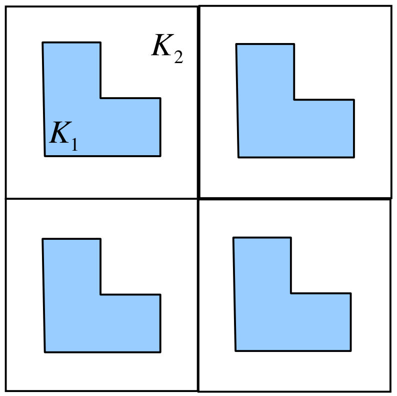
$K_1 = E \qquad K_2 = 10000E$

Periodic cubic inclusion



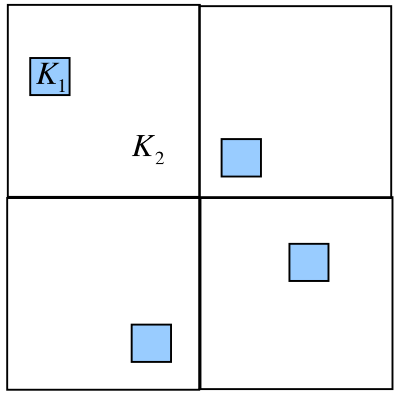
Coarse grid 8x8x8,
fine grid 8x8x8
TGM iter = 13

Periodic L-shaped inclusion



Coarse grid 8x8x8,
fine grid 12x12x12
TGM iter = 23

Random inclusion

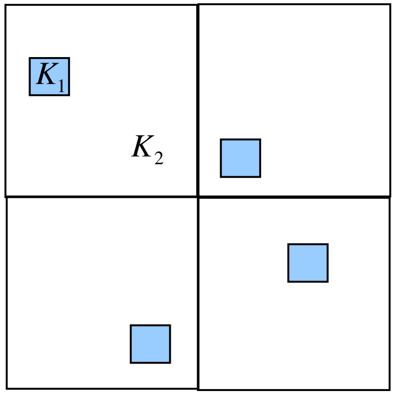


Coarse grid 8x8x8,
fine grid 8x8x8
TGM iter = 12

TGM acc = 1E-5

TGM for different geometries

Small inclusions



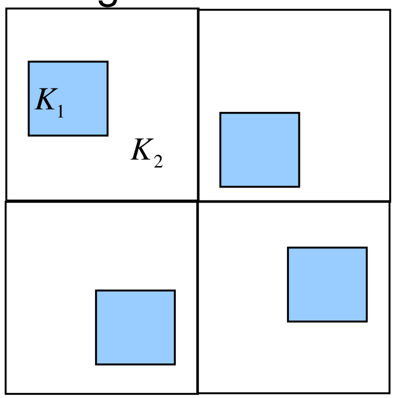
inc = 1x1x1

TGM iter = 11

Coarse grid 8x8x8,
fine grid 8x8x8

TGM acc = 1E-5

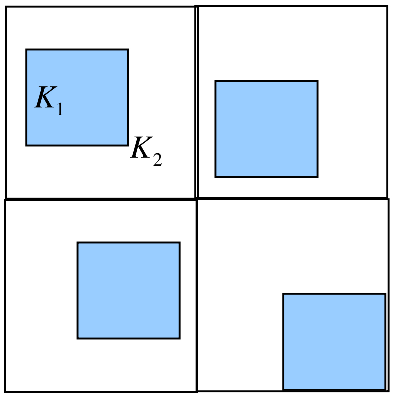
Larger inclusions



inc = 2x2x2

TGM iter = 11

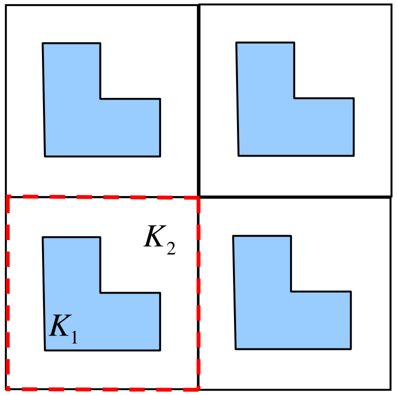
Large inclusions



inc = 4x4x4

TGM iter = 11

Oversampling

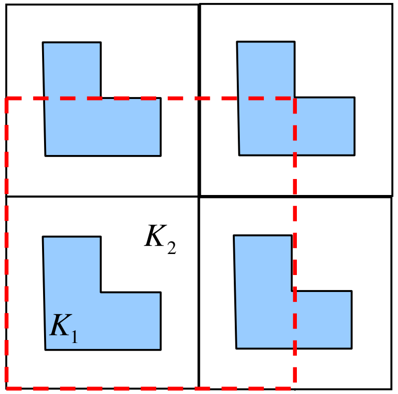


$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 10000 \end{pmatrix}$$

TGM acc = 1E-4

$$K^* = \begin{pmatrix} 6569.1 & -192.6 & 1.2\times10^{-5} \\ -192.6 & 6569.1 & 8.1\times10^{-6} \\ 1.2\times10^{-5} & 8.1\times10^{-6} & 7126.0 \end{pmatrix}$$

TGM iter = 7



$$K^* = \begin{pmatrix} 6411.6 & -256.0 & 1.5\times10^{-4} \\ -256.0 & 6411.6 & 8.1\times10^{-6} \\ 1.5\times10^{-4} & 8.1\times10^{-6} & 6957.5 \end{pmatrix}$$

TGM iter = 7

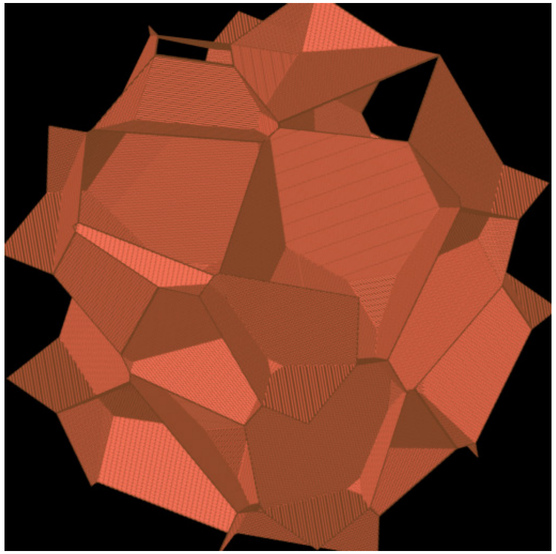
3D upscaling

Fine grid permeability tensor

$$K_1 = E, \quad K_2 = \alpha\, E$$

Effective permeability **contrast 1:3**

$$K^* = \begin{pmatrix} 1.1232 & 8.66 \cdot 10^{-5} & -2.12 \cdot 10^{-4} \\ 8.66 \cdot 10^{-5} & 1.1218 & 4.16 \cdot 10^{-4} \\ -2.12 \cdot 10^{-4} & 4.16 \cdot 10^{-4} & 1.1219 \end{pmatrix}$$



Foam

Effective permeability **contrast 1:1000**

$$K^* = \begin{pmatrix} 44.55 & -0.14 & -0.05 \\ -0.14 & 43.30 & -0.31 \\ -0.05 & -0.31 & 43.90 \end{pmatrix}$$

acc = 10-E5

Conclusions

- Finite volume discretization for the case of highly varying anisotropic permeability tensor
- Additive and multiplicative Schwarz as a smoother withing two-level preconditioner
- Coarse scale operator obtained from numerical upscaling
- Influence of the overlapping, smoother, number of subdomains on the convergence of TGM
- Applicability for non-periodic media

Future work

- Two-level DD as a preconditioner for Krylov subspace methods
- Study the influence of cell-problem formulation on the convergence of the preconditioned CG
- Develop further approaches for two-phase flows
- Theoretical analysis