Numerical Methods for Optical Tomography

Dipl.-Comp. Math. Matthias Schlottbom (schlottbom@aices.rwth-aachen.de)
Advisors: Dr. Herbert Egger, Prof. Dr. Wolfgang Marquardt

Background

Optical Tomography (OT) is a medical imaging method using near infrared light to probe (highly) scattering media, in order to recover optical parameters from boundary measurements. The spatial distribution of the recovered parameters gives information about physiologically important properties, such as the oxygenation state of blood and tissue. These functional information, the relatively low costs, and the harmless radiation are the main advantages of OT. For a review of this topic we refer to ([1]) and the references therein.

[1] A. P. Gibson, J. C. Hebden, and S. R. Arridge.
Recent advances in diffuse optical imaging.
Physics in Medicine and Biology, 50(4):R1–R43, 2005.

Physical Model

The transport of photons is usually modelled by the linear *Boltzmann* equation

$$\left(\frac{1}{c}\frac{d}{dt} + \hat{s}\cdot\nabla + \mu_{tr}\right)\phi(\hat{s}) = \mu_s \int_{S^{n-1}} \Theta(\hat{s}\cdot\hat{s}')\phi(\hat{s}')\,d\hat{s}' + q(\hat{s}) \qquad (1)$$

where $\mu_{tr}(r) = \mu_a(r) + \mu_s(r)$ is the transport cross-section describing the decay of intensity of light by absorption (μ_a) and scattering (μ_s), and $q(r, \hat{s}, t)$ is a source term. $\Theta(a\cdot b)$ is the normalized phase function representing the probability of scattering from direction $\hat{a}$ to direction $\hat{b}$. The angular density $\phi(r, \hat{s}, t)$ describes the number of photons per unit volume at position r at time t with velocity in direction $\hat{s}$, c is the speed of light in vacuum. The *vacuum boundary condition*, which states that there is no energy flux crossing the boundary, is the simplest boundary condition for the Boltzmann equation

$$\hat{s}\cdot\nu\phi = 0, \quad \text{on } \partial\Omega \times \{\hat{s}\cdot\nu < 0\}$$

where Ω denotes the domain of interest and ν the outward unit normal.

In medical applications the density, and hence the scattering, is relatively large, so that one uses the so-called *diffusion approximation* of the Boltzmann equation

$$\frac{1}{c}\frac{d}{dt}\Phi - \operatorname{div}(\kappa\nabla\Phi) + \mu_a\Phi = f \qquad (2)$$

where the derived quantity Φ is the photon density defined by

$$\Phi(r, t) = \int_{S^{n-1}} \phi(r, \hat{s}, t)\,d\hat{s}.$$

An appropriate boundary condition of *Robin* type is the following one

$$\Phi + A\kappa\frac{\partial\Phi}{\partial n} = q$$

where A incorporates refractive index mismatches at the boundary and q represents a sources on the boundary $\partial\Omega$.

Comparison Model (1) and Model (2)

- Model (2) simplifies the (numerical) solution of the inverse problem
- Model (2) not valid near boundaries, sources or in non-scattering regions
- Model (2) delivers no quantitative information in general
- Model (1) bigger computational complexity

Both equations can be transformed into frequency domain where $\frac{d}{dt}$ is replaced by $i\omega$ where ω is the modulation frequency.

Numerical Schemes for the Forward Problem

Boltzmann Equation - Model (1)
For discretization of model (1) we use a *Discontinuous Galerkin* (DG) method with the ansatz $\phi(\hat{s}, r) = \sum u_i(r)\chi_i(\hat{s})$ where the spatial varying coefficients u_i satisfy

$$\sum_Q -(u_i, \hat{s}\cdot\nabla v)_Q + ((\mu_{tr} + i\frac{\omega}{c})u_i - \mathcal{S}\underline{u})_Q + \sum_E \langle\mathcal{F}\cdot\nu, v\rangle_E = (q_i, v)$$

for all $v \in V_h^{DG}$ where $\mathcal{S}$ represents the scattering operator, $\underline{u} = (u_1, \ldots, u_N)$ and $\mathcal{F}$ is a numerical flux.

Diffusion Approximation - Model (2)
For this second order parabolic/elliptic equation the variational formulation is

$$(\kappa\nabla u, \nabla v) + ((\mu_a + i\frac{\omega}{c})u, v) + \langle\frac{1}{A}u, v\rangle_{\partial\Omega} = (f, v) + \langle\frac{1}{A}q, v\rangle_{\partial\Omega}$$

for all $v \in H^1$. *Finite Elements* are a natural discretization.

Acknowledgement

Financial support from the Deutsche Forschungsgemeinschaft (German Research Association) through grant GSC 111 is gratefully acknowledged.

Deutsche Forschungsgemeinschaft
DFG

Inverse Problem and Regularization

A typical workflow in OT consists of

- illuminating the tissue with a light source,
- measuring the outward flux of light,
- reconstructing the scattering and absorption coefficient.

The link between (noisy) measurements and parameters is mathematically given by

$$F(\mu) = y^\delta$$

where $\mu = (\mu_a, \mu_s)$ and F models the physics of the underlying problem. In order to obtain a theoretical understanding of the mathematical problem one can reformulate the equation as a constrained optimization problem

$$J(\mu) = \|B(\phi) - y^\delta\|^2 + R(\alpha, \mu) \to \min! \text{ subject to } e(\phi, \mu) = 0$$

where $e(\phi, \mu)$ is the governing partial differential equation, B is the observation operator with $B(\phi) = F(\mu)$ and R is a regularization method.

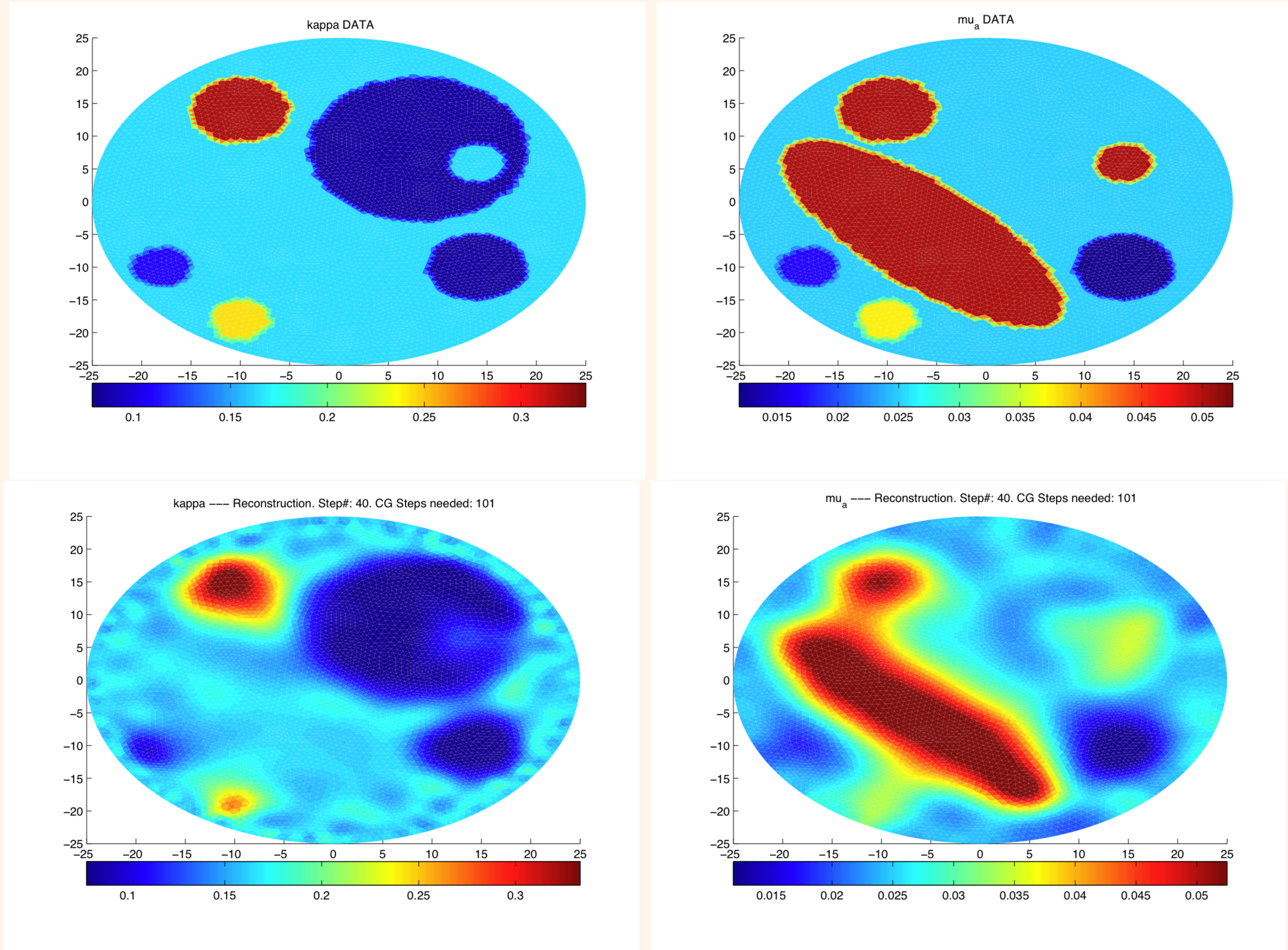

Figure: Left column: Target and reconstructed scattering distribution. Right column: Target and reconstructed absorbtion distribution. Data are contaminated with 1% additive gaussian noise. Reconstruction performed with an iteratively Tikhonov-regularized Gauss-Newton method.

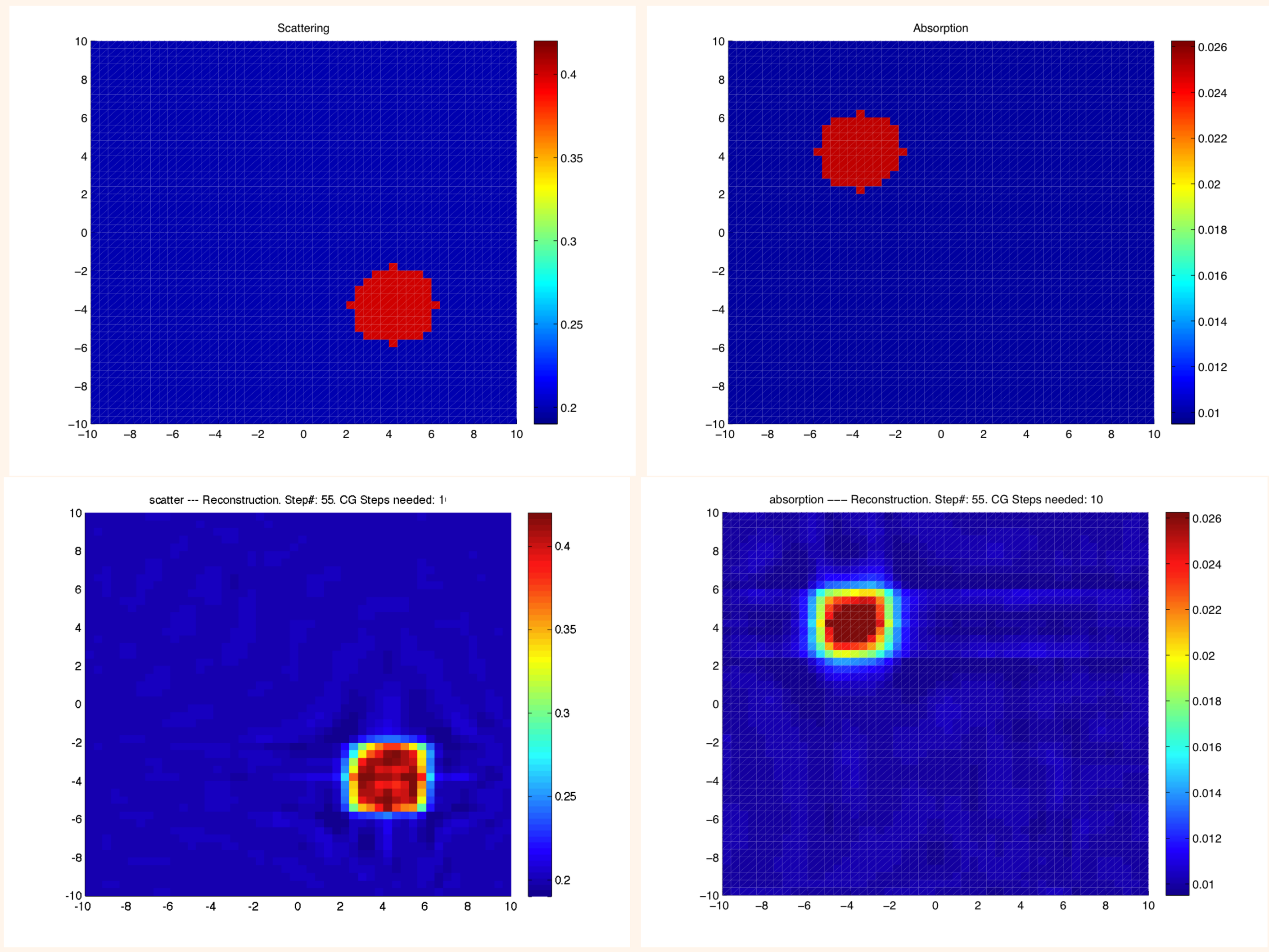

Figure: Left column: Target and reconstructed scattering distribution. Right column: Target and reconstructed absorbtion distribution. No noise. Reconstruction performed with a truncated Newton-CG algorithm.

Research Directions

The aim of this project is to investigate analytical and numerical methods for the solution of the direct and inverse problem in OT. Research directions are

- High-Order DG methods for the forward problem.
- Formulation of error estimates that allow a moment-adaptive strategy.
- Analytical investigation of the inverse problem: existence and uniqueness of solution, conditional stability estimates.
- Numerical solution of the inverse problem with validation through data obtained by experiments.
- Development or use of appropriate regularization techniques.

AACHEN INSTITUTE FOR ADVANCED STUDY
IN COMPUTATIONAL ENGINEERING SCIENCE