Thin cylindrical conductivity inclusions in a 3-dimensional domain: polarization tensor and unique determination from boundary data

Elisa Francini

in collaboration with Elena Beretta, Yves Capdeboscq and Frédéric de Gournay

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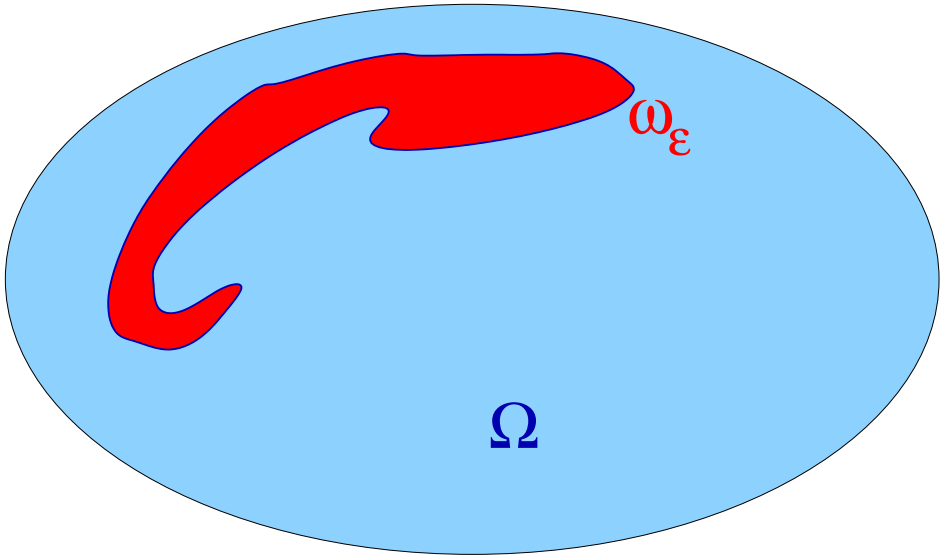
Low volume fraction inclusions

Ω is a bounded smooth domain,
$\omega_\epsilon \subset \Omega$ is an inclusion
$|\omega_\epsilon| \longrightarrow 0$ as $\epsilon \longrightarrow 0$
γ_0 and γ_1 are smooth functions defined in Ω.
$\gamma_\epsilon = \gamma_0 + (\gamma_1 - \gamma_0)\mathbf{1}_{\omega_\epsilon}$ is the conductivity of the body



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- Recover information on the inclusion from boundary measurements of the potential.

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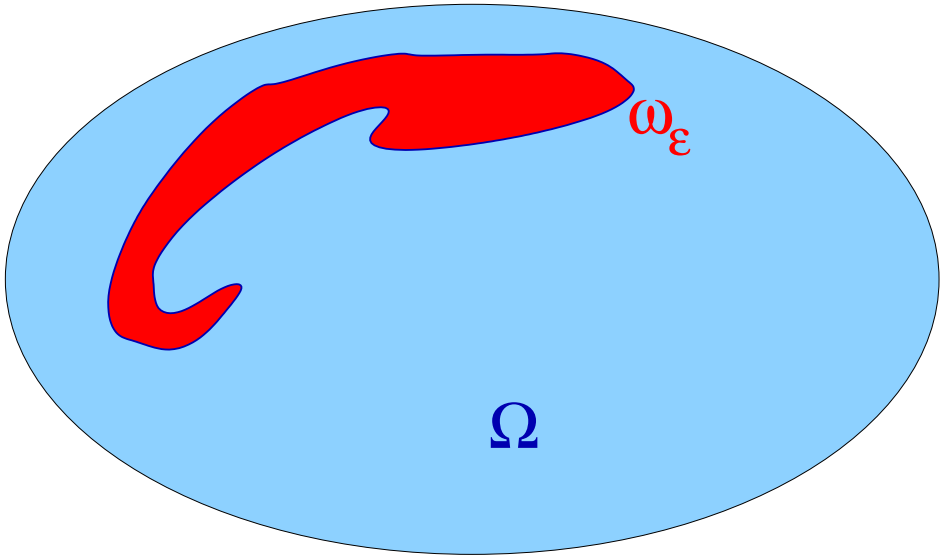

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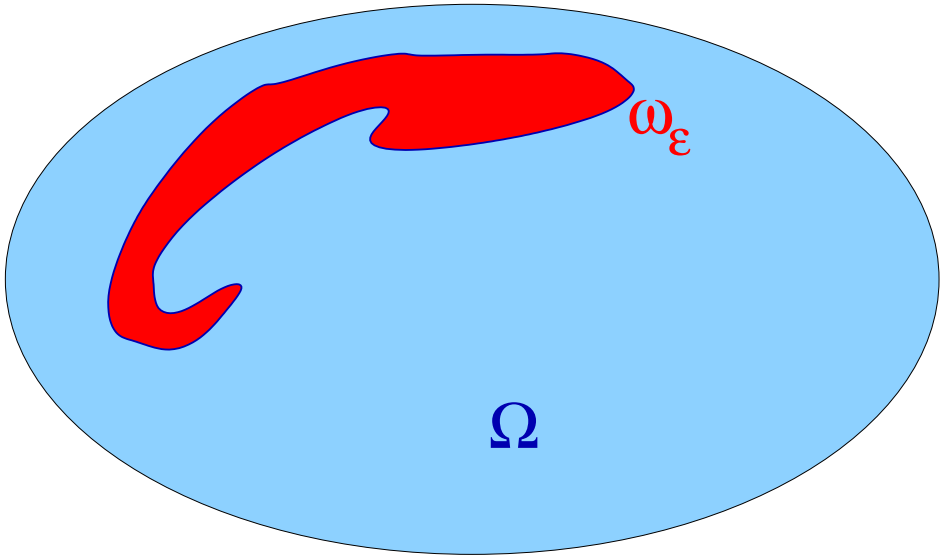

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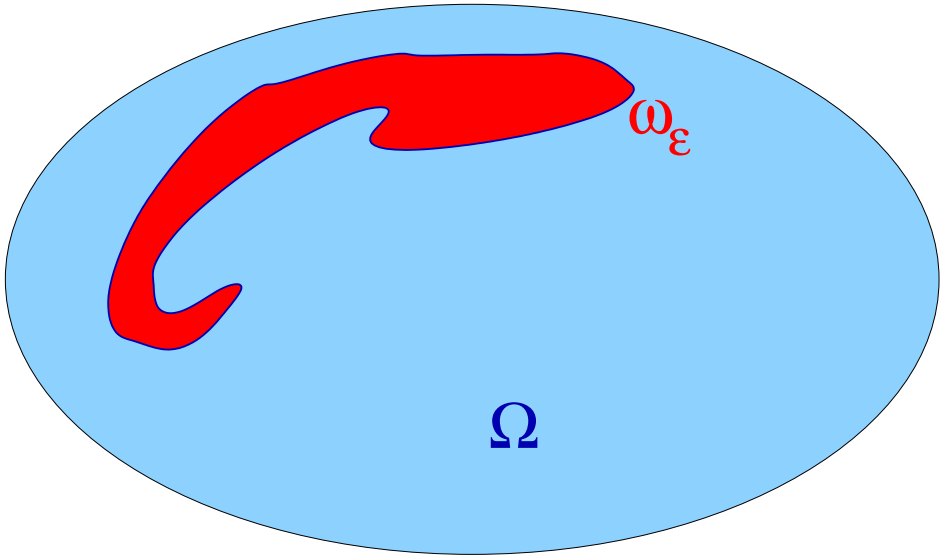

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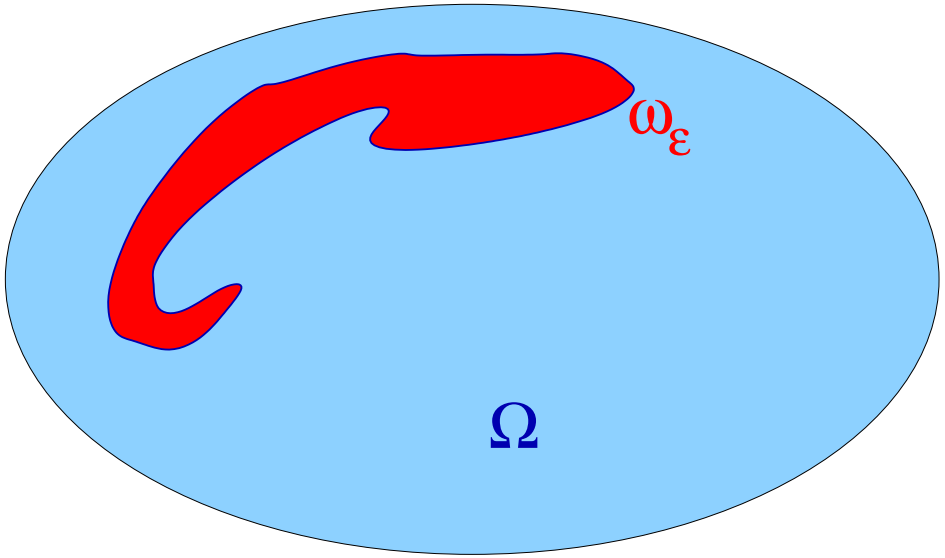

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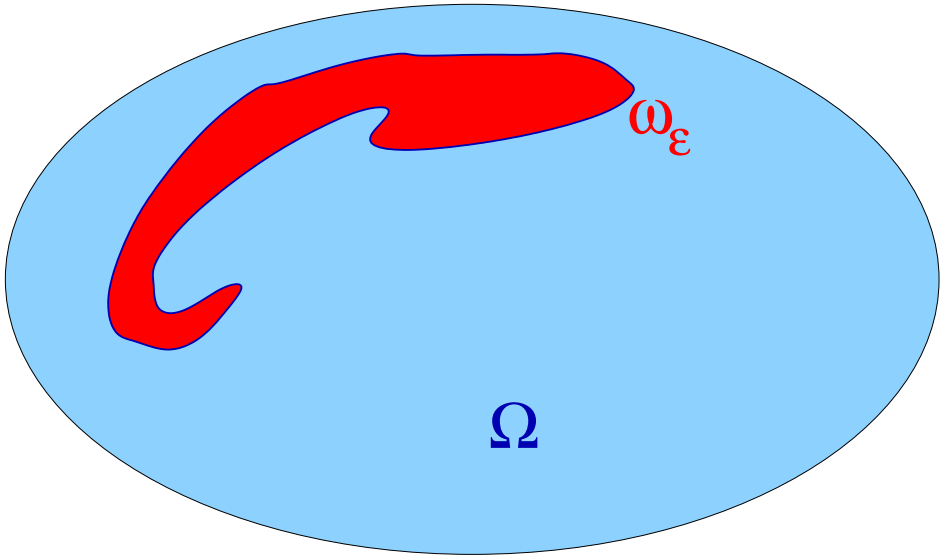

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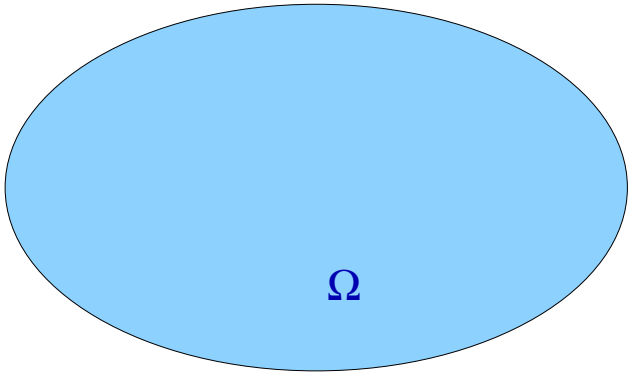
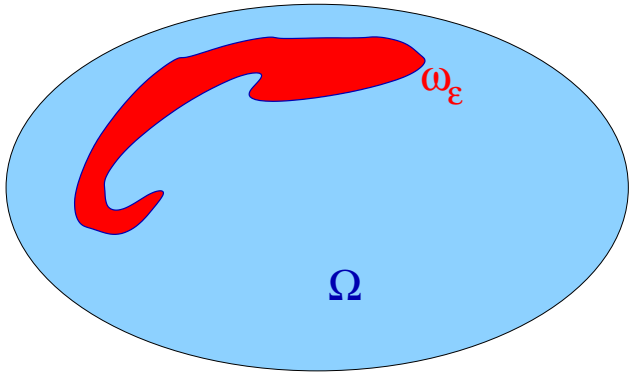
Compare two potentials

A current $g \in H^{-1/2}(\partial\Omega)$ such that $\int_{\partial\Omega} g\, d\sigma = 0$ induces two different potentials:

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) &= 0 \text{ in } \Omega, \\ \gamma_0 \dfrac{\partial u_0}{\partial n} &= g \text{ on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_0\, d\sigma &= 0, \end{cases} \qquad \begin{cases} \operatorname{div}(\gamma_\epsilon \nabla u_\epsilon) &= 0 \text{ in } \Omega, \\ \gamma_\epsilon \dfrac{\partial u_\epsilon}{\partial n} &= g \text{ on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_\epsilon\, d\sigma &= 0. \end{cases}$$





Can we evaluate $u_\epsilon - u_0$ on $\partial\Omega$?

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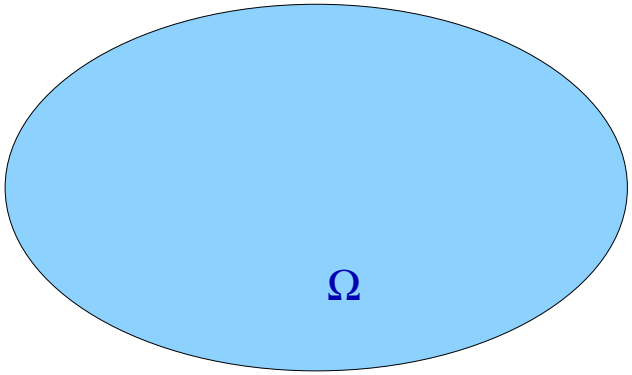

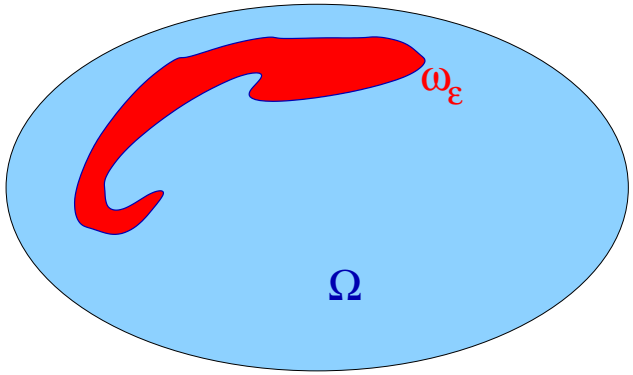

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Assumptions and notation

Assumption on the behavior of the inclusion : $d(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$ for every ϵ and

$$|\omega_\epsilon|^{-1}\, \mathbf{1}_{\omega_\epsilon}(\cdot) \text{ converges in the sense of measure to } \mu \text{ when } \epsilon \to 0.$$

Assumption on the conductivities

$$\gamma_i \quad \text{smooth}, \quad c_0 < \gamma_i(x) < c_0^{-1}, \quad \text{for} \quad x \in \Omega, \quad i = 0, 1.$$

Let N denote the Neumann function of the unperturbed domain: given $y \in \Omega$, let $N(\cdot, y)$ be the solution to

$$\begin{cases} \operatorname{div}_x\left(\gamma_0(x)\nabla_x N(x,y)\right) &= \delta_y(x) \text{ for } x \in \Omega, \\ \gamma_0(x)\dfrac{\partial N}{\partial n_x}(x,y) &= \dfrac{1}{|\partial\Omega|} \text{ for } x \in \partial\Omega, \\ \displaystyle\int_{\partial\Omega} N(x,y)\, d\sigma_x &= 0. \end{cases}$$

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The general result by Capdeboscq and Vogelius

Theorem

There exists a tensor $\{M_{ij}\}_{i,j=1}^{n} \in L^2(\Omega, d\mu)$ *such that, for* $y \in \partial\Omega$,

$$\begin{aligned}(u_\epsilon - u_0)(y) &= |\omega_\epsilon| \sum_{i,j=1}^{n} \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial N}{\partial x_j}(x,y) d\mu(x) \\ &\quad + o\left(|\omega_\epsilon|\right).\end{aligned}$$

Here $|\omega_\epsilon|^{-1} \left\| o\left(|\omega_\epsilon|\right) \right\|_{L^\infty(\partial\Omega)} \to 0$ *as* $\epsilon \to 0$ *uniformly with respect to* $\{g \in H^{-1/2}(\partial\Omega) \ : \ \int_{\partial\Omega} g \ d\sigma = 0 \ , \ \ \|g\|_{L^2(\partial\Omega)} \leq 1\}$.

Properties of the polarization tensor

One of the possible definition of the polarization tensor is the following:

$$\int_\Omega (\gamma_1 - \gamma_0) M\xi \cdot \xi \, \phi \, d\mu = \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \frac{\gamma_0}{\gamma_1} |\xi|^2 \, \phi \, dx$$
$$+ \frac{1}{|\omega_\epsilon|} \min_{w \in H_0^1(\Omega)} \int_\Omega \gamma_\epsilon \left| \nabla w + \mathbf{1}_{\omega_\epsilon} \frac{\gamma_1 - \gamma_0}{\gamma_1} \xi \right|^2 \phi \, dx + o(1).$$

where ξ is a vector, ϕ is any smooth positive function and $o(1)$ tends to zero with ϵ.

Moreover

$$M_{ij}(x) = M_{ji}(x),$$

$$\min \left\{ 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right\} |\xi|^2 \leq M_{ij}\xi_i\xi_j \leq \max \left\{ 1, \frac{\gamma_0(x)}{\gamma_1(x)} \right\} |\xi|^2$$

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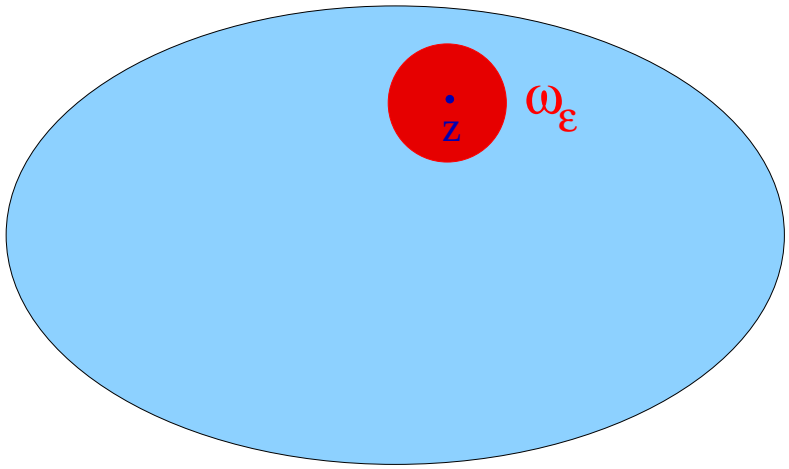
Special case: diametrically small inclusions

Let z be a point in Ω and B a bounded domain of volume 1. If

$$\omega_\epsilon = z + \epsilon B$$

then $\mu = \delta_z$ and M is the tensor of Pólya-Szegö associated to B. If B is a ball

$$M = \frac{n\gamma_0}{\gamma_1 + (n-1)\gamma_0} Id_n$$



In this case

$$(u_\epsilon - u_0)(y) = \epsilon^n M \nabla u_0(z) \cdot \nabla N(z, y) + 0(\epsilon^n)$$

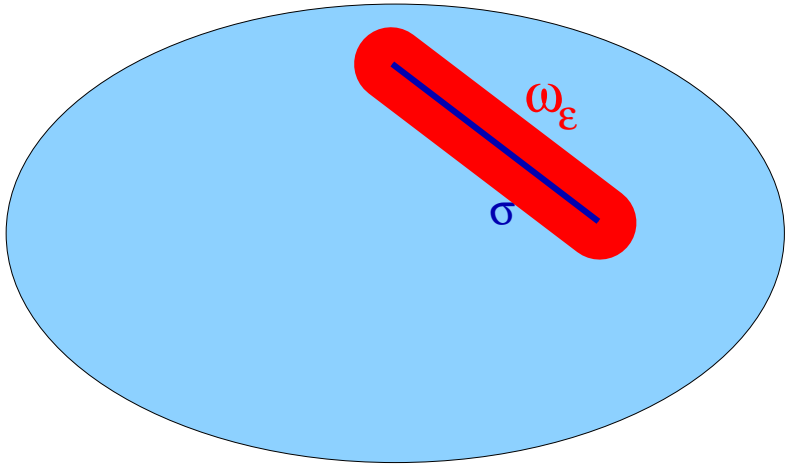
Special case: thin inclusions

Let σ be a curve in $\mathbb{R}^2$

$$\omega_\epsilon = \{x \,:\, d(x,\sigma) < \epsilon\}$$

The limit measure μ is concentrated on σ and

$$M = \left(\tau \otimes \tau + \frac{\gamma_0}{\gamma_1} n \otimes n\right).$$

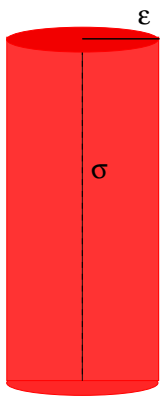


In this case

$$(u_\epsilon - u_0)(y) \;=\; 2\epsilon \int_\sigma (\gamma_1 - \gamma_0)\left(\frac{\partial u_0}{\partial \tau}(x)\frac{\partial N}{\partial \tau}(x,y) + \frac{\gamma_0}{\gamma_1}\frac{\partial u_0}{\partial n}(x)\frac{\partial N}{\partial n}(x,y)\right) d\sigma$$
$$+0(\epsilon)$$

Cylinders in $\mathbb{R}^3$

$\Omega \subset \mathbb{R}^3$ and ω_ϵ is a cylindrical inclusion with axis σ and basis of the form ϵB_2 where B_2 is a bidimensional domain. The limit measure μ is concentrated on σ. Let τ be the direction of σ.



Theorem

The tensor M has eigenvalue 1 corresponding to the eigendirection τ:

$$M\tau \cdot \tau = 1,$$

and, if $v \cdot \tau = 0$, then

$$Mv \cdot v = mv \cdot v,$$

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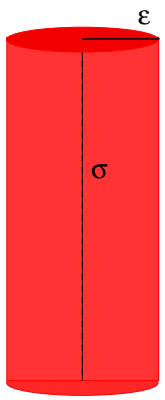

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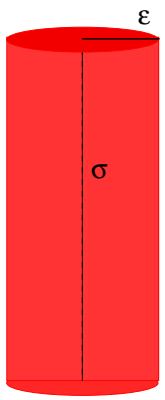

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Idea of proof...

Let us take the cylinder centered at the origin with $\tau = e_3$. In order to evaluate $M\tau \cdot \tau$ we need to find $\Phi^3_\epsilon(x)$ such that

$$\mathrm{div}\left(\gamma_\epsilon \phi \nabla \Phi^3_\epsilon\right) = \mathrm{div}\left((\gamma_0 - \gamma_1)\mathbf{1}_{\omega_\epsilon} e_3 \phi\right) \text{ in } \Omega.$$

By using regularity estimates it is possible to show that

$$\begin{aligned} \frac{1}{|\omega_\epsilon|}\int_\Omega \gamma_\epsilon \left|\nabla \Phi^3_\epsilon + \mathbf{1}_{\omega_\epsilon}\frac{\gamma_1-\gamma_0}{\gamma_1}\xi\right|^2 \phi\, dx &= \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\gamma_1-\gamma_0)\left(1-\frac{\gamma_0}{\gamma_1}\right)|\xi|^2 \\ &+ o(1). \end{aligned}$$

..Idea of proof

In order to evaluate $Me_i \cdot e_i$ for $i = 1, 2$ the idea is to approximate the "correctors" Φ_ϵ^i for the 3-dimensional problem by correctors of the corresponding bidimensional section.

Asymptotic expansion

Let m be the bidimensional tensor associated with an inclusion of the form ϵB_2. Then, if we denote be τ the tangent direction to the axis σ, we have, for $y \in \partial\Omega$,

$$(u_\epsilon - u_0)(y) = \epsilon^2 u_\sigma(y) + o(\epsilon^2)$$

where

$$u_\sigma(y) = \int_\sigma (\gamma_1 - \gamma_0)(x) \left[\frac{\partial u_0}{\partial \tau}(x) \frac{\partial N}{\partial \tau}(x,y) + m(x) \widetilde{\nabla u_0}(x) \cdot \widetilde{\nabla N}(x,y) \right] d\sigma_x.$$

where $\tilde{v} = v - (v \cdot \tau)\tau$ denotes the non tangential part of v

Information from the correction term

The function u_σ can be extended from $\partial\Omega$ to $\Omega \setminus \sigma$ and solves

$$\operatorname{div}(\gamma_0 \nabla u_\sigma) = 0 \quad \text{in} \quad \Omega \setminus \sigma.$$

The correction term is singular at every point of segment σ. Moreover, $\gamma_0 \frac{\partial u_\sigma}{\partial n} = 0$ on $\partial\Omega$.

Proposition

Let γ_0 and γ_1 be smooth positive functions. Let Σ be an open subset of $\partial\Omega$ and let σ and σ' be two segments strictly contained in Ω. Let u_0 be a smooth solution to $div(\gamma_0 \nabla u_0) = 0$ in Ω such that $\nabla u_0 \neq 0$ in Ω, and let u_σ and $u_{\sigma'}$ be the correction terms for segments σ and σ' respectively. If

$$u_\sigma = u_{\sigma'} \quad on \quad \Sigma,$$

then

$$\sigma = \sigma'.$$

Information from the correction term

The function u_σ can be extended from $\partial\Omega$ to $\Omega \setminus \sigma$ and solves

$$\operatorname{div}(\gamma_0 \nabla u_\sigma) = 0 \quad \text{in} \quad \Omega \setminus \sigma.$$

The correction term is singular at every point of segment σ. Moreover, $\gamma_0 \frac{\partial u_\sigma}{\partial n} = 0$ on $\partial\Omega$.

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then

$$\sigma = \sigma'.$$

Reconstruction algorithm

- Assumptions: $\Omega = B_R$ ball of radius R centered at the origin.
- Linear background potentials: $u_0^i = x_i$ for $i = 1, 2, 3$.
- Measurements: u_ϵ^i boundary values of the solutions corresponding to $g^i = \frac{\partial x_j}{\partial n}$.
- Conductivities: $\gamma_0 = 1$, γ_1 constant (say $\gamma_1 > 1$).

First step: identification of the direction τ of the axis of the cylinder. By the asymptotic expansion, the power perturbation can be calculated as,

$$\begin{aligned}
\delta W_{ij} &= \int_{\partial B_R} \left({u_\epsilon}^i - u_0^i \right) \frac{\partial x_j}{\partial n} ds \\
&= |\omega_\epsilon| \int_\Omega (1 - \gamma_1) M \nabla x^i \cdot \nabla x^j d\mu + o\left(|\omega_\epsilon|\right), \\
&= |\omega_\epsilon| (1 - \gamma_1) M e^i \cdot e^j + o\left(|\omega_\epsilon|\right).
\end{aligned}$$

The eigenvector of the matrix δW_{ij} corresponding to the biggest eigenvalue is the direction τ

Reconstruction algorithm: second step

Median plane

Consider the background $u_0^\tau = \tau \cdot x$ and the corresponding solution u_ϵ^τ that is completely determined by the previous ones by $u_\epsilon^\tau = \sum_{i=1}^3 (\tau \cdot e^i) u_\epsilon^i$. Let A and B be the endpoints of σ.

$$\delta W_\tau = \int_{\partial B_R} \left({u_\epsilon}^\tau - u_0^\tau\right)(\tau \cdot n) ds = |\omega_\epsilon| \left(1 - \gamma_1\right) M\tau \cdot \tau + o\left(|\omega_\epsilon|\right).$$

$$\begin{aligned} D_1 &= \int_{\partial B_R} \left({u_\epsilon}^\tau - u_0^\tau\right) \frac{\partial}{\partial n} \left(\frac{(x \cdot \tau)^2 - (x \cdot e_1)^2}{2} \right) ds \\ &= \frac{1}{2} |\omega_\epsilon| \left(1 - \gamma_1\right) M\tau \cdot \tau \left((A \cdot \tau) + (B \cdot \tau)\right) + o\left(|\omega_\epsilon|\right). \end{aligned}$$

Hence

$$\frac{D_\psi}{\delta W_\tau} = \frac{1}{2} \left((A \cdot \tau) + (B \cdot \tau)\right) + o(1).$$

Reconstruction algorithm: final steps

By using harmonic test function of the form $\phi_a(x) = |x-a|^{-1}$ with a on a plane orthogonal to τ and outside B_R.

$$\begin{aligned} D_a &= \int_{\partial B_R} \left({u_\epsilon}^\tau - x \cdot \tau \right) \frac{\partial \phi_a}{\partial n} ds \\ &= |\omega_\epsilon| \left(1-\gamma_1\right) M\tau \cdot \tau \left| \frac{1}{|A-a|} - \frac{1}{|B-a|} \right| + o\left(|\omega_\epsilon|\right) \end{aligned}$$

By a minimization process with respect to a we identify A and B.

Numerical simulation

$\Omega = B_{1.5}$, $\gamma_0 = 1$, $\gamma_1 = 3$, $|\sigma| = 1$, circular cross-section, $\epsilon = (10\sqrt{\pi})^{-1}$,

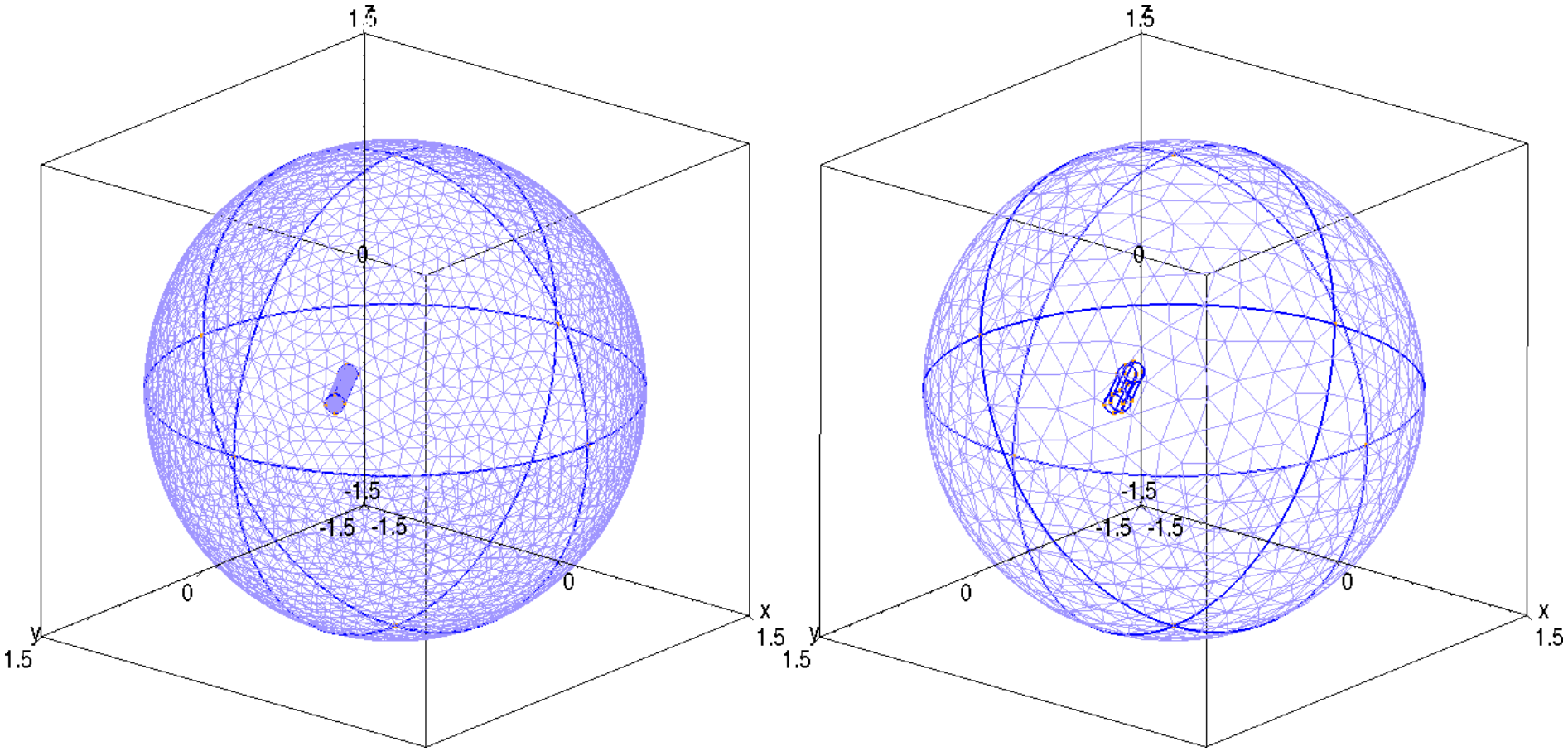

Figure: On the left, the boundary elements of the mesh used for the simulation (Gmsh and Getfem++). On the right, the reconstructed inclusion together with the original one.

Numerical simulation

$\Omega = B_{1.5}$, $\gamma_0 = 1$, $\gamma_1 = 3$, $|\sigma| = 1$, circular cross-section, $\epsilon = (10\sqrt{\pi})^{-1}$,
The midpoint is $X_m := (x_m, y_m, z_m) \approx (0.006, 0.506, 0.150)$.,

$$\tau = (\sin(\pi/3)\cos(\pi/4), \sin(\pi/3)\sin(\pi/4), \cos(\pi/3)).$$

The three dimensional mesh has 12625 nodes.
The eigenvalues of δW are $\lambda_0 = -1.83777 \times 10^{-2}$,
$\lambda_1 = -1.04347 \times 10^{-2}$, and $\lambda_2 = -1.04434 \times 10^{-2}$. Thus, $\gamma_1 > 1$ and the first eigenvalue corresponds to the direction of the cylinder. Its eigenvector is $\tilde{\tau} = (-0.612494, -0.612253, -0.499998)$, and $|\tilde{\tau} + \tau| \approx 2 \times 10^{-4}$.
The computed median plane abscissa is $\tilde{x}_\tau = -0.387759$, and
$|\tilde{x}_\tau + X_m \cdot \tau| = 1 \times 10^{-3}$.
$\tilde{L} = 0.984$.
Finally, since $\lambda_1 = \lambda_2$, it is clear that the equivalent ellipse is a disk: this leads to an estimation of $\tilde{\gamma}_1 \approx 2.5$, and of the radius of the disk of $\tilde{\epsilon} \approx \frac{11}{100\sqrt{\pi}}$.

Numerical simulation 2

$\Omega = B_{1.5}$, $\gamma_0 = 1$, $\gamma_1 = 0.5$, $|\sigma| = 6/5$, elliptic cross-section $a/b = 2$, $\epsilon = (10\sqrt{\pi})^{-1}$,

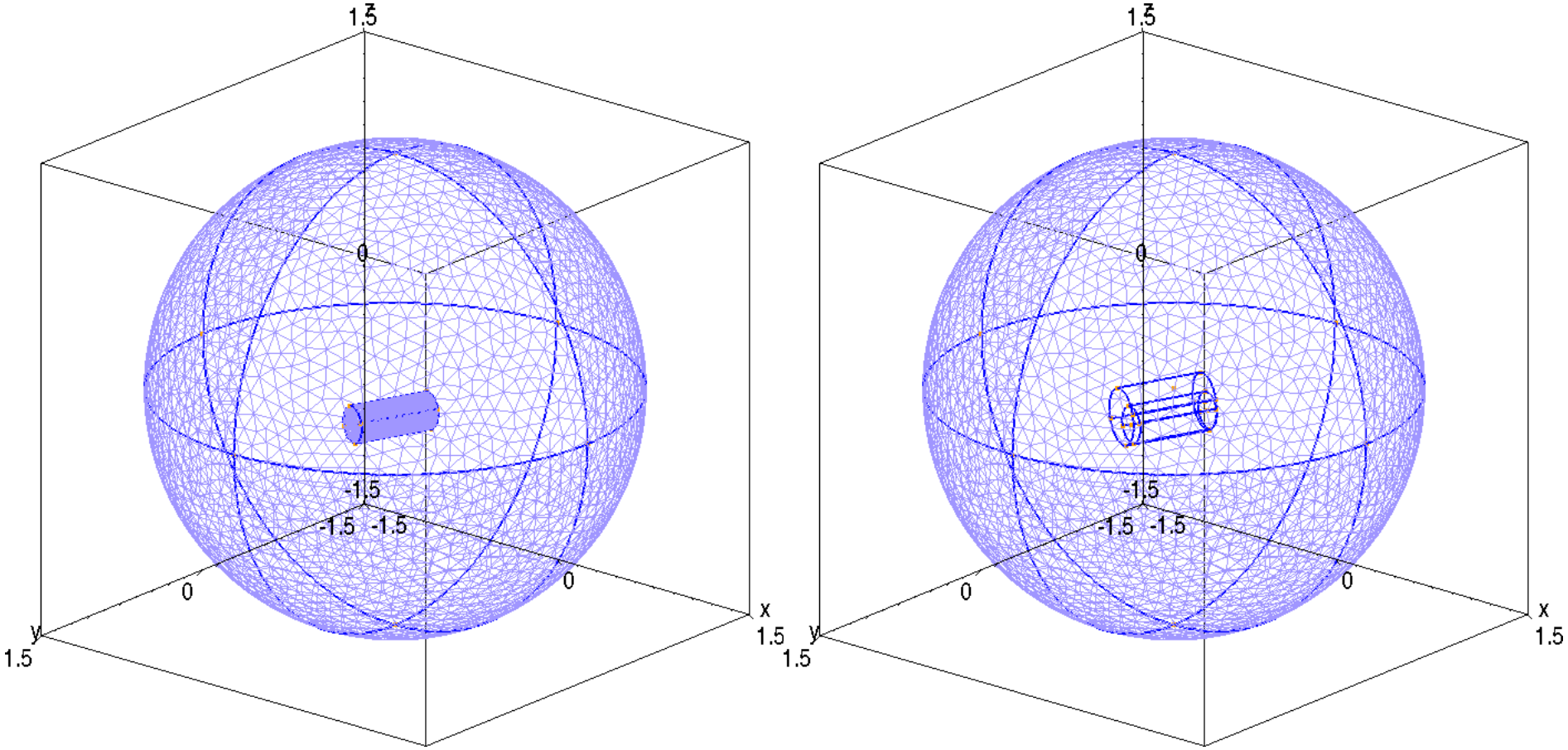

Figure: Simulation for a cylinder with elliptic cross-section. On the left, the mesh. On the right, the reconstructed inclusion together with the original one.

Numerical simulation 2

$\Omega = B_{1.5}$, $\gamma_0 = 1$, $\gamma_1 = 0.5$, $|\sigma| = 6/5$, elliptic cross-section $a/b = 2$, $\epsilon = (10\sqrt{\pi})^{-1}$, Main directions: $\tau = (0.8324, 0.4009, 0.3827)$ axis of the cylinder, $\nu_1 = (-0.3448, 0.1660, 0.9239)$ major axis, $\nu_2 = (-0.4339, 0.9010)$ minor axis of the ellipse. The midpoint $X_m = (-0.001, 0.041, -0.170)$. The three dimensional mesh has 15306 nodes.
The eigenvalues of δW are $\lambda_0 = 1.72031 \times 10^{-2}$, $\lambda_1 = 1.22152 \times 10^{-2}$, and $\lambda_2 = 1.39706 \times 10^{-2}$, Thus, $\gamma_1 > 1$ and the second eigenvalue corresponds to the direction of the cylinder, the first one to the minor axis, and the third to the major axis. The error in the main direction is 2×10^{-4} and in the cross section, it is 8×10^{-4}. Finally, turning to the polarization tensor, from the ratios λ_0/λ_1 and λ_2/λ_1, we obtain $a \approx \frac{31}{100\sqrt{\pi}}$ and $b \approx \frac{13}{100\sqrt{\pi}}$ and $\gamma_1 = 0.58$

Thank you!