Reconstruction of discontinuous functions by perimeter penalization

Luca RONDI

Università degli Studi di Trieste
Dipartimento di Matematica e Informatica

AIP 2009
Wien, 20 – 24 July 2009

Electrical Impedance Tomography

Reconstruction of discontinuous conductivities

Electrical Impedance Tomography: the forward problem

$\Omega \subset \mathbb{R}^N$ ($N \geq 2$) bounded domain, $\partial\Omega$ Lipschitz
conductivity $\sigma : \Omega \longrightarrow \mathbb{R}$ measurable and uniformly elliptic

$$0 < \lambda \leq \sigma \leq \lambda^{-1} \quad \text{a.e. in } \Omega$$

prescribed current density $f \in H_*^{-1/2}(\partial\Omega)$ where

$$H_*^{-1/2}(\partial\Omega) = \{f \in H^{-1/2}(\partial\Omega) : f \text{ has zero mean}\}$$

electrostatic potential u solution to the forward problem

$$(1) \qquad \begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ \sigma \nabla u \cdot \nu = f & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} u = 0 & \end{cases}$$

Electrical Impedance Tomography: the inverse problem

Unknown

- conductivity σ

Available data

- Neumann-to-Dirichlet map $\Lambda(\sigma) : H_*^{-1/2}(\partial\Omega) \longrightarrow H_*^{1/2}(\partial\Omega)$

$$\Lambda(\sigma)[f] = u|_{\partial\Omega} \in H_*^{1/2}(\partial\Omega) \quad \text{for any } f \in H_*^{-1/2}(\partial\Omega),$$

u solution to the forward problem (1)

Inverse conductivity problem (Calderón 1980)

Determine the conductivity σ from electrostatic measurements on the boundary, that is by measuring the Neumann-to-Dirichlet map $\Lambda(\sigma)$

Main issues

Uniqueness

Does the Neumann-to-Dirichlet map $\Lambda(\sigma)$ uniquely determine the conductivity σ? Is the forward function $\sigma \longrightarrow \Lambda(\sigma)$ injective?

$N \geq 3$ Kohn & Vogelius (1984) — Sylvester & Uhlmann (1987) (σ regular)
Isakov (1988) (σ piecewise regular)

$N = 2$ Nachman (1995) (σ regular) — Astala & Päivärinta (2006) ($\sigma \in L^\infty$)

Reconstruction

Numerically reconstruct σ from (an approximation of) $\Lambda(\sigma)$

Main difficulties

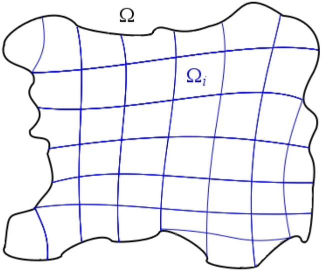
- nonlinear problem
- severely ill-posed problem (Mandache (2001), σ regular, $N \geq 2$)

Discontinuous conductivities: two particular cases (1)

Domain discretization

$$\Omega = \bigcup_{i=1}^{n} \Omega_i \qquad \Omega_i \text{ assigned}$$

$$\sigma = \sum_{i=1}^{n} k_i \chi_{\Omega_i} \ (\sigma \text{ piecewise constant})$$



Problem: determine the unknown constants k_i

- Lipschitz stability (Alessandrini & Vessella (2005))
- Lipschitz constant exponentially exploding as $n \longrightarrow +\infty$ (Rondi (2006))

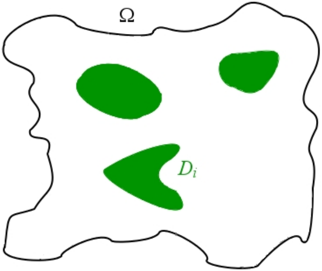
Discontinuous conductivities: two particular cases (2)

Inverse inclusion problem

$$\sigma = k_0 + \sum_{i=1}^{n} (k_i - k_0)\chi_{D_i}$$

$D_i \Subset \Omega$ $\qquad$ D_i unknown inclusions

k_0 known background conductivity
k_i constants or functions known or not



Problem: determine the unknown inclusions D_i (and in case the coefficients k_i)

- optimal stability of logarithmic type (Alessandrini & Di Cristo (2005))
- optimality (Di Cristo & Rondi (2003))

Setup of the inverse problem

Let X be the set of measurable functions σ such that

- $0 < \lambda \leq \sigma \leq \lambda^{-1}$ a.e. in Ω
- σ belongs to some class of discontinuous functions (for example piecewise constant or piecewise regular)

Let $Y = \mathcal{L}(H_*^{-1/2}(\partial\Omega), H_*^{1/2}(\partial\Omega))$

Let $\Lambda : X \longrightarrow Y$ be the forward function such that, for any $\sigma \in X$, $\Lambda(\sigma)$ is the Neumann-to-Dirichlet map associated to σ

Let $\sigma_0 \in X$ be the unknown conductivity, $\Lambda_0 = \Lambda(\sigma_0)$ and Λ_ε be the approximate (measured) Neumann-to-Dirichlet map such that

$$d_Y(\Lambda_0, \Lambda_\varepsilon) \leq \varepsilon$$

Setup of the inverse problem: reconstruction

Let $0 < \beta \le \alpha$ and let us consider the regularized problem

$$\min_{\sigma\in X} \quad d_Y(\Lambda(\sigma),\Lambda_\varepsilon)^\alpha + a\varepsilon^\beta R(\sigma)$$

$R : X \longrightarrow [0,+\infty]$ regularization operator (usually R is a norm or a seminorm)
$a\varepsilon^\beta > 0$ regularization coefficient

Main issue

How to choose the regularization operator R, and the distances d_X and d_Y, so that the following is guaranteed?

(1) existence of a minimizer: σ_ε minimizer (regularized solution)

(2) convergence of minimizers: $\lim_{\varepsilon\to 0^+} \sigma_\varepsilon = \sigma_0$

Examples of regularization operators

Total variation

$$R(\sigma) = TV(\sigma) = |D\sigma|(\Omega)$$

- Dobson & Santosa (1994) (discretization)
- Chan & Tai (2003) — Chung, Chan & Tai (2005) (level set)

Examples of regularization operators

Total variation

$$R(\sigma) = TV(\sigma) = |D\sigma|(\Omega)$$

- Dobson & Santosa (1994) (discretization)
- Chan & Tai (2003) — Chung, Chan & Tai (2005) (level set)

Mumford-Shah functional

$$R(\sigma) = MS(\sigma) = \int_\Omega |\nabla\sigma|^2 + \mathcal{H}^{N-1}(J(\sigma))$$

- Rondi & Santosa (2001) (Γ-convergence)

Examples of regularization operators

Total variation

$$R(\sigma) = TV(\sigma) = |D\sigma|(\Omega)$$

- Dobson & Santosa (1994) (discretization)
- Chan & Tai (2003) — Chung, Chan & Tai (2005) (level set)

Mumford-Shah functional

$$R(\sigma) = MS(\sigma) = \int_\Omega |\nabla\sigma|^2 + \mathcal{H}^{N-1}(J(\sigma))$$

- Rondi & Santosa (2001) (Γ-convergence)

Particular cases

Examples of regularization operators

Total variation

$$R(\sigma) = TV(\sigma) = |D\sigma|(\Omega)$$

- Dobson & Santosa (1994) (discretization)
- Chan & Tai (2003) — Chung, Chan & Tai (2005) (level set)

Mumford-Shah functional

$$R(\sigma) = MS(\sigma) = \int_\Omega |\nabla\sigma|^2 + \mathcal{H}^{N-1}(J(\sigma))$$

- Rondi & Santosa (2001) (Γ-convergence)

Particular cases

Domain discretization

$$TV(\sigma) = \sum_{i<j} |k_i - k_j| \mathcal{H}^{N-1}(\partial\Omega_i \cap \partial\Omega_j)$$

Examples of regularization operators

Total variation

$$R(\sigma) = TV(\sigma) = |D\sigma|(\Omega)$$

- Dobson & Santosa (1994) (discretization)
- Chan & Tai (2003) — Chung, Chan & Tai (2005) (level set)

Mumford-Shah functional

$$R(\sigma) = MS(\sigma) = \int_\Omega |\nabla\sigma|^2 + \mathcal{H}^{N-1}(J(\sigma))$$

- Rondi & Santosa (2001) (Γ-convergence)

Particular cases

Inverse inclusion problem (inclusions D_i pairwise disjoint)

$$TV(\sigma) = \sum_i |k_i - k_0| P(D_i) \quad \text{or} \quad MS(\sigma) = \sum_i P(D_i)$$

$P(D) = \mathcal{H}^{N-1}(\partial D)$ perimeter of D

Convergence result (Rondi 2008)

Let $d_X(\sigma_1, \sigma_2) = \|\sigma_1 - \sigma_2\|_{L^1}$ and the regularization operator R be either the total variation or the Mumford-Shah functional

Theorem

Let us assume that $R(\sigma_0) < +\infty$ and $d_Y(\Lambda_1, \Lambda_2) = \|\Lambda_1 - \Lambda_2\|_{\mathcal{L}(L^2_*, L^2_*)}$

Let $\sigma_n = \sigma_{\varepsilon_n}$ be regularized solutions $(\varepsilon_n \rightarrow 0^+)$

If $N = 2$, then $\lim_n \sigma_n = \sigma_0$ in L^1

If $N \geq 3$ and $\beta < \alpha$, then, up to a subsequence, $\lim_n \sigma_n = \overline{\sigma}$ in L^1 where $\Lambda(\overline{\sigma}) = \Lambda_0 = \Lambda(\sigma_0)$ and

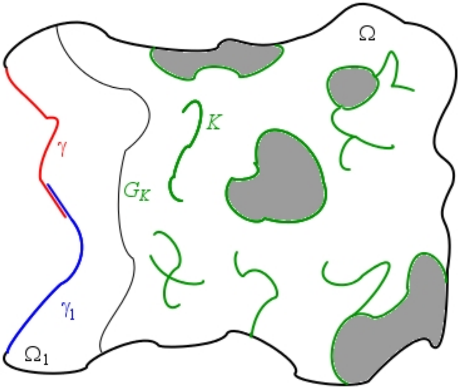
$$R(\overline{\sigma}) = \min\{R(\sigma) : \Lambda(\sigma) = \Lambda_0\}$$

Remark: for the case $N = 2$, it is crucial the uniqueness of the inverse problem guaranteed by Astala & Päivärinta (2006)

The inverse crack or cavity problem

A variational approach

Geometry of the forward problem



$\Omega \subset \mathbb{R}^N$ $(N \geq 2)$
bounded domain; $\partial\Omega$ Lipschitz
K defect
G_K partially known domain
(connected component of
$\Omega \backslash K$ containing Ω_1)

K may consist simultaneously of (interior or surface-breaking) cracks, cavities, material losses at the boundary and other kinds of defects

The forward problem

K is a perfectly insulating defect in a (homogeneous and isotropic) conducting body Ω

$f \in L^s(\partial\Omega)$, $s > N-1$, is the prescribed current density on $\partial\Omega$ such that

$$\operatorname{supp}(f) \subset \gamma_1 \quad \text{and} \quad \int_{\gamma_1} f = 0$$

$u = u(f, K)$ is the electrostatic potential in Ω, (unique) solution to

$$(2) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \backslash K \\ \nabla u \cdot \nu = 0 & \text{on } \partial(\Omega \backslash K) \backslash \gamma_1 \\ \nabla u \cdot \nu = f & \text{on } \gamma_1 \\ \int_\gamma u = 0 & \end{cases}$$

with the following normalization

$$(3) \qquad u = 0 \quad \text{in } \Omega \backslash G_K$$

The inverse crack or cavity problem

K unknown perfectly insulating defect in Ω

f prescribed current density on γ_1

u electrostatic potential in Ω, solution to (2) with the normalization (3)

Measurement: $g = u|_\gamma$ is the measured voltage on γ

Remark: $g \in L^2(\gamma)$ such that

$$\int_\gamma g = \int_\gamma u = 0$$

Aim of the inverse problem

From one or more voltage and current measurements (prescribing the current f on γ_1 and measuring the corresponding voltage $g = u|_\gamma$ on γ), reconstruct the unknown defect K (or better ∂G_K)

uniqueness for cracks in 2D

Friedman & Vogelius (1989) — Bryan & Vogelius (1992)
Alessandrini & Diaz Valenzuela (1996) — Kim & Seo (1996)

uniqueness for surface-breaking cracks in 2D

Elcrat, Isakov & Neculoiu (1996) — Andrieux, Ben Abda & Jaoua (1998)

uniqueness for cavities and material losses at the boundary in 2D

Andrieux, Ben Abda & Jaoua (1993)

uniqueness for cracks in 3D

Alessandrini & Di Benedetto (1997) (planar cracks)
Eller (1996) (infinitely many measurements)

Solving the inverse problem: a two-steps procedure

Step 1: continuation from the Cauchy data

Cauchy data (g, f) $\longrightarrow$ electrostatic potential u in Ω

Step 2: determination of the defect from the electrostatic potential

electrostatic potential u in Ω $\longrightarrow$ defect K

Solving the inverse problem: a two-steps procedure

Step 1: continuation from the Cauchy data

Cauchy data (g, f) $\longrightarrow$ electrostatic potential u in Ω

Step 2: determination of the defect from the electrostatic potential

electrostatic potential u in Ω $\longrightarrow$ defect K

Determination of (a part of) K from features of the potential u (for example level sets, critical points, singularities, ...)

perfectly insulating defect K

$$\Downarrow$$

$J(u) \subset \partial G_K$ $\quad$ $J(u)$ jump set of u

Solving the inverse problem: a two-steps procedure

Step 1: continuation from the Cauchy data

Cauchy data (g, f) $\longrightarrow$ electrostatic potential u in Ω

Step 2: determination of the defect from the electrostatic potential

electrostatic potential u in Ω $\longrightarrow$ defect K

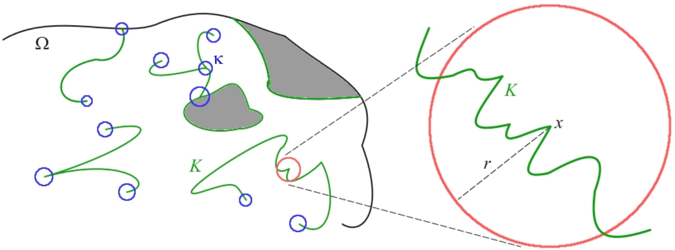
Step 1 by variational methods!

Admissible defects

K is an admissible defect if

$$K \subset \overline{\Omega} \text{ compact}, \quad \operatorname{dist}(K, \overline{\Omega}_1) \geq \rho, \quad \mathcal{H}^{N-1}(\partial G_K) < +\infty$$

and it is Lipschitz if, up to a closed set κ with $\mathcal{H}^{N-2}(\kappa) < +\infty$, ∂G_K is locally the graph of a Lipschitz function



Remark: K admissible defect $\Rightarrow$ $u = u(f, K) \in SBV(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \leq C_1 \|f\|_{L^s(\gamma_1)} \qquad \mathcal{H}^{N-1}(J(u) \backslash \partial G_K) = 0$$

The uniqueness result (Rondi 2006)

Uniqueness Theorem

Let K_1 be a Lipschitz admissible defect and K_2 be an admissible defect

Let $u_1 = u(f, K_1)$ and $u_2 = u(f, K_2)$, and let $g_1 = u_1|_\gamma$ and $g_2 = u_2|_\gamma$

If $g_1 = g_2$, then

$$u_1 = u_2 \text{ almost everywhere in } \Omega$$

Corollary

Under the same assumptions, if $g_1 = g_2$, then

$$|G_{K_1} \triangle G_{K_2}| = 0$$

A defect K is a material loss if G_K is the interior of its own closure

Remark: A material loss may contain cavities and material losses at the boundary but not cracks

Theorem

Under the same assumptions of the uniqueness theorem, let K_1 and K_2 be two material losses
If $g_1 = g_2$, then $G_{K_1} = G_{K_2}$

Remark: a single measurement may not be enough to uniquely determine a crack

Reconstruction for the cavities case: unknowns and data

Let K_0 be an unknown material loss, for example $K_0 = \partial\sigma_0$, $\sigma_0 = \bigcup_{i=1}^{n} D_i^0$

$D_i^0 \Subset \Omega$ $\quad$ D_i^0 pairwise disjoint unknown cavities, regular enough

Let $u_0 = u(f_0, K_0)$ and let $g_0 = u_0|_\gamma$

Unknowns: the material loss K_0 and the harmonic function u_0

Exact Cauchy data: Cauchy data (g_0, f_0)

Available data: noisy Cauchy data $(g_\varepsilon, f_\varepsilon)$ $(0 < \varepsilon \leq 1)$ such that

$$f_\varepsilon \in L^s(\gamma_1),\ \int_{\gamma_1} f_\varepsilon = 0 \qquad g_\varepsilon \in L^2(\gamma),\ \int_\gamma g_\varepsilon = 0$$

$$\|f_\varepsilon - f_0\|_{L^s(\gamma_1)} \leq \varepsilon \quad \text{and} \quad \|g_\varepsilon - g_0\|_{L^2(\gamma)} \leq \varepsilon$$

Aim

From the noisy Cauchy data, reconstruct u_0 and its jump set $J(u_0)$

Reconstruction: least-squares formulation

We look for $K = \partial\sigma$ solving, in a suitable class of admissible cavities, the minimization problem

$$\min_{K=\partial\sigma} \int_{\gamma} |u(f_\varepsilon, K) - g_\varepsilon|^2$$

$\sigma = \bigcup_i D_i$ $\qquad D_i \Subset \Omega$ $\qquad D_i$ pairwise disjoint cavities

Reconstruction: least-squares formulation

We look for $K = \partial\sigma$ solving, in a suitable class of admissible cavities, the minimization problem

$$\min_{K=\partial\sigma} \int_{\gamma} |u(f_\varepsilon, K) - g_\varepsilon|^2$$

$\sigma = \bigcup_i D_i \qquad D_i \Subset \Omega \qquad D_i$ pairwise disjoint cavities

Main difficulties

- Numerically handling functionals depending on the shape of a set

Reconstruction: least-squares formulation

We look for $K = \partial\sigma$ solving, in a suitable class of admissible cavities, the minimization problem

$$\min_{K=\partial\sigma} \int_{\gamma} |u(f_{\varepsilon}, K) - g_{\varepsilon}|^2$$

$\sigma = \bigcup_i D_i \qquad D_i \Subset \Omega \qquad D_i$ pairwise disjoint cavities

Main difficulties

- Numerically handling functionals depending on the shape of a set

$\rightsquigarrow$ phase-field functions approach

Reconstruction: least-squares formulation

We look for $K = \partial\sigma$ solving, in a suitable class of admissible cavities, the minimization problem

$$\min_{K=\partial\sigma} \int_{\gamma} |u(f_\varepsilon, K) - g_\varepsilon|^2$$

$\sigma = \bigcup_i D_i$ $\quad D_i \Subset \Omega$ $\quad D_i$ pairwise disjoint cavities

Main difficulties

- Numerically handling functionals depending on the shape of a set

  $\rightsquigarrow$ phase-field functions approach

- Instability

Reconstruction: least-squares formulation

We look for $K = \partial\sigma$ solving, in a suitable class of admissible cavities, the minimization problem

$$\min_{K=\partial\sigma} \int_\gamma |u(f_\varepsilon, K) - g_\varepsilon|^2 + \sum_i P(D_i)$$

$\sigma = \bigcup_i D_i$ $\quad$ $D_i \Subset \Omega$ $\quad$ D_i pairwise disjoint cavities

Main difficulties

- Numerically handling functionals depending on the shape of a set

  $\rightsquigarrow$ phase-field functions approach

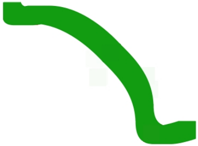
- Instability

  $\rightsquigarrow$ regularization by perimeter penalization

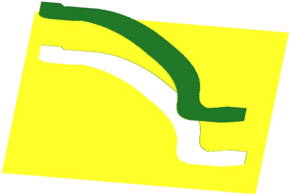
Phase-field functions approach: from the set to the characteristic function

$$u = u(f_\varepsilon, K), \qquad K = \partial\sigma, \qquad \sigma \Subset \Omega$$

$$K = \partial\sigma \quad \begin{cases} \Delta u = 0 & \text{in } \Omega\backslash\sigma \\ \nabla u \cdot \nu = 0 & \text{on } \partial\sigma \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_\gamma u = 0 & \end{cases}$$

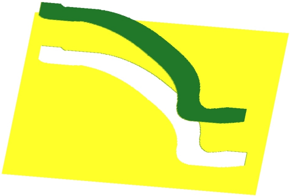


$$\chi_\sigma \quad \begin{cases} \mathrm{div}((1-\chi_\sigma)\nabla u) = 0 & \text{in } \Omega \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_\gamma u = 0 & \end{cases}$$

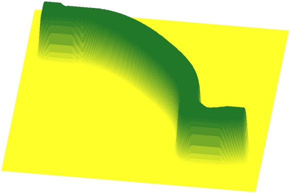


Phase-field functions approach: from the characteristic function to the phase-field function

χ_σ
$$\begin{cases} \mathrm{div}((1-\chi_\sigma)\nabla u) = 0 & \text{in } \Omega \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_\gamma u = 0 \end{cases}$$



$0 \leq \tilde{v} \leq 1$
$$\begin{cases} \mathrm{div}((1-\tilde{v})\nabla u) = 0 & \text{in } \Omega \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_\gamma u = 0 \end{cases}$$



phase-field function $\tilde{v}$ $\longrightarrow$ $v = 1 - \tilde{v}$ $\longrightarrow$

$\longrightarrow$ $v_\varepsilon = (1-\varepsilon^2)\psi(v) + \varepsilon^2$ where $\psi(t) = -2t^3 + 3t^2$

The least-squares problem for phase-field functions

Given a phase-field function $\tilde{v}$, determine $u = u(f_\varepsilon, v_\varepsilon)$ solution to

$$\begin{cases} \operatorname{div}(v_\varepsilon \nabla u) = 0 & \text{in } \Omega \\ \nabla u \cdot \nu = f_\varepsilon & \text{on } \partial\Omega \\ \int_\gamma u = 0 & \end{cases}$$

We look for a phase-field function $\tilde{v} \in H^1(\Omega, [0,1])$ solving

$$\min_{\tilde{v}} \int_\gamma |u(f_\varepsilon, v_\varepsilon) - g_\varepsilon|^2 + \text{regularization}$$

Perimeter penalization $\rightsquigarrow$ Modica-Mortola

Let $W : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous, nonnegative function such that $W(t) = 0$ if and only if $t \in \{0, 1\}$, $\beta > 0$ small enough

We look for a phase-field function $\tilde{v} \in H^1(\Omega, [0, 1])$ solving

$$\min_{\tilde{v}} \mathcal{F}_\varepsilon(v)$$

where

$$\mathcal{F}_\varepsilon(v) = \frac{1}{\varepsilon^\beta} \int_\gamma |u(f_\varepsilon, v_\varepsilon) - g_\varepsilon|^2 + \int_\Omega \left(v_\varepsilon |\nabla u(f_\varepsilon, v_\varepsilon)|^2 + \frac{W(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right)$$

Convergence result (Rondi 2009)

Let $\tilde{v}(\varepsilon)$, $\varepsilon > 0$, satisfy

$$\mathcal{F}_{\varepsilon}(v(\varepsilon)) \leq C$$

where $v(\varepsilon) = 1 - \tilde{v}(\varepsilon)$

Then, under a technical assumption on $\tilde{v}(\varepsilon)$, as $\varepsilon \longrightarrow 0^{+}$ we have

- $v(\varepsilon)_{\varepsilon} u(f_{\varepsilon}, v(\varepsilon)_{\varepsilon}) \longrightarrow u_0$ in $L^p(\Omega)$ for any $1 \leq p < +\infty$
- $v(\varepsilon)_{\varepsilon} \nabla u(f_{\varepsilon}, v(\varepsilon)_{\varepsilon}) \longrightarrow \nabla u_0$ weakly in $L^2(\Omega)$
- $\overline{\{\tilde{v}(\varepsilon) > 1 - a\}} \longrightarrow \sigma_0$ in the Hausdorff distance $(0 < a < 1)$