UNIVERSITY OF MANCHESTER
School of Mathematics

Reconstructing thin shapes by a level set technique

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joint with:

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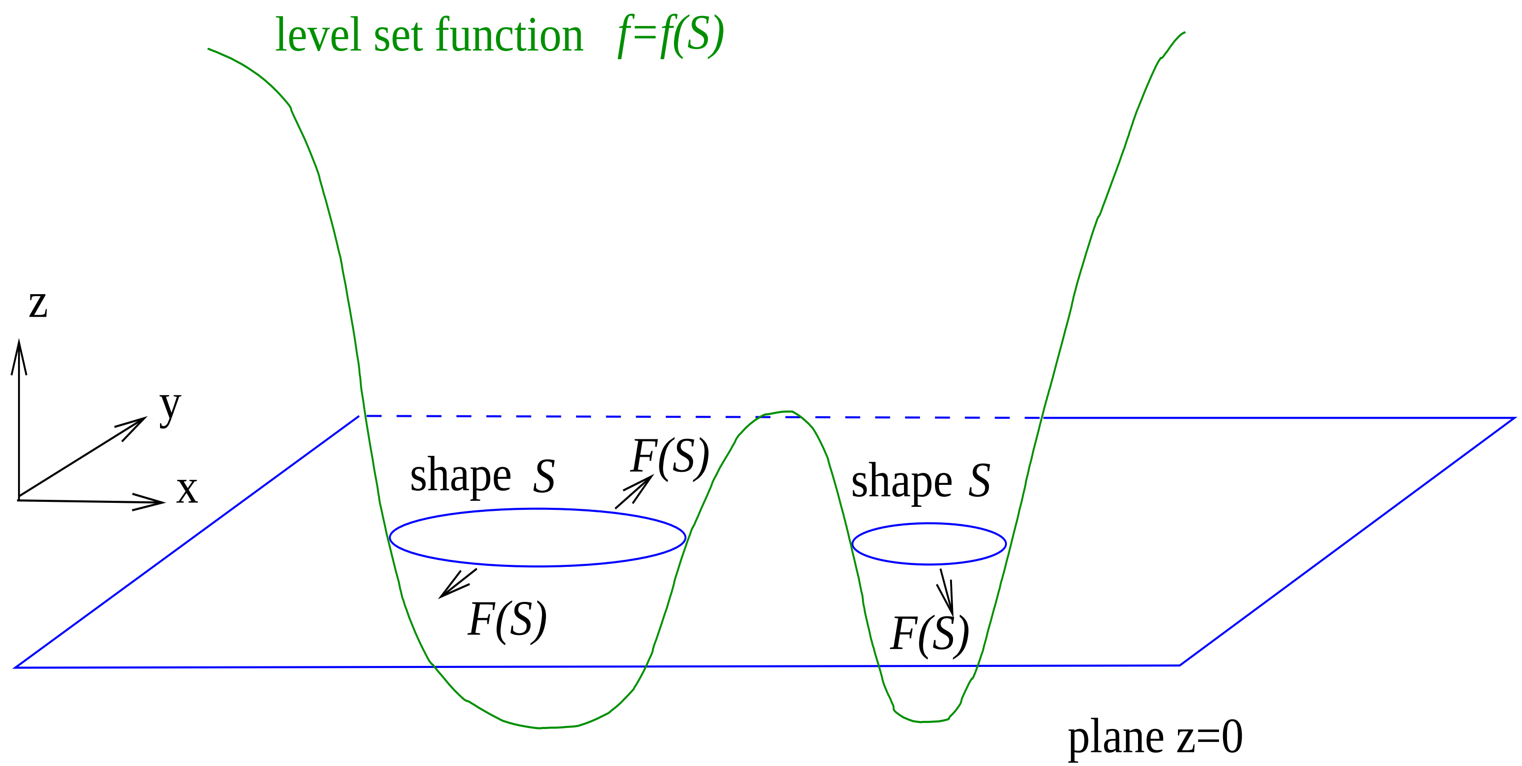
OUTLINE

New application for level sets: crack detection

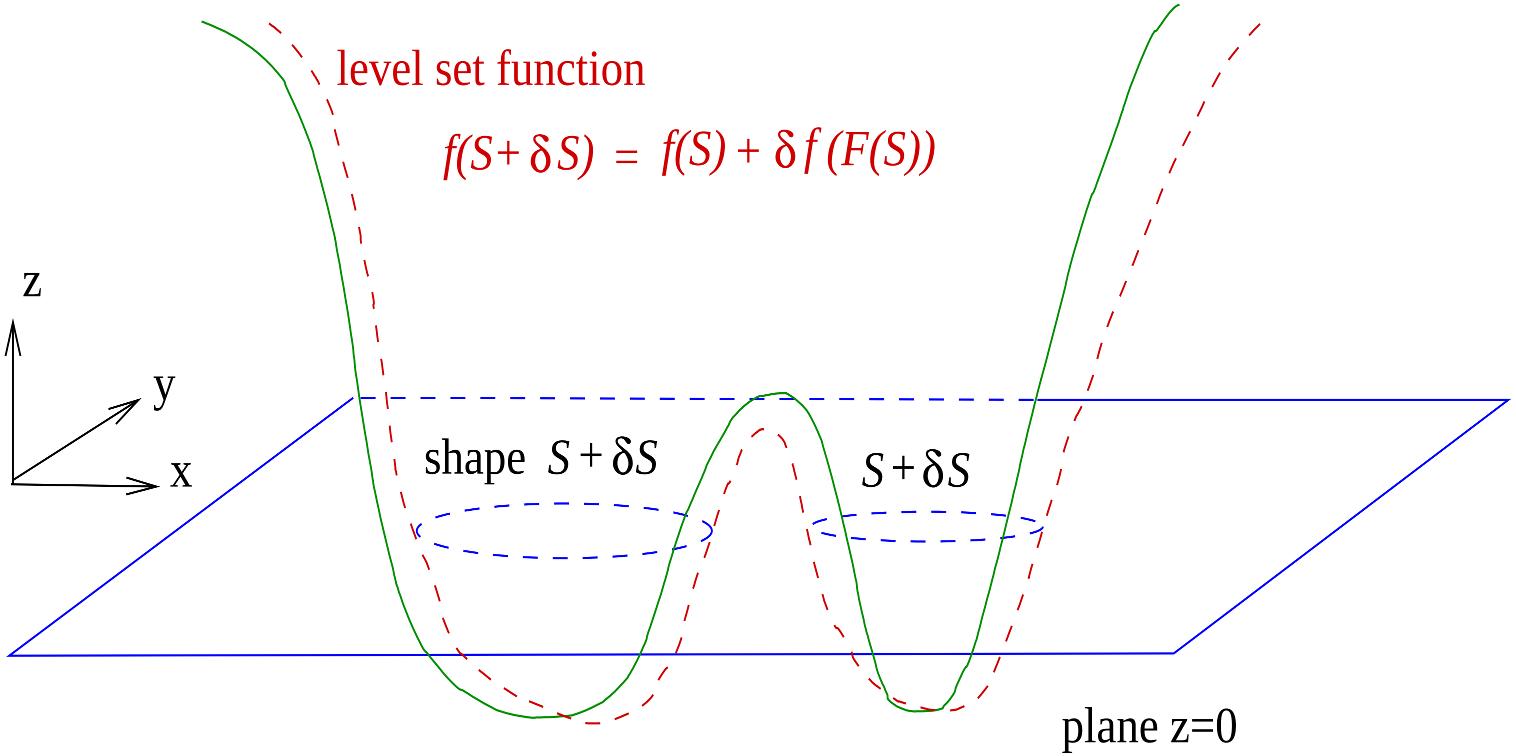
- Classical shape evolution
- Representing thin shapes by two level set functions
- Deforming thin shapes with level sets
- The numerical algorithm
- Numerical experiments
- Summary and future work

Joint work with Diego Álvarez and Miguel Moscoso, UC3M.

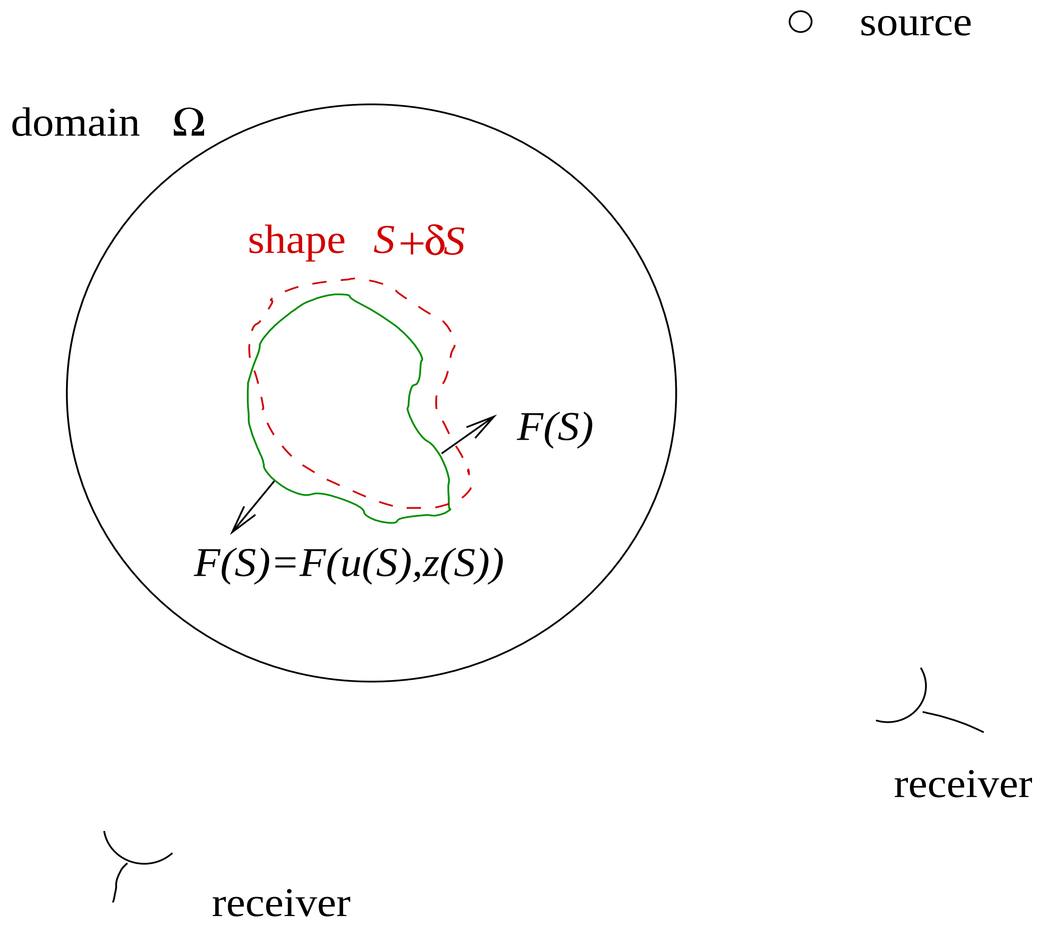
CLASSICAL SHAPE EVOLUTION



CLASSICAL SHAPE EVOLUTION



CLASSICAL SHAPE EVOLUTION

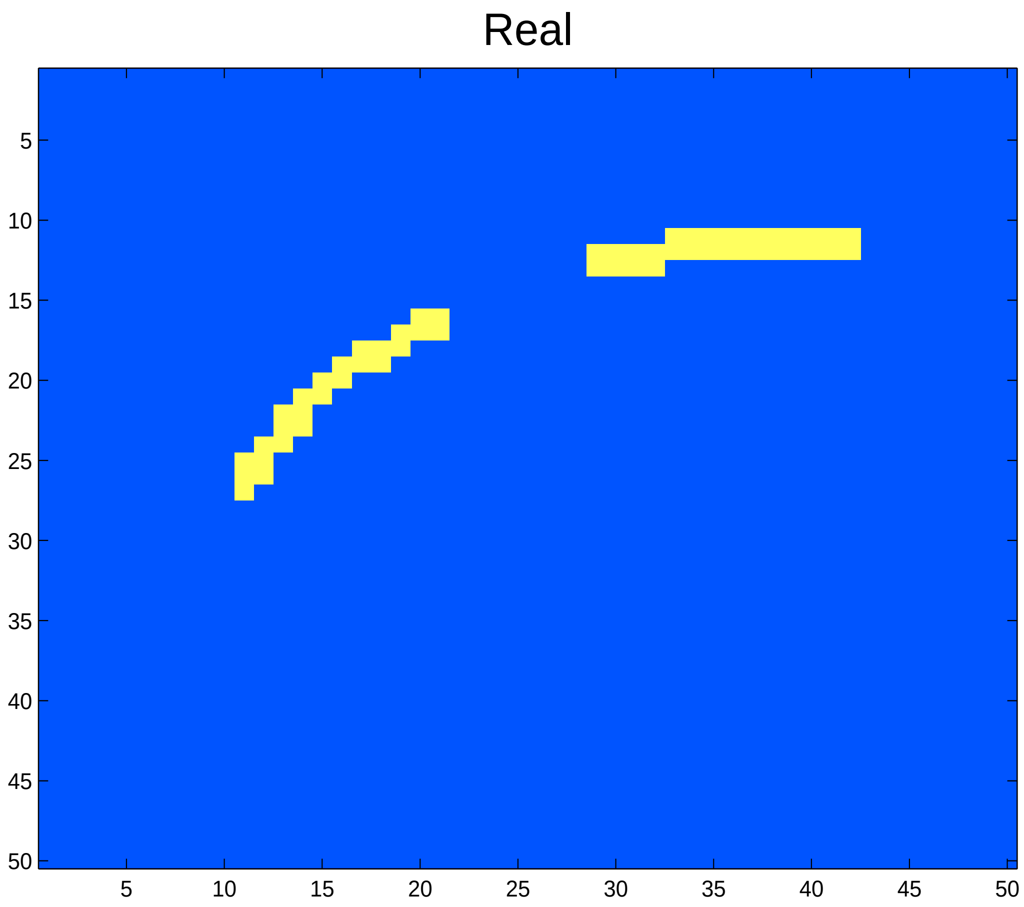


CLASSICAL SHAPE EVOLUTION

Level set approach: In order to evolve the level set function f such that the zero level set follows the flow F, we need to numerically solve the following **Hamilton-Jacobi type equation** (Osher and Sethian, 1988):

$$\frac{\partial f}{\partial t} + F|\nabla f| = 0.$$

CRACK-TYPE STRUCTURES

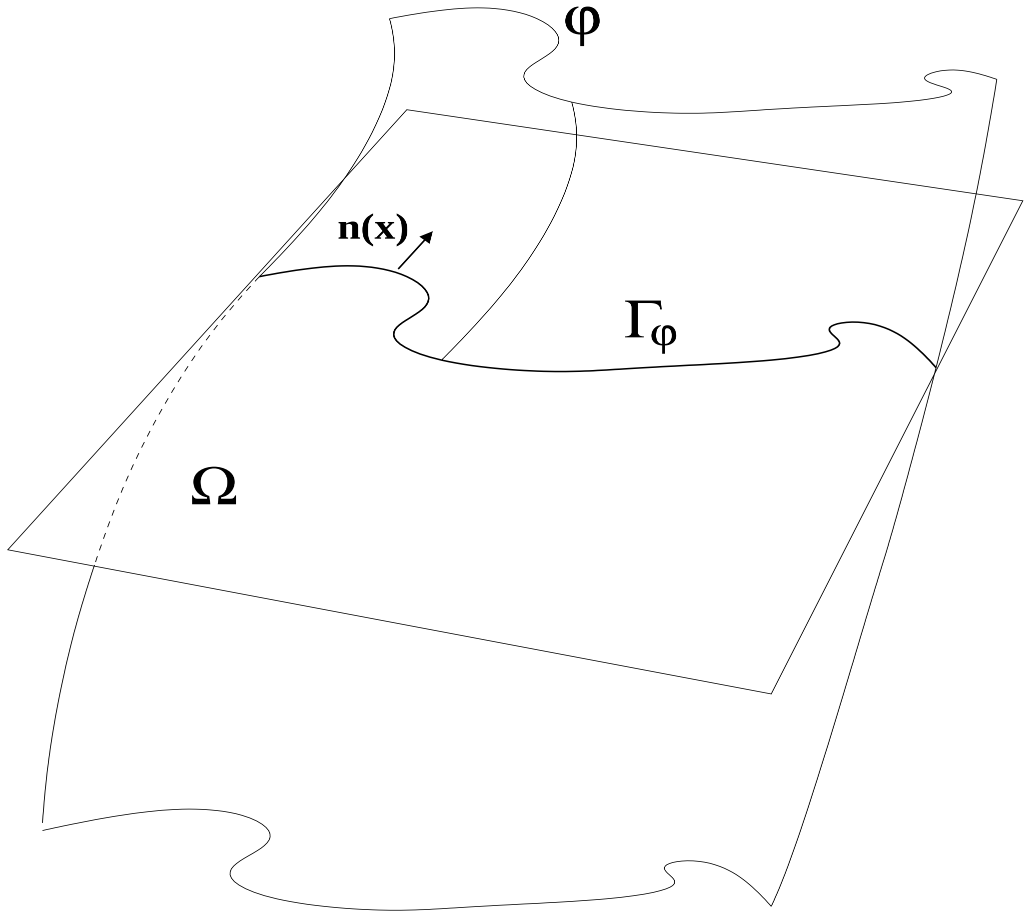


PROBLEMS OF CLASSICAL LEVEL SET TECHNIQUES

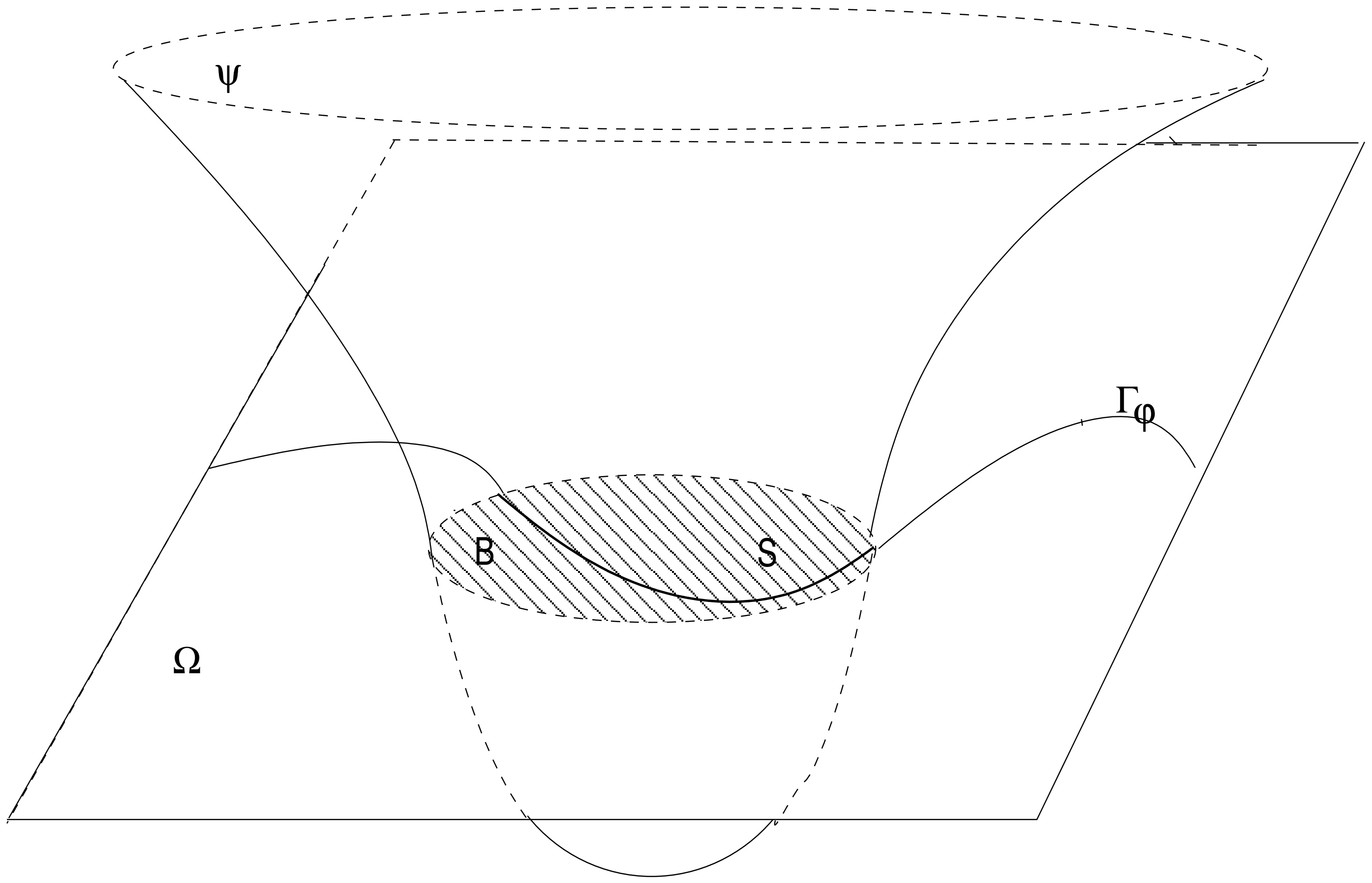
- Classical level set techniques describe ’volumetric’ shapes
- Cracks are ’thin’ shapes, ideally without volume
- Cracks are not ’closed’
- Cracks might be ’broken’, consisting of several pieces
- ’Crack evolution’: Propagating cracks vs. changing topology of cracks

Idea: use two level set functions for describing cracks (thin shapes)!

First level set function φ



Second level set function ψ



THE PHYSICAL PROBLEM

The electric potential u_j satisfies

$$\nabla \cdot b(\mathsf{x}) \nabla u_j = 0 \quad \text{in } \Omega, \tag{1}$$

and the boundary condition

$$u_j = \gamma_j \quad \text{on} \quad \partial\Omega. \tag{2}$$

The corresponding physical measurements are

$$\mathcal{A}_j(b) = g_j = \frac{\partial u_j}{\partial n}(d_l) \tag{3}$$

taken at positions $d_l \in \partial\Omega$ for $l = 1, \ldots, \underline{l}$.

Residual operator

$$\mathcal{R}_j(b) = \mathcal{A}_j(b) - \tilde{g}_j. \tag{4}$$

x

CLASSICAL GRADIENT DIRECTION

Least squares cost functional:

$$\mathcal{J}_j(b) = \frac{1}{2}\|\mathcal{R}_j(b)\|_Z^2 \tag{5}$$

Classical gradient direction

$$\mathrm{grad}_b \mathcal{J}_j(\mathbf{x}) = \mathcal{R}_j'(b)^* \mathcal{R}_j(b) \tag{6}$$

Adjoint formulation:

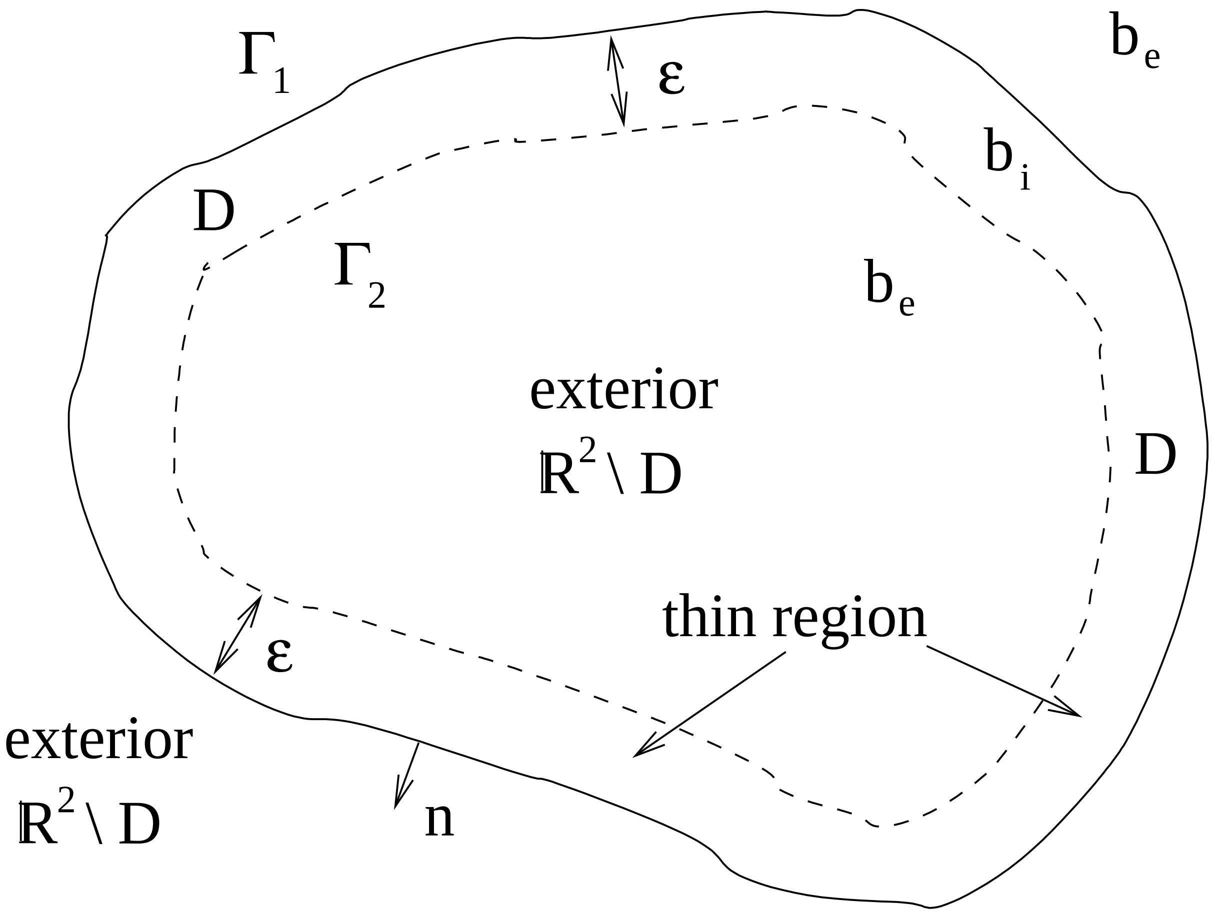
$$\mathrm{grad}_b \mathcal{J}_j(\mathbf{x}) = \nabla u_j \cdot \nabla z_j, \tag{7}$$

Adjoint equation: u_j solves (1)-(2) and z_j solves the following adjoint equation

$$\nabla \cdot b \nabla z_j = 0 \quad \text{in } \Omega, \tag{8}$$

$$b z_j = \mathcal{R}_j(b) \quad \text{on } \partial\Omega. \tag{9}$$

REPRESENTING THIN SHAPES (CRACKS)



REPRESENTING THIN SHAPES (CRACKS)

First level set function $\varphi(x,y)$:

$$\Omega_1 = \{(x,y) \in \mathbb{R}^2 \ : \ \varphi(x,y) \leq 0\}, \tag{10}$$

$$\Gamma_1 = \{(x,y) \in \mathbb{R}^2 \ : \ \varphi(x,y) = 0\}. \tag{11}$$

Γ_1 is the zero level set.

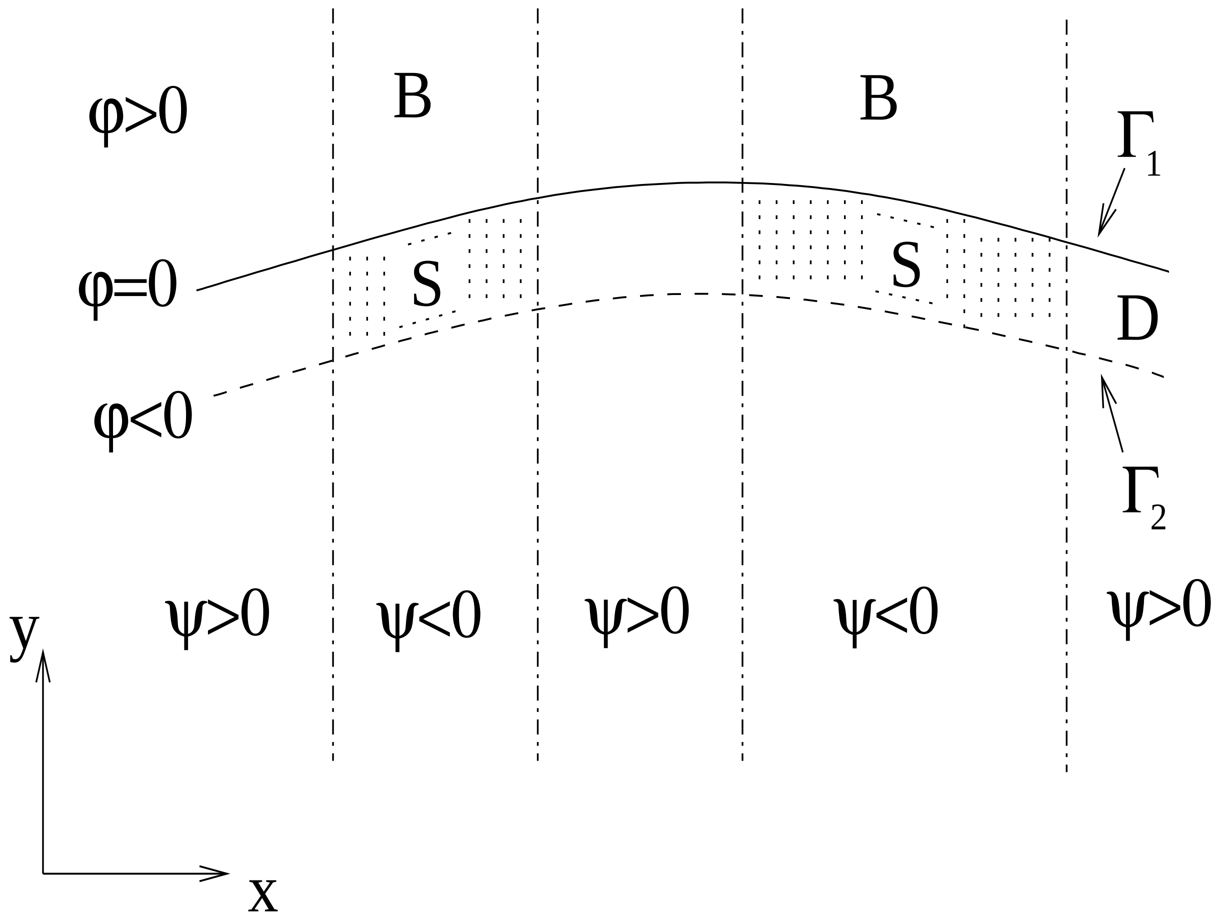
The outward normal n to Γ_1 is

$$\mathbf{n}(\mathbf{x}) = \frac{\nabla \varphi(\mathbf{x})}{|\nabla \varphi(\mathbf{x})|} \quad \text{on} \quad \Gamma_1. \tag{12}$$

Thin region D of width $\epsilon > 0$:

$$D = \Omega_1 \cap \{\mathbf{y} \in \mathbb{R}^2 \ : \quad \text{there exist} \quad \mathbf{x} \in \Gamma_1 \quad \text{such that} \quad \mathbf{y} = \mathbf{x} - \alpha \mathbf{n}(\mathbf{x}), \quad \text{for some} \quad 0 \leq \alpha \leq \epsilon \,\}. \tag{13}$$

REPRESENTING THIN SHAPES (CRACKS)



REPRESENTING THIN SHAPES (CRACKS)

Second level set function $\psi(x)$ defines pieces of the crack:

$$B = \{(x, y) \in \mathbb{R}^2 \; : \; \psi(x, y) < 0 \}. \tag{14}$$

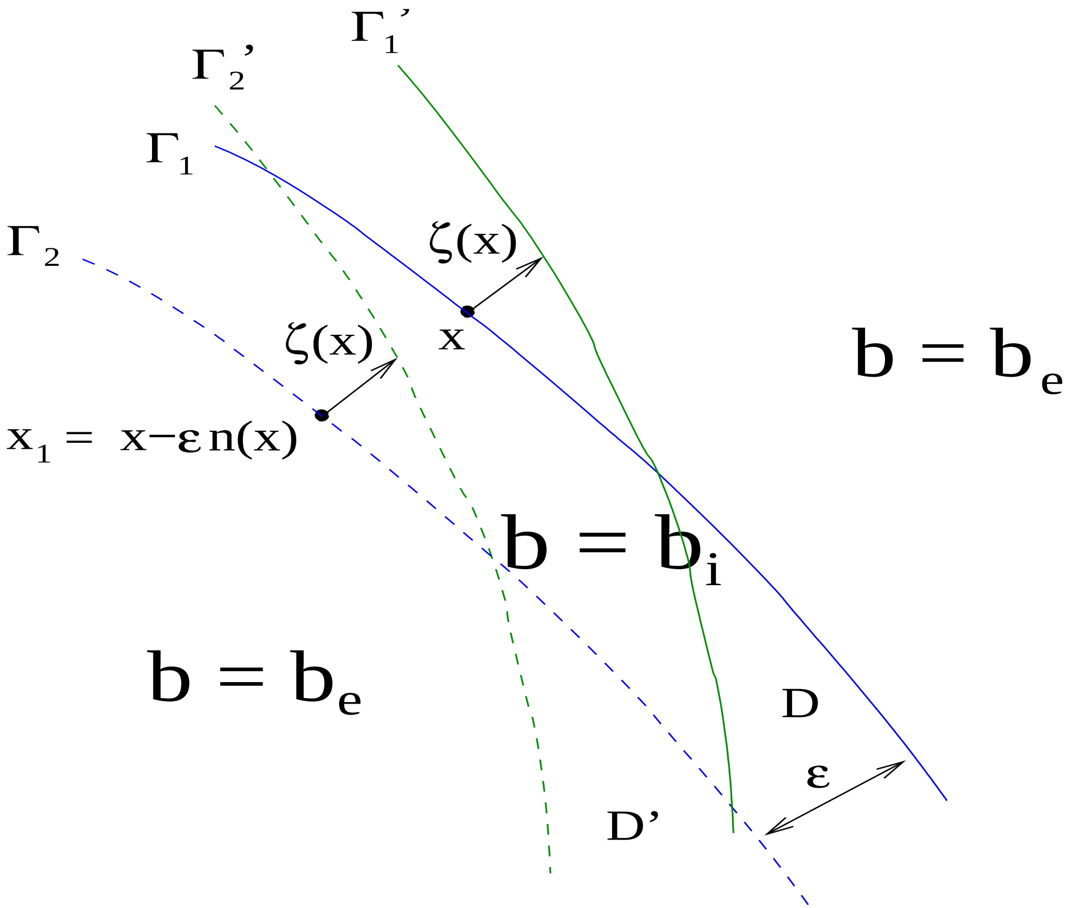
The thin shapes S are given as

$$S = (D \cap B) \cap \Omega. \tag{15}$$

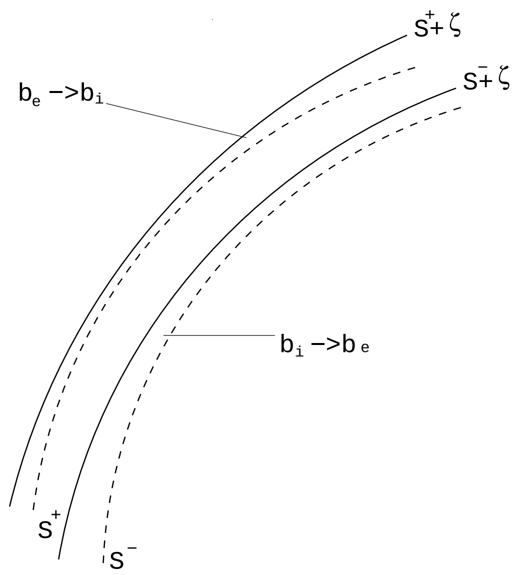
The conductivity distribution in Ω is given as

$$b(\mathrm{x}) = \begin{cases} b_i & \text{for} \quad \mathrm{x} \in S[\varphi, \psi], \\ b_e & \text{for} \quad \mathrm{x} \in \Omega \backslash S[\varphi, \psi]. \end{cases} \tag{16}$$

SHAPE EVOLUTION (CRACKS)



Propagating the first level set function



$$\delta_\varphi \mathcal{J}_j \approx \int_{S^+} (b_i - b_e)\mathrm{grad}_b \mathcal{J}_j(\mathbf{x})\zeta(\mathbf{x})\mathbf{n}(\mathbf{x})ds(\mathbf{x}) - \int_{S^-} (b_i - b_e)\mathrm{grad}_b \mathcal{J}_j(\mathbf{x})\zeta(\mathbf{x})\mathbf{n}(\mathbf{x})ds(\mathbf{x}) \; . \tag{17}$$

Propagating the first level set function

Descent direction for first level set function:

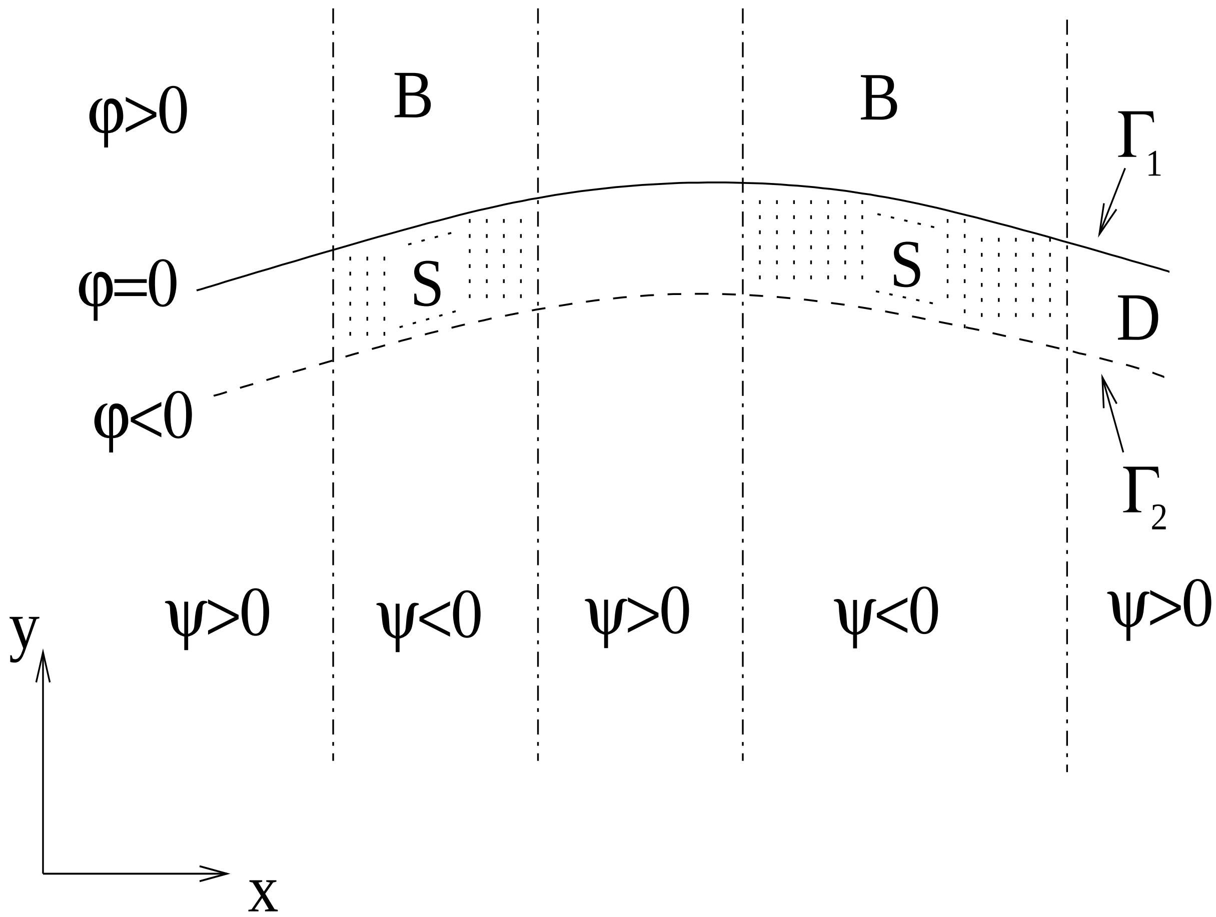
$$F_\varphi(\mathbf{x}) \;=\; -(b_i - b_e)\left[\mathrm{grad}_b \mathcal{J}_j(\mathbf{x}) - \mathrm{grad}_b \mathcal{J}_j(\mathbf{x} - \epsilon \mathbf{n}(\mathbf{x}))\right] \chi_B(\mathbf{x})$$

with $\mathbf{x} = (x, y)$.

Hamilton-Jacobi type formulation of shape evolution:

$$\frac{\partial \varphi}{\partial t} + F_\varphi |\nabla \varphi| \;=\; 0 \qquad (18)$$

Propagating the second level set function (example)



Propagating the second level set function (example)

Descent direction for second level set function:

$$F_\psi(\mathbf{x}) = -(b_i - b_e) \int \mathrm{grad}_b \mathcal{J}_j(\mathbf{x}) \delta_{\Gamma_B}(\mathbf{x}) \chi_D(\mathbf{x})\, dy$$

with $\mathbf{x} = (x, y)$.

Hamilton-Jacobi type formulation of shape evolution:

$$\frac{\partial \psi}{\partial t} + F_\psi |\nabla \psi| = 0 \qquad (19)$$

SHAPE EVOLUTION (CRACKS)

Iteration rule for $\varphi^{(n)}$ (moving the cracks):

$$\varphi^{(n+1)} = \varphi^{(n)} + \tau_\varphi F_\varphi^{(n)} |\nabla \varphi^{(n)}| \tag{20}$$

$$\varphi^{(0)} = \varphi_0. \tag{21}$$

Iteration rule for $\psi^{(n)}$ (breaking and merging cracks, changing lengths):

$$\psi^{(n+1)} = \psi^{(n)} + \tau_\psi F_\psi^{(n)} |\nabla \psi^{(n)}|, \tag{22}$$

$$\psi^{(0)} = \psi_0 \, . \tag{23}$$

Step n of the numerical algorithm

1. For each source γ_j, we calculate the residuals $\zeta_j = g_j^{(n)} - \widetilde{g}_j$ for the electrical conductivity $b^{(n)}(\mathrm{x})$.
2. We solve the forward and adjoint problem and calculate

$$\mathcal{R}_j'(b)^* \mathcal{R}_j(b) \;=\; \nabla u_j \cdot \nabla z_j \,.$$

3. We calculate $F_\varphi(\mathrm{x})$ and $F_\psi(x)$ as described before.
4. We choose appropriate extension velocities and apply some additional regularization.
5. We correct the level set functions $\varphi^{(n)}$ and $\psi^{(n)}$ according to the above derived iteration rules. The step-sizes are chosen empirically prior to starting the algorithm.
6. We determine the new parameter function $b^{(n+1)}(\mathrm{x})$.

Numerical examples

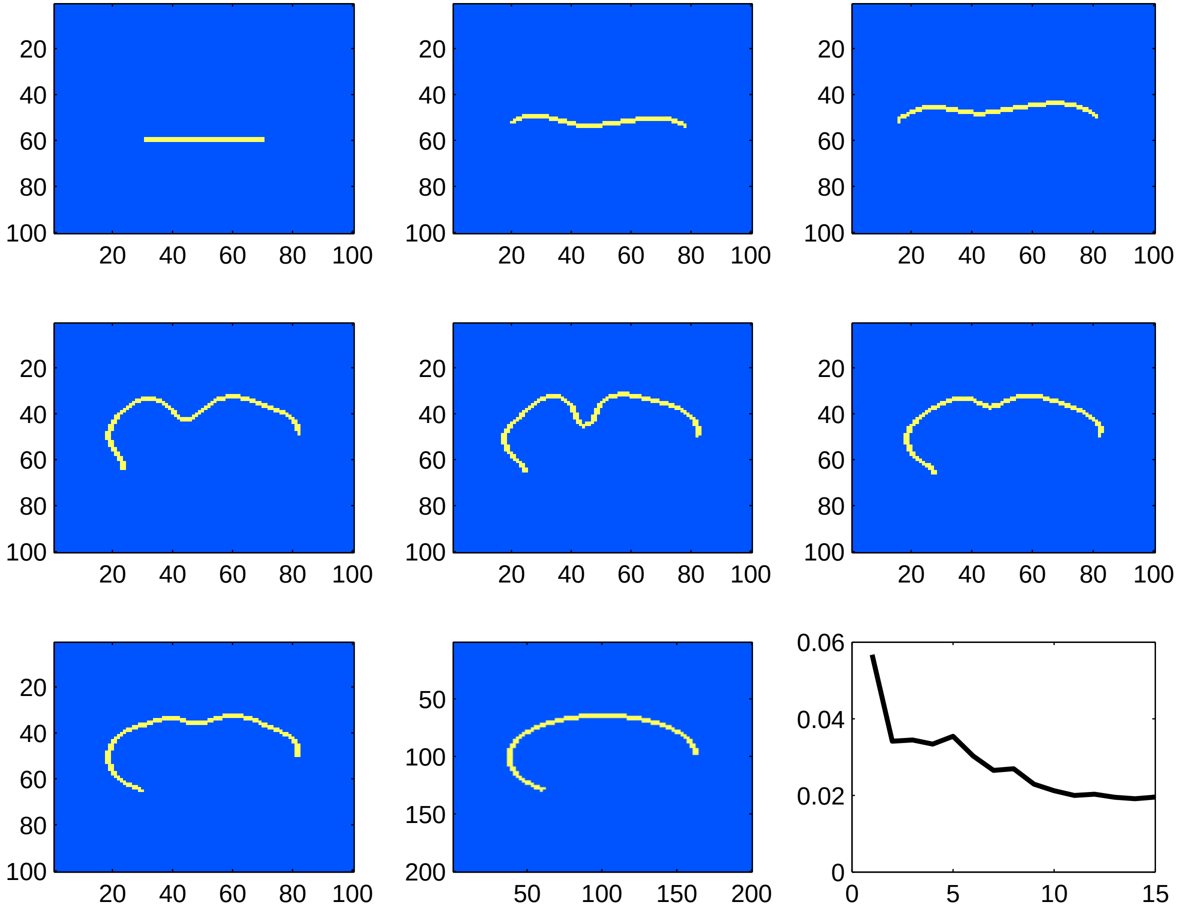


Figura 1: First numerical experiment: reconstruction of a single crack. Top row (from left to right): Initial profile, profile after **10** source activations, and profile after **20** source activations. Center row (from left to right): profiles after **144**, **252** and **324** source activations. Bottom row (from left to right): Reconstructed profile (after **540** source activations), true profile, and evolution of the cumulative cost $\boldsymbol{J_{loop}}$ versus number of loops.

Numerical examples

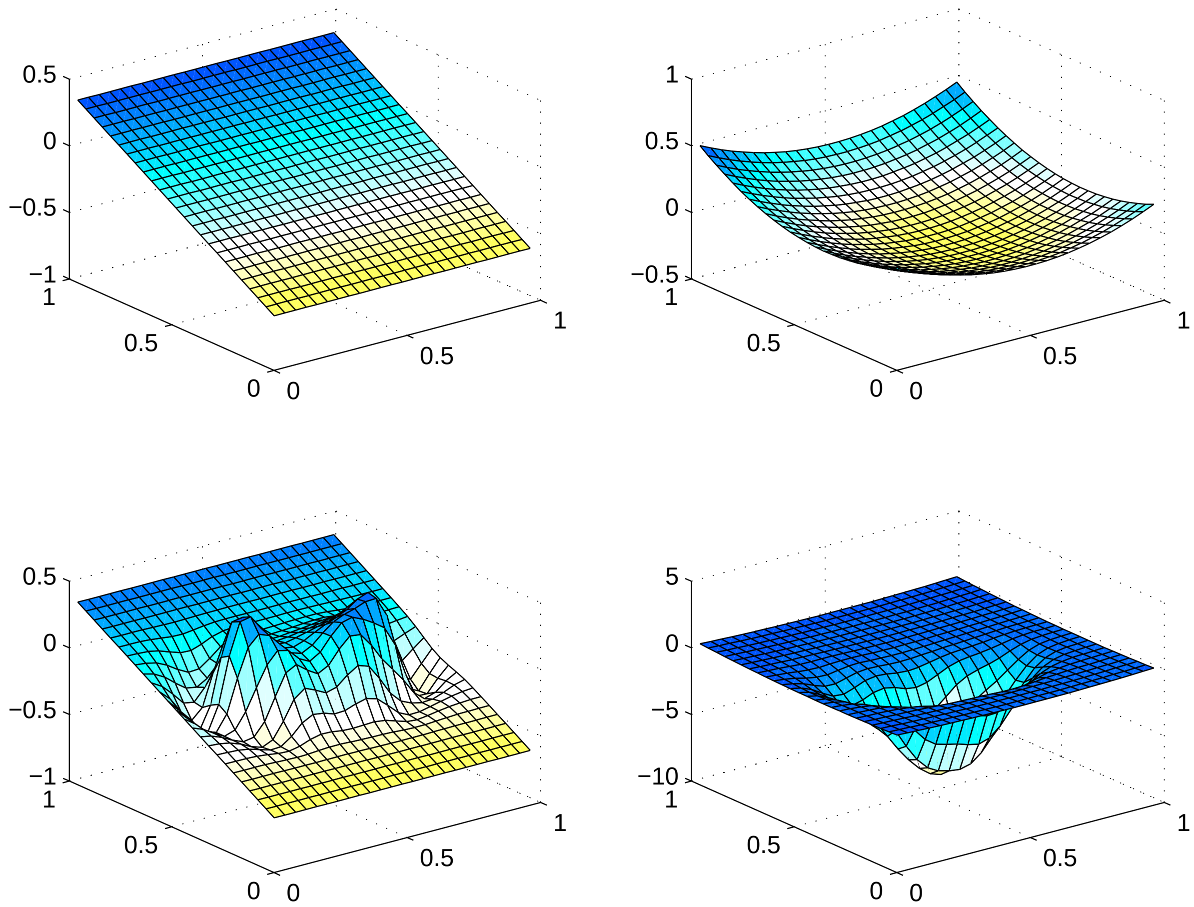


Figura 2: Initial and final level set function for the reconstruction of a single crack. Initial on top row: $\varphi^{(0)}(\mathbf{x})$ and $\psi^{(0)}(\mathbf{x})$. Final on bottom row: $\varphi^{(f)}(\mathbf{x})$ and $\psi^{(f)}(\mathbf{x})$

Numerical examples

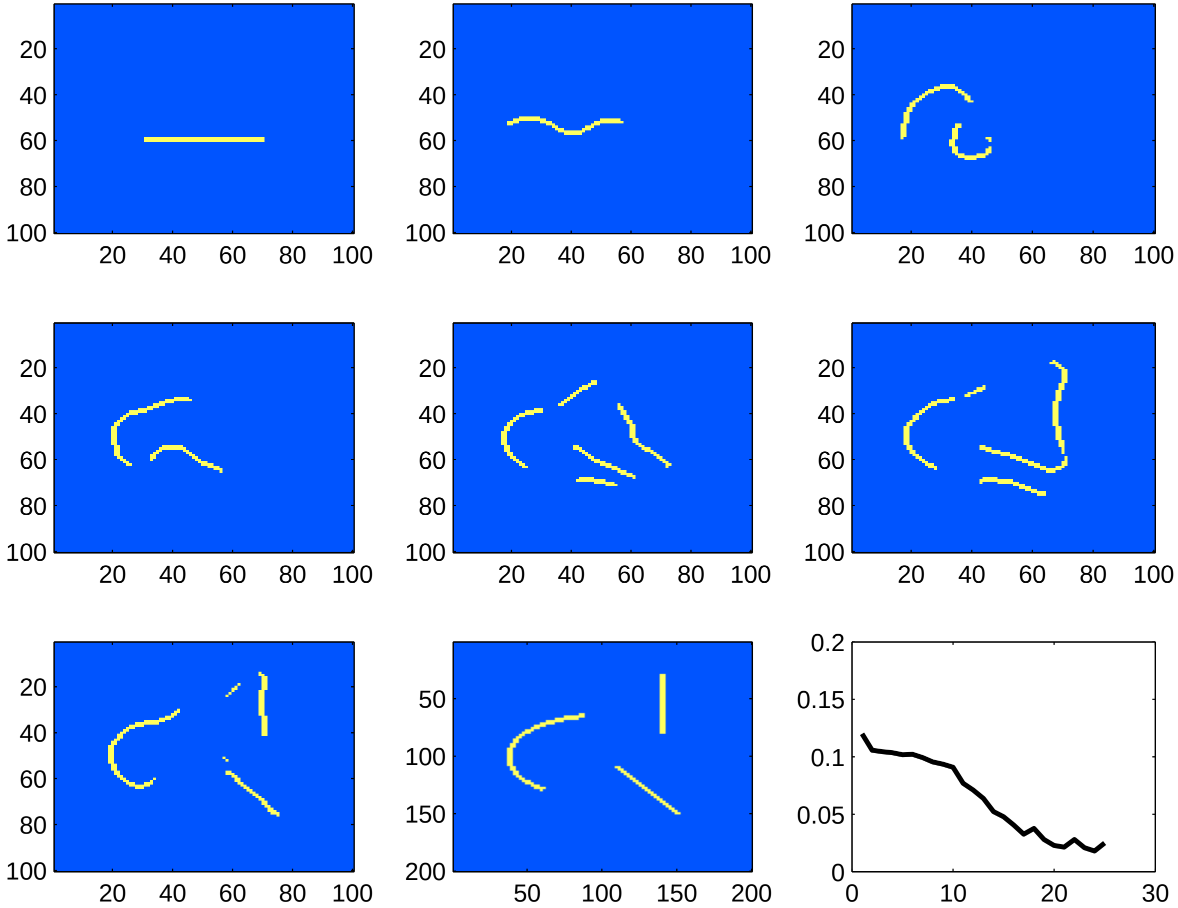


Figura 3: Second numerical experiment: reconstruction of three cracks. On the two first rows from left to right: initial guess, and reconstruction after 10,40,252,360,504 iterations. On the third row from left to right: final reconstruction (900 iteration), real crack and evolution of $\boldsymbol{J_{loop}}$ versus the number of loops.

Numerical examples

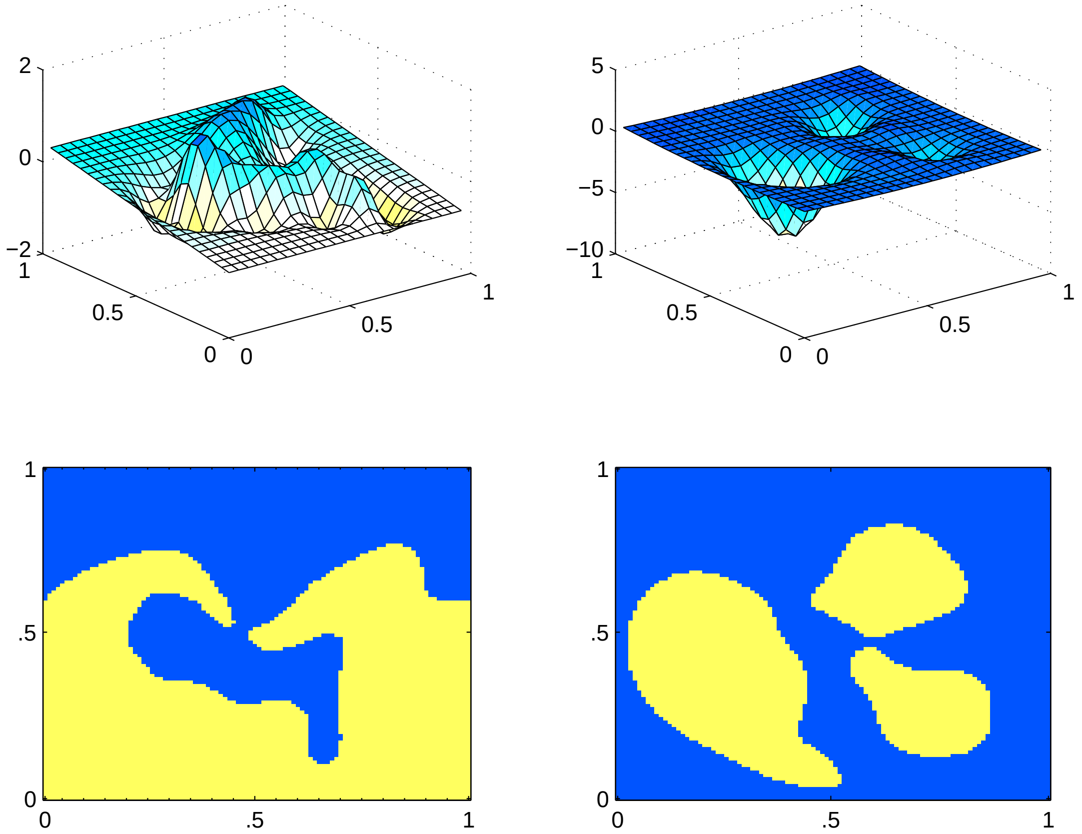


Figura 4: Final level set function, in the case of reconstructing three cracks. Left column: surface and contour plot of sign of $\varphi(\mathbf{x})$. Right column: surface and contour plot of sign of $\psi(\mathbf{x})$

Numerical examples

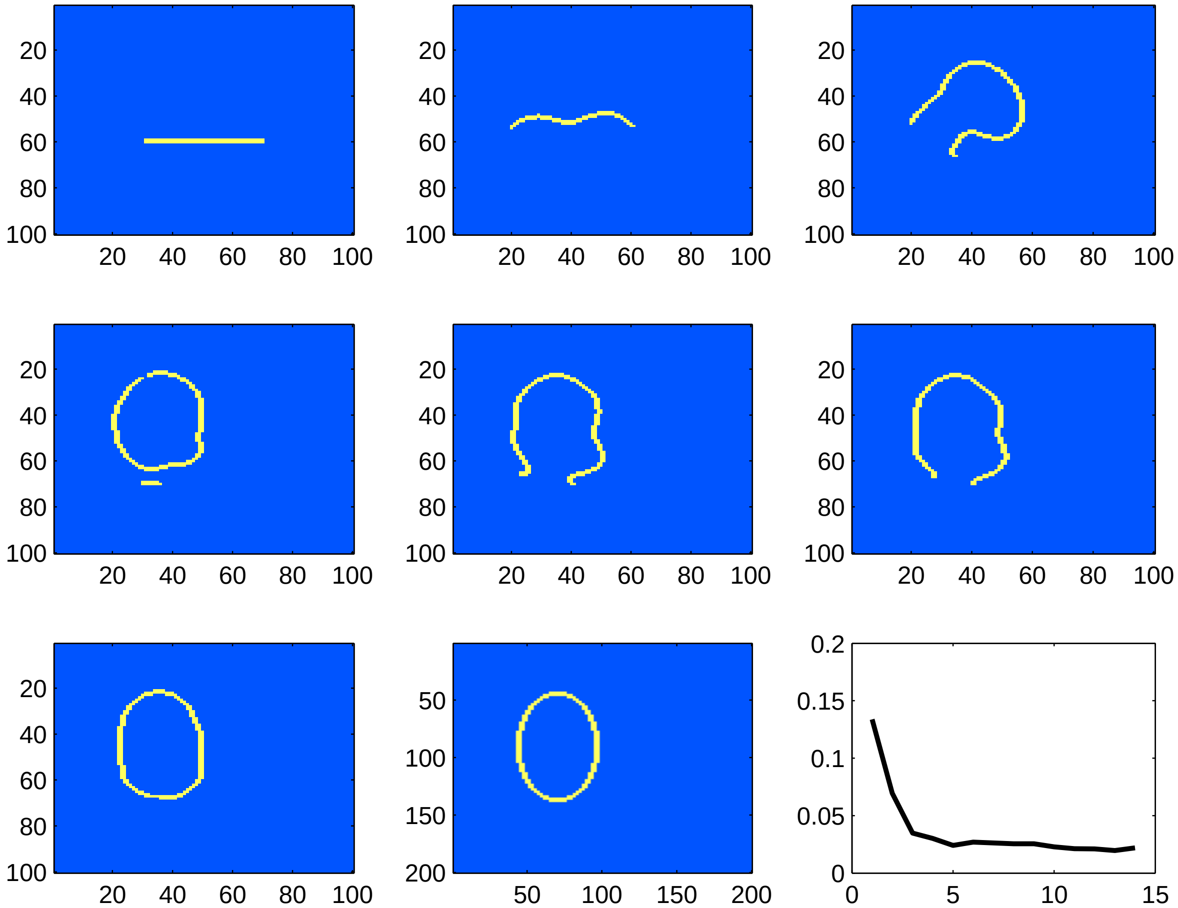


Figura 5: Thrid numerical experiment: Reconstructing a closed curve. On the two first rows from left to right: initial guess, reconstruction after 10,30,40,80,160 iterations. On the third row from left to right: final reconstruction (360 iteration), real crack and evolution of J_{loop} versus the number of loops.

Numerical examples

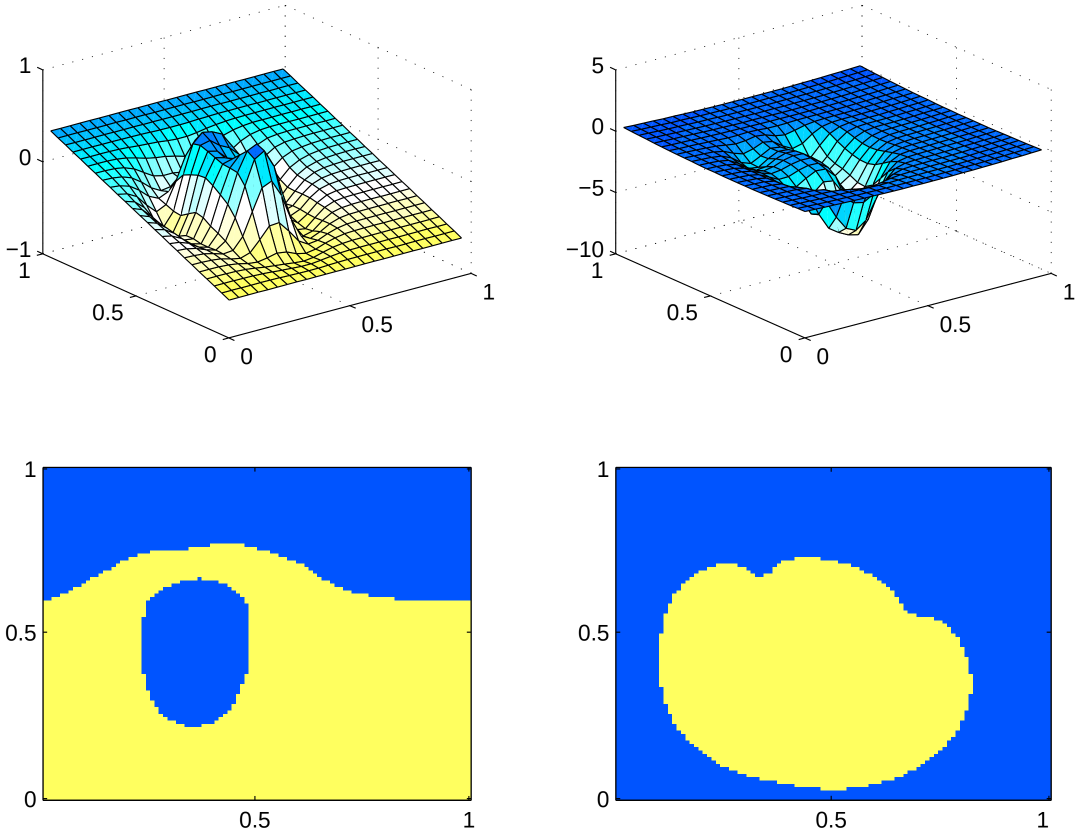


Figura 6: Final level set function, in the case of reconstructing three cracks. Left column: surface and contour plot of sign of $\varphi(\mathbf{x})$. Right column: surface and contour plot of sign of $\psi(\mathbf{x})$

Numerical examples

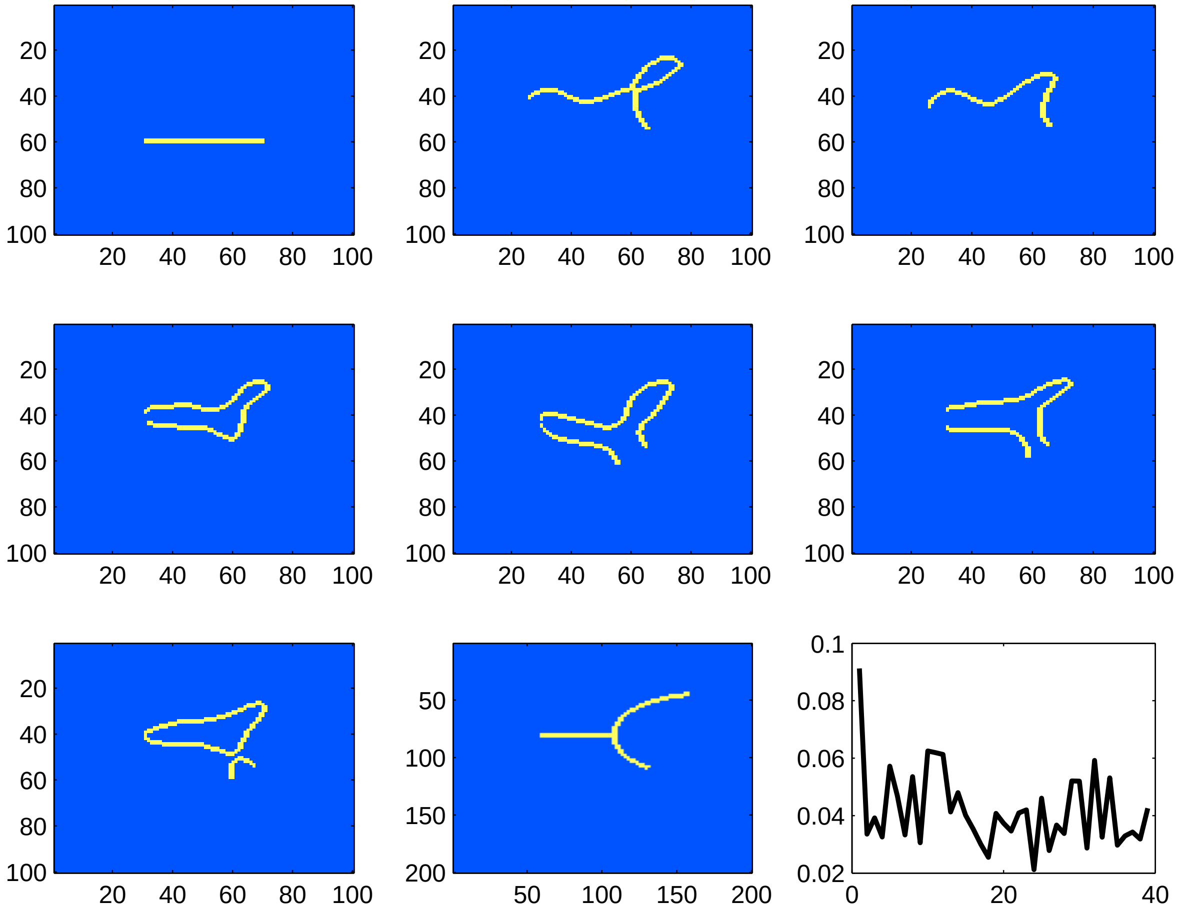


Figura 7: Forth numerical experiment: a pitchfork shape. On the two first rows from left to right: initial guess, reconstruction after 72,144,360,684,864 iterations. On the third row from left to right: final reconstruction (1552 iteration), real crack and evolution of J_{loop} versus the number of loops.

Numerical examples

- **Novel level set technique** for finding and characterizing **thin shapes (cracks)** from electrical boundary data.
- **Two level set functions** employed, one being responsible for the location and form of the thin shape, the other one for the length and connectivity
- **Adjoint formulation** for gradient calculation
- Applicable to the reconstruction of penetrable cracks
- Future extensions to the simultaneous recovery of interior parameter values and thickness of cracks seem possible.

Thanks!