Identification of generalized impedance boundary conditions in inverse scattering problems

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About the Generalized Impedance Boundary Conditions

- Context : scattering problems in the harmonic regime
- GIBCs : correspond to models involving small parameters

⟶ For example, perfect conductor coated with a layer for TE polarization (order 1),

$$\partial_\nu u + Zu = 0 \text{ on } \Gamma, \quad Z = \delta(\partial_{ss} + k^2 n),$$

with δ : width of the layer, s : curvilinear abscissa, k : wave number, n : mean value of the thin coating index along ν

We consider the following model of GIBC :

$$\partial_\nu u + \mu \Delta_\Gamma u + \lambda u = 0 \text{ on } \Gamma,$$

with μ : complex constant, λ : complex function.

Outline of the talk

Typical inverse problem : the obstacle being known, determine $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ from the far field $\boldsymbol{u^\infty}$ associated to one incident wave at fixed frequency

Nonlinear operator of interest : $\boldsymbol{T : (\lambda, \mu) \longrightarrow u^\infty}$

- The forward problem
- Uniqueness for the inverse problem
- Stability for the inverse problem
- Numerical experiments
- Perspectives

The forward problem

Obstacle $D \subset \mathbb{R}^3$, $\Omega := \mathbb{R}^3 \setminus \overline{D}$

Incident wave $u^i(x) = e^{ik\, d.x}$

Governing equations for $u^s = u - u^i$:

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u^s}{\partial \nu} + \mu \Delta_\Gamma u^s + \lambda u^s = f & \text{on} \quad \Gamma, \\ \lim\limits_{R \to +\infty} \displaystyle\int_{\partial B_R} |\partial u^s / \partial r - iku^s|^2 \, ds(x) = 0, \end{cases}$$

with

$$f := -\left(\frac{\partial u^i}{\partial \nu} + \mu \Delta_\Gamma u^i + \lambda u^i \right) |_\Gamma$$

The forward problem

- <u>**Classical impedance problem $\mu = 0$**</u>:

uniquely solvable in $V_{0R} = \{H^1(\Omega \cap B(0,R))\}$ provided $\lambda \in L^\infty(\Gamma)$ with $\mathrm{Im}(\lambda) \geq 0$

- <u>**Generalized impedance problem $\mu \neq 0$**</u>:

uniquely solvable in $V_R = \{v \in V_{0R},\ v|_\Gamma \in H^1(\Gamma)\}$ provided $\lambda \in L^\infty(\Gamma)$ with $\mathrm{Im}(\lambda) \geq 0$, $\mathrm{Re}(\mu) > 0$ and $\mathrm{Im}(\mu) \leq 0$.

<u>Remark</u> : $\Delta_\Gamma v$ is defined in $H^{-1}(\Gamma)$ by

$$\langle \Delta_\Gamma v, w \rangle_{H^{-1}(\Gamma), H^1(\Gamma)} = -\int_\Gamma \nabla_\Gamma v . \nabla_\Gamma w \, ds, \quad \forall w \in H^1(\Gamma)$$

Uniqueness for inverse problem (the obstacle is known)

- **Classical impedance problem $\mu = 0$** (Colton and Kirsch 81): uniqueness for piecewise continuous λ

Proof : assume $T(\lambda_1) = T(\lambda_2) = u^\infty$. Rellich Lemma + unique continuation $\Rightarrow u_1 = u_2$ in Ω, then $(u_1 - u_2)|_\Gamma = 0$ and $\partial_\nu (u_1 - u_2)|_\Gamma = 0$.

$$\partial_\nu u_1 + \lambda_1 u_1 = \partial_\nu u_1 + \lambda_2 u_1 = 0 \text{ on } \Gamma$$

Then $(\lambda_1 - \lambda_2) u_1 = 0$ on Γ. For $x_0 \in \Gamma$ not on a curve of discontinuity s.t. $(\lambda_1 - \lambda_2)(x_0) \neq 0$, then $|(\lambda_1 - \lambda_2)(x)| > 0$ on $B(x_0, \eta) \cap \Gamma$.

As a result $u_1 = 0$, $\partial_\nu u_1 = 0$ on $B(x_0, \eta) \cap \Gamma$, and unique continuation $\Rightarrow u_1 = 0$ in Ω. This contradicts the fact that u^i is a plane wave. Hence $\lambda_1(x) = \lambda_2(x)$ a.e. on Γ. ∎

Uniqueness for the inverse problem

- <u>**Generalized impedance problem $\mu \neq 0$**</u> : non uniqueness

A counterexample in 2D : $D = B(0,1)$, $d = (1,0)$, $k = 1$,
u_0 : solution of the classical impedance problem with $\lambda_0 = i$
$\alpha := \Delta_\Gamma u_0 / u_0$ is a smooth function on Γ

- $\mu_1 \neq \mu_2$ s.t.

$|\mu_i| \max_\Gamma |\alpha| \leq 1, \quad \mathrm{Re}(\mu_i) > 0, \, \mathrm{Im}(\mu_i) \leq 0$

- $\lambda_1 \neq \lambda_2$ s.t. $\lambda_i := \lambda_0 - \alpha \mu_i$ on Γ

$\longrightarrow$ We have on Γ:

$$\mathrm{Im}(\lambda_i) = \mathrm{Im}(\lambda_0) - \mathrm{Im}(\alpha \mu_i) \geq \mathrm{Im}(\lambda_0) - |\mu_i| \max_\Gamma |\alpha| \geq 0$$

$$\partial_\nu u_0 + \mu_i \Delta_\Gamma u_0 + \lambda_i u_0 = (-\lambda_0 + \alpha \mu_i + \lambda_i) u_0 = 0$$

As a result, $u_0^\infty = T(i,0)$ is the far field associated to the generalized impedance problem with both (λ_1, μ_1) and (λ_2, μ_2)

Uniqueness for the inverse problem

We can restore uniqueness with restrictions : two examples

- λ and μ two complex constants +

Geometric assumption : there exists $x_0 \in \Gamma$, $\eta > 0$ such that $\Gamma_0 := \Gamma \cap B(x_0, \eta)$ is portion of a plane, cylinder or sphere and $\{x + \gamma\nu(x),\ x \in \Gamma_0,\ \gamma > 0\} \subset \Omega$

- λ piecewise continuous, and μ complex constant : $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\mu)$ are fixed and known, the unknown being $\mathrm{Im}(\lambda)$ and $\mathrm{Re}(\mu)$ +

Geometric assumption : both D, λ are invariant by reflection against a plane which does not contain d or by a rotation around an axis which is not directed by d

- More general conditions in Bourgeois & Haddar (2009, submitted)

Uniqueness for the inverse problem

Second case : sketch of the proof

$$\partial_\nu u + \mu_1 \Delta_\Gamma u + \lambda_1 u = \partial_\nu u + \mu_2 \Delta_\Gamma u + \lambda_2 u = 0 \text{ on } \Gamma$$

If $\mu_1 \neq \mu_2$, then

$$\int_\Gamma |\nabla_\Gamma u|^2 \, ds = \frac{1}{\mu_2 - \mu_1} \int_\Gamma (\lambda_2 - \lambda_1)|u|^2 \, ds$$

Hyp. : $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\mu)$ are fixed and known

Then $(\lambda_2 - \lambda_1)/(\mu_2 - \mu_1) \in i\mathbb{R} \Rightarrow u = C$ on Γ, and $\lambda_1 = \lambda_2 = \lambda$.

$$u^s + u^i = C \quad \text{and} \quad \partial_\nu u^s + \partial u^i_\nu = -C\lambda \quad \text{on} \quad \Gamma$$

Uniqueness for the inverse problem

Second case : sketch of the proof (cont.):

Representation formulas for u^s and u^i on Γ :

$$\begin{cases} u^s(x)/2 = \mathcal{T}(u^s)(x) - \mathcal{S}(\partial_\nu u^s(x)) \\ u^i(x)/2 = -\mathcal{T}(u^i)(x) + \mathcal{S}(\partial_\nu u^i(x)) \end{cases}$$

with

$\mathcal{S} := \gamma^- \mathbf{SL} = \gamma^+ \mathbf{SL}, \quad \mathcal{T} = (\gamma^+ \mathbf{DL} + \gamma^- \mathbf{DL})/2$

(SL : single layer potential, DL : double layer potential)

We obtain

$$u^i(x) = \frac{C}{2}(1 - 2\mathcal{T}(1)(x) - 2\mathcal{S}(\lambda)(x)) \text{ on } \Gamma$$

This is forbidden by the geometric assumption. ■

Stability for the inverse problem

The classical impedance problem: many results in the litterature (Labreuche 99, Sincich 06, ...)

Some proprieties of operator $T : \lambda \in L^{\infty}_{+}(\Gamma) \to u^{\infty} \in L^2(S^2)$:

- Injective (piecewise continuous λ)
- Differentiable in the sense of Fréchet

$dT_{\lambda} : h \to v_h^{\infty}$ is defined by

$$v_h^{\infty}(\hat{x}) = \int_{\Gamma} p(y, \hat{x}) u(y, d) h(y)\, ds(y) \quad \forall \hat{x} \in S^2$$

where $p(., \hat{x})$ is the solution associated to $\Phi^{\infty}(., \hat{x})$.

- dT_{λ} injective (piecewise continuous λ)

$\Rightarrow$ Some simple Lipschitz stability results can be derived in compact subsets of finite dimensional spaces

Stability for the inverse problem

The generalized impedance problem:

Some proprieties of operator $T : (\lambda, \mu) \in V(\Gamma) \rightarrow u^\infty \in L^2(S^2)$:

- Injective
- Differentiable in the sense of Fréchet

$dT_{\lambda,\mu} : (h, l) \longrightarrow v^\infty_{h,l}$ is defined by

$$v^\infty_{h,l}(\hat{x}) = \langle p(.,\hat{x}), l\Delta_\Gamma u(.,d) + u(.,d)h \rangle_{H^1,H^{-1}} \quad \forall \hat{x} \in S^2$$

where $p(.,\hat{x})$ is the solution associated to $\Phi^\infty(.,\hat{x})$.

- $dT_{\lambda,\mu}$ injective

$\Rightarrow$ Some simple Lipschitz stability results can be derived in compact subsets of finite dimensional spaces

Numerical experiments in $2D$

- Minimize the cost function (classical impedance)

$$F(\lambda) = \frac{1}{2}||T(\lambda) - u^{\infty}_{\text{obs}}||^2_{L^2(S^1)}$$

- Artificial data u^{∞}_{obs} obtained with a Finite Element Method
- Projection of λ along the trace on Γ of the FE basis
- Computation of gradient (classical impedance): $h_1 = \text{Re}(h)$, $h_2 = \text{Im}(h)$

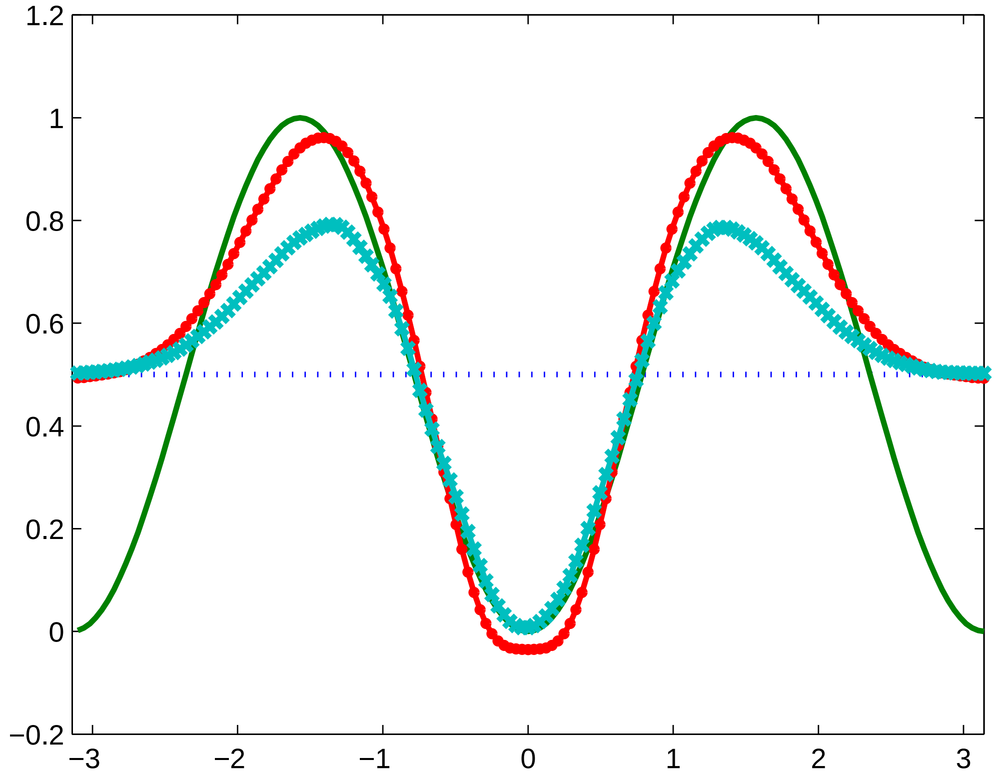
$$(dF(\lambda), h) = \text{Re} \int_{\Gamma} \{(h_1(y) + ih_2(y))u(y) \int_{S^1} p(y, \hat{x})\overline{(T(\lambda) - u^{\infty}_{\text{obs}})(\hat{x})}d\hat{x}\}ds(y)$$

- $H^1(\Gamma)$ regularization of gradient
- Obstacle : $B(0, 1)$, incident wave $d = (-1, 0)$, $k = 9$

Numerical experiments : classical impedance problem

Initial guess, exact solution, retrieved solutions with **0** and **2%** noise

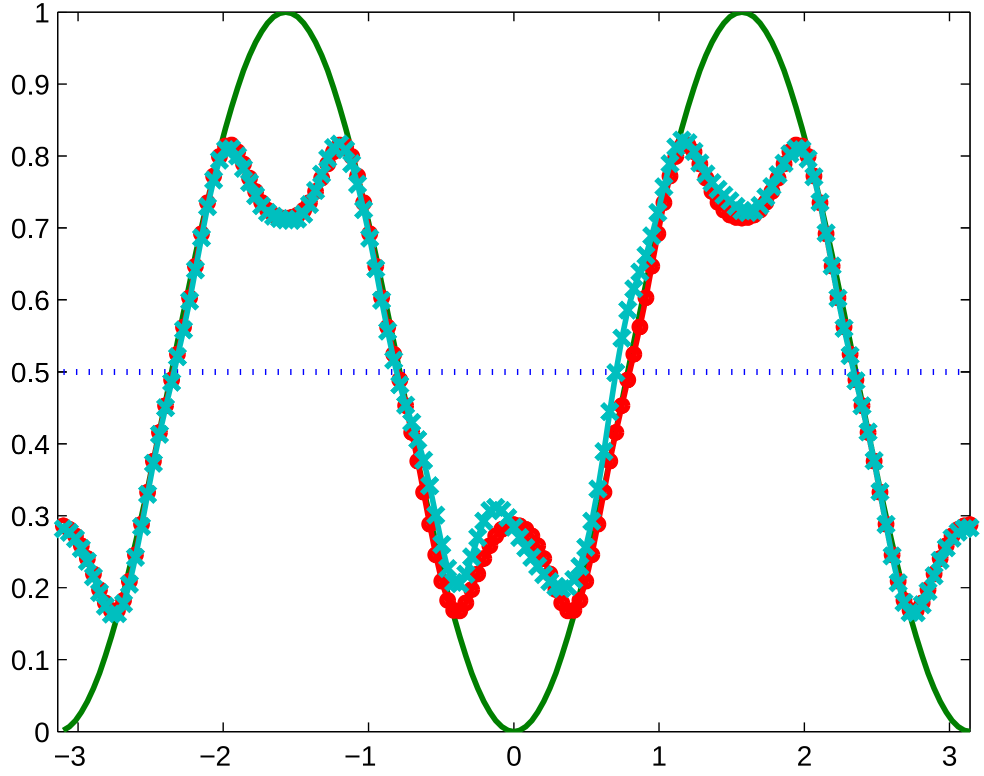
$$\mathbf{Re}(\boldsymbol{\lambda}) = \mathbf{0}$$
$$\mathbf{Im}(\boldsymbol{\lambda}) = \mathbf{sin^2}(\boldsymbol{\theta})$$



Numerical experiments : classical impedance problem

4 directions of incident wave, measurements limited to **1/4**-th of $\boldsymbol{S^1}$

$$\mathbf{Re}(\boldsymbol{\lambda}) = \mathbf{0}$$
$$\mathbf{Im}(\boldsymbol{\lambda}) = \mathbf{sin}^2(\boldsymbol{\theta})$$



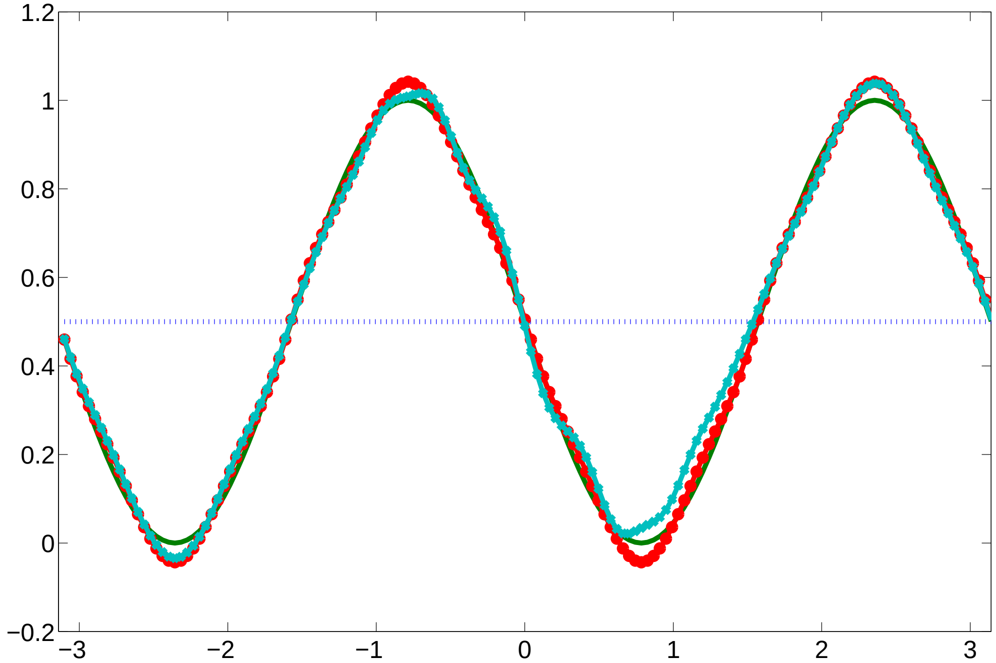
Numerical experiments : classical impedance problem

4 directions of incident wave, measurements limited to **1/4**-th of $\boldsymbol{S^1}$

$\mathbf{Re}(\boldsymbol{\lambda}) = \mathbf{0}$

$\mathbf{Im}(\boldsymbol{\lambda}) =$

$\mathbf{sin^2}(\boldsymbol{\theta} - \boldsymbol{\pi}/4)$



Numerical experiments : generalized impedance problem

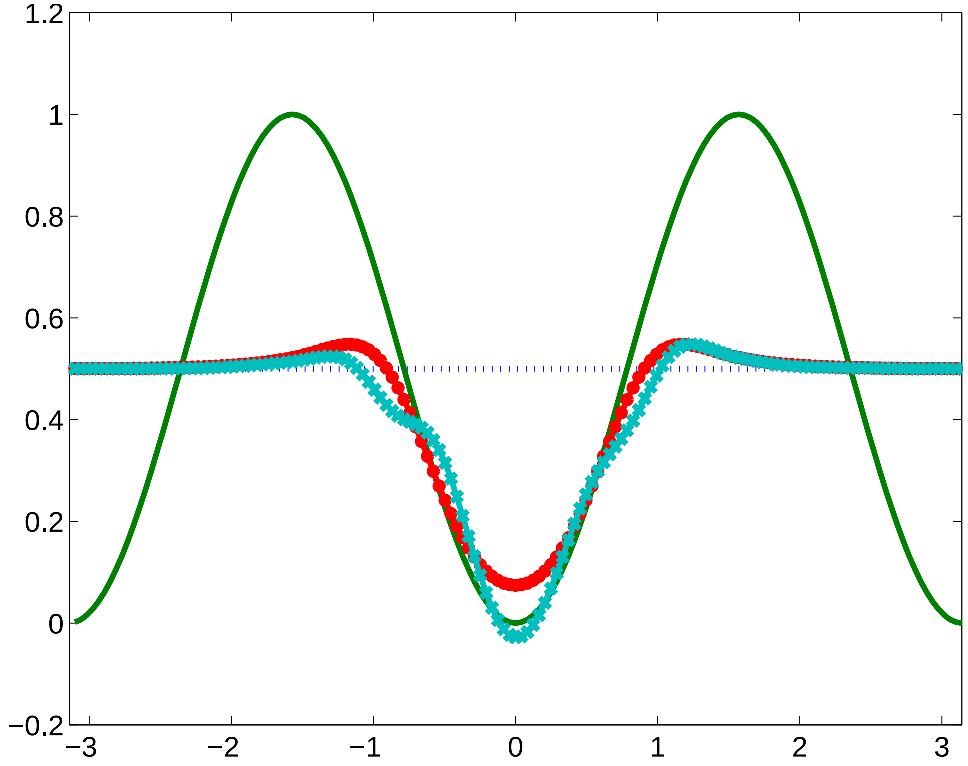
Second example :

$\mathbf{Re(\lambda) = 0}$

$\mathbf{Im(\mu) = 0}$

$\mathbf{Im(\lambda) = \sin^2(\theta)}$

$\mathbf{Re(\mu) = 0.5}$



Perspectives

- Improve uniqueness results for our GIBC
- Obtain logarithmic stability results for our GIBC without restriction on the set of parameters
- Other GIBCs, for example involving $\mathbf{div}_{\Gamma}(\boldsymbol{\mu}(\boldsymbol{x})\boldsymbol{\nabla}_{\Gamma}\boldsymbol{u})$
- Uniqueness from backscattering data : an open problem