Numerical Inverse Scattering Transform for Solving the Nonlinear Schrödinger Equation

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  Numerical method

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Numerical experiments
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Nonlinear Schrödinger equation (NLS)

We are interested in the IVP for the cubic Schrödinger equation (NLS)

\[
\begin{align*}
\text{i} q_t &= q_{xx} + 2q|q|^2, \quad x \in \mathbb{R}, \ t > 0, \\
q(x; 0) &\text{ given} \quad x \in \mathbb{R},
\end{align*}
\]

where \( q = q(x; t) \); we follow the path of the *Inverse Scattering Transform* (IST):

1. **Initial potential** \( q(x; 0) \)
2. **Time evolution of scattering data**
   \[ \text{i} S_t = \pm 4S_{\alpha\alpha} \]
3. **Evolved scattering data** \( S(\alpha; t) \)
4. **Evolved potential** \( q(x; t) \)

Numerical solution to the cubic NLS - AIP09 · Vienna
Nonlinear Schrödinger equation (NLS)

We are interested in the IVP for the cubic Schrödinger equation (NLS)

\[
\begin{cases}
iq_t = q_{xx} + 2|q|^2, & x \in \mathbb{R}, \ t > 0, \\
q(x; 0) & \text{given} \quad x \in \mathbb{R},
\end{cases}
\]

where \( q = q(x; t) \); we follow the path of the \textit{Inverse Scattering Transform} (IST):

\[\text{initial potential} \quad q(x; 0) \quad \text{direct scattering} \quad \Rightarrow \quad \text{initial scattering data} \quad S(\alpha; 0)\]

\[\text{NLS:} \quad \begin{align*}
iq_t &= q_{xx} + 2|q|^2 \\
\end{align*}\]

\[\text{time evolution of scattering data:} \quad iS_t &= \pm 4S_{\alpha \alpha}\]

\[\text{evolved potential} \quad q(x; t) \quad \text{inverse scattering} \quad \Rightarrow \quad \text{evolved scattering data} \quad S(\alpha; t)\]

\[\text{"The IST solves exactly the IVP for the NLS; all the physical properties are then preserved."}\]
Numerical method

Here we validate the last two step of the procedure.

A further work will be the implementation of the direct scattering, thanks to a result due to van der Mee.

Proposed algorithm

1. assume that we know $S(\lambda; 0)$; choose \{$t_1, \ldots, t_k$\}
2. for each $t_i$:
   - choose a discretization step $h$ and sample $S(\lambda; 0)$ with step $h$: $S_h(0)$
   - apply a numerical method to $S_h(0)$ and compute $S_h(t_i)$
   - apply a numerical method to $S_h(t_i)$ and compute [twice] $q(0; t_i)$
   - check if $q(0; t_i)$ is accurate:
     - YES compute $q(x_k; t_i)$ for all $x_k$
     - NO halve $h$ and restart the computation for the same $t_i$
Issues related to the application of the IST to the NLS

- eigenvalue system + FT + nonlinear least square problem;
- linear evolution problem;
- two systems of two Marchenko integral equations
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Scattering data

- “reflection coefficients” from the left and from the right: $L(\lambda)$, $R(\lambda)$;
- “transmission coefficients”: $T(\lambda)$;
- “bound states” $\lambda_j \in \mathbb{C}^+$, and their “norming constants” $\Gamma_{lj}$, $\Gamma_{rj}$.

In the numerical experiments for the Inverse Scattering, we assume to know the initial scattering data and use them to compute the potential.
Issues related to the application of the IST to the NLS

- eigenvalue system + FT + nonlinear least square problem;
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Time evolution of the relevant scattering data

\[
\begin{align*}
L(\lambda) & \quad R(\lambda) & \quad \lambda_j & \quad \Gamma_{lj} & \quad \Gamma_{rj} \\
\downarrow e^{-4i\lambda^2 t} & \quad \downarrow e^{+4i\lambda^2 t} & \quad \downarrow 1 & \quad \downarrow e^{+4i\lambda^2 t} & \quad \downarrow e^{-4i\lambda^2 t} \\
L(\lambda; t) & \quad R(\lambda; t) & \quad \lambda_j(t) & \quad \Gamma_{lj}(t) & \quad \Gamma_{rj}(t)
\end{align*}
\]

The scattering data determines the Marchenko integral kernels, i.e. the kernels of the integral system related to the inverse scattering step.
Issues related to the application of the IST to the NLS

- eigenvalue system + FT + nonlinear least square problem;
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Kernels of the Marchenko integral systems

Marchenko kernel from the left

\[ \Omega_l(\alpha) = \hat{R}(\alpha) + \sum_j \Gamma_{lj} e^{i\lambda_j \alpha} \]

\[ i(\Omega_l)_t = 4(\Omega_l)_{\alpha \alpha} \]

\[ \Omega_r(\alpha; t) = \hat{R}(\alpha; t) + \sum_j \Gamma_{rj}(t) e^{i\lambda_j \alpha} \]

Marchenko kernel from the right

\[ \Omega_r(\alpha) = \hat{L}(\alpha) + \sum_j \Gamma_{rj} e^{i\lambda_j \alpha} \]

\[ i(\Omega_r)_t = -4(\Omega_r)_{\alpha \alpha} \]

\[ \Omega_r(\alpha; t) = \hat{L}(\alpha; t) + \sum_j \Gamma_{rj}(t) e^{i\lambda_j \alpha} \]
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Marchenko integral systems:

For a fixed value $t \geq 0$, and for any value of $x \in \mathbb{R}$, solve: \( (B(\mathbb{R}, \mathbb{R}^+; t)) \)

\[
\begin{cases}
B_{l1}(x, \gamma; t) - \int_{0}^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l2}(x, \beta; t) \, d\beta = 0 \\
\int_{0}^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l1}(x, \beta; t) \, d\beta + B_{l2}(x, \gamma; t) = -\Omega_l(2x + \gamma; t)
\end{cases}
\]

and/or

\[
\begin{cases}
B_{r1}(x, \gamma; t) + \int_{0}^{+\infty} \Omega_r(-2x + \gamma + \beta; t)B_{r2}(x, \beta; t) \, d\beta = 0, \\
- \int_{0}^{+\infty} \Omega_r(-2x + \gamma + \beta; t)B_{r1}(x, \beta; t) \, d\beta + B_{r2}(x, \gamma; t) = \Omega_r(-2x + \gamma; t).
\end{cases}
\]
Issues related to the application of the IST to the NLS

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The Marchenko system returns the solution:

Potential

\[ q(x; t) = \begin{cases} 
2B_{l2}(x, 0^+; t), & x \geq 0, \\
-2B_{r2}(x, 0^+; t), & x \leq 0;
\end{cases} \]

Density energy:

\[ \begin{align*}
B_{l1}(x, 0^+; t) &= -\frac{1}{2} \int_x^{+\infty} |q(y; t)|^2 \, dy, \\
B_{r1}(x, 0^+; t) &= -\frac{1}{2} \int_{-\infty}^{x} |q(y; t)|^2 \, dy,
\end{align*} \]
Time evolution of the scattering data

- The time evolution of each bound state term is straightforward:

\[
\Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}, \quad \Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}.
\]
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- About the time evolution of the reflection coefficients \( L(\lambda), R(\lambda) \):

\[
\begin{align*}
L(\lambda) & \quad \xrightarrow{e^{-4i\lambda^2 t_i}} \quad L(\lambda; t_i) \xrightarrow{\mathcal{F}} \hat{L}(\alpha; t_i) \\
R(\lambda) & \quad \xrightarrow{e^{4i\lambda^2 t_i}} \quad R(\lambda; t_i) \xrightarrow{\mathcal{F}} \hat{R}(\alpha; t_i)
\end{align*}
\]
Time evolution of the scattering data

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\[ \Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}, \quad \Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}. \]

- About the time evolution of the reflection coefficients \( L(\lambda), R(\lambda) \):

\[
\begin{align*}
L(\lambda) &\xleftarrow{\mathcal{F}^{-1}} \hat{L}(\alpha) \quad & R(\lambda) &\xleftarrow{\mathcal{F}^{-1}} \hat{R}(\alpha) \\
\text{e}^{-4i\lambda^2 t_i} &\xrightarrow{\mathcal{F}} \quad & \text{e}^{4i\lambda^2 t_i} &\xrightarrow{\mathcal{F}} \\
L(\lambda; t_i) &\quad & R(\lambda; t_i) &\quad \\
\text{i} \hat{L}_t = -4\hat{L}_{\alpha\alpha} &\quad & \text{i} \hat{R}_t = 4\hat{R}_{\alpha\alpha} &\quad \\
\end{align*}
\]
Time evolution of the scattering data

- The time evolution of each bound state term is straightforward:

\[ \Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}, \quad \Gamma_{lj} e^{i\lambda_j \alpha} \rightarrow (\Gamma_{lj} e^{i\lambda_j^2 t_i}) e^{i\lambda_j \alpha}. \]

- About the time evolution of the reflection coefficients \( L(\lambda), R(\lambda) \):

\[
\begin{align*}
\mathcal{F}^{-1} &\quad \hat{L}(\alpha) \quad \mathcal{F}^{-1} &\quad \hat{R}(\alpha) \\
L(\lambda) &\quad \hat{L}(\alpha) \\
\text{e}^{-4i\lambda^2 t_i} &\quad \text{i} \hat{L}_t = -4 \hat{L}_{\alpha\alpha} \\
L(\lambda; t_i) &\quad \hat{L}(\alpha; t_i) \\
\mathcal{F} &\quad \mathcal{F} \\
R(\lambda) &\quad \hat{R}(\alpha) \\
\text{e}^{4i\lambda^2 t_i} &\quad \text{i} \hat{R}_t = 4 \hat{R}_{\alpha\alpha} \\
R(\lambda; t_i) &\quad \hat{R}(\alpha; t_i)
\end{align*}
\]

1. \([\alpha_0, \alpha_N]\) support of \( \hat{R}(\alpha) \), i.e. at time \( t = 0 \);
2. approximate \( \hat{R}(\alpha; t_i) \) on \([\alpha_0, \alpha_N]\):
   - apply IFSTT; integrate (multiply by \( \text{e}^{4i\lambda^2 t_i} \)); apply FFT;
3. check if \( \hat{R}(\alpha; t_i) \) is negligible at the boundary;
4. enlarge \([\alpha_0, \alpha_N]\) and recompute \( \hat{R}(\alpha; t_i) \) with the same stepsize.
Solution to the Marchenko systems

- one routine to solve both the Marchenko systems
- discretization of the “left” system by the Nyström method
- numerical method to solve the linear system
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\[
\begin{align*}
\text{left} \quad \begin{cases} 
B_{l1}(x, \gamma; t) - \int_0^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l2}(x, \beta; t) \, d\beta = 0 \\
\int_0^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l1}(x, \beta; t) \, d\beta + B_{l2}(x, \gamma; t) = -\Omega_l(2x + \gamma; t)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{right} \quad \begin{cases} 
B_{r1}(x, \gamma; t) + \int_0^{+\infty} \Omega_r(-2x + \gamma + \beta; t)B_{r2}(x, \beta; t) \, d\beta = 0, \\
- \int_0^{+\infty} \Omega_r(-2x + \gamma + \beta; t)B_{r1}(x, \beta; t) \, d\beta + B_{r2}(x, \gamma; t) = \Omega_r(-2x + \gamma; t),
\end{cases}
\end{align*}
\]

1) conjugate both equations;
2) change of sign in the first integral and in the second equation;
3) replace $x$ by $-x$
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\int_0^{+\infty} \Omega_l(2x + \gamma + \beta; t) B_{l1}(x, \beta; t) \, d\beta + B_{l2}(x, \gamma; t) = -\Omega_l(2x + \gamma; t)
\end{cases} \\
\text{right} & \quad \begin{cases} 
\overline{B_{r1}(-x, \gamma; t)} - \int_0^{+\infty} \Omega_r(2x + \gamma + \beta; t) \left( -\overline{B_{r2}(-x, \beta; t)} \right) \, d\beta = 0, \\
\int_0^{+\infty} \Omega_r(2x + \gamma + \beta; t) \overline{B_{r1}(-x, \beta; t)} \, d\beta + \left( -\overline{B_{r2}(-x, \gamma; t)} \right) = -\Omega_r(2x + \gamma; t),
\end{cases}
\end{align*}
\]

1) conjugate both equations;
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\end{cases}
\]

Given \( x_k \) and \( t \), we collocate the left Marchenko system in each point of the grid \( \{ \beta_i = ih, \; i = 0, \ldots, N \} \), and approximate the integrals by applying the repeated Simpson’s quadrature rule in the same points.

- \( h \) is the stepsize used in the time evolution routine
- \( N \) is the number of points where \( \hat{R}(\alpha; t_i) \) has been approximated
- \( x_k \) belongs to the same interval \([ \alpha_{i/2}, \alpha_{N/2} ]\), with half stepsize

\[
x_k = \frac{\alpha_k}{2} = \frac{\alpha_0}{2} + \frac{h}{2}k \quad \Rightarrow \quad 2x_k + \gamma_i + \beta_j = \alpha_{k+i+j}
\]
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\[
\begin{cases}
  B_{l1}(x, \gamma; t) - \int_0^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l2}(x, \beta; t) \, d\beta = 0 \\
  \int_0^{+\infty} \Omega_l(2x + \gamma + \beta; t)B_{l1}(x, \beta; t) \, d\beta + B_{l2}(x, \gamma; t) = -\Omega_l(2x + \gamma; t)
\end{cases}
\]

Given \( x_k \) and \( t \), we collocate the left Marchenko system in each point of the grid \( \{\beta_i = ih, \ i = 0, \ldots, N\} \), and approximate the integrals by applying the repeated Simpson’s quadrature rule in the same points.

\[
\begin{cases}
  B_{l1}(x_k, \beta_i; t) - \sum_{j=0}^N \Omega_l(\alpha_{k+i+j}; t)d_j B_{l2}(x_k, \beta_j; t) = 0 \\
  \sum_{j=0}^N \Omega_l(\alpha_{k+i+j}; t)d_j B_{l1}(x_k, \beta_i; t) + B_{l2}(x_k, \beta_i; t) = -\Omega(\alpha_{k+i}; t)
\end{cases}
\]
Solution to the Marchenko systems

- one routine to solve both the Marchenko systems
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The support of $\Omega_l(\alpha; t)$ is essentially $(-\infty, \alpha_N]$, so we put $\Omega_l(\alpha; t) = 0$ whenever $\alpha > \alpha^t_N$. In matrix notation, the previous system takes the form

$$\begin{bmatrix}
  I & -H_l^* D \\
  H_l D & I
\end{bmatrix}
\begin{bmatrix}
  b_{l1} \\
  b_{l2}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  -\omega
\end{bmatrix}$$

where, for $i, j = 0, \ldots, N$,

- $D = \frac{h}{3} \text{diag}(1, 4, 2, 4, 2, \ldots)$
- $(H_l)_{ij} = \Omega_l(\alpha_{k+i+j}; t)$ (upper-left triangular + low rank)
- $(b_{l1})_i = B_{1l}(x_k, \beta_i; t)$
- $(b_{l2})_i = B_{2l}(x_k, \beta_i; t)$
Solution to the Marchenko systems

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If the reflection coefficient vanishes \( L(\lambda) = R(\lambda) = 0 \), the system matrix is “Identity + Low Rank”. \( \text{Low} = \text{twice the number of bound states} \).

- we have an exact formula for the inverse
- we can compute the limit of the solution, wrt \( h \), and get the analytical solution of the Marchenko systems
- this has been implemented, and improves the overall results (precision, time complexity)

If the reflection coefficients do not vanish, we apply the CG to the Schur system, i.e. \((I + HDH^* D)b_{l2} = -\omega\), but in a symmetryzed version:

\[
(D + DHDH^* D)b_{l2} = -D\omega.
\]

Since the \( B(x, \gamma; t) \) are (...) regular, we can also reduce the number of required iterations.
One soliton

- **parameters:** 
  \[ \eta, \xi, x_0, \phi \in \mathbb{R}, \quad \eta > 0 \]

- **initial potential:** 
  \[ q(x; 0) = -2i\eta e^{-i(2\xi x + \phi)} \text{sech}(x_0 - 2\eta x) \]

- **scattering data:** 
  \[ \lambda_1 = i\eta - \xi, \quad \Gamma_{l1} = 2i\eta e^{x_0 - i\phi} \quad \Gamma_{r1} = -2i\eta e^{-x_0 + i\phi} \]

  \[ T(\lambda) = \frac{\lambda + \lambda_1}{\lambda - \lambda_1}, \quad L(\lambda) = 0 \quad R(\lambda) = 0, \]

- **solution to the NLS:** 
  \[ q(x; t) = -2i\eta e^{-i(2\xi x + 4(\eta^2 - \xi^2)t + \phi)} \text{sech}(x_0 - 2\eta x + 8\eta \xi t) \]

  The amplitude is \(2\eta\) and the peak point at time \(t\) is \(\frac{x_0}{2\eta} + 4\xi t\).
One soliton

Parameters: \( x \in [-12, 7], \quad \eta \in \{1, 3, 6\}, \quad \xi = -\frac{1}{3}, \quad x_0 = 1, \quad \phi = \frac{\pi}{4}. \)

**IST** \((h \simeq 0.075)\)

\[
\begin{align*}
|u| & \quad \times 10^{-15} & |e| \\
\text{Re } u & \quad \text{Im } u & \text{Re } u & \quad \text{Im } u
\end{align*}
\]

**FSS** \((\Delta t \simeq 2.6 \cdot 10^{-4})\)

\[
\begin{align*}
|u| & \quad x 10^{-6} & |u-u_{\text{true}}| \\
\text{Re } u & \quad \text{Im } u & \text{Re } u & \quad \text{Im } u
\end{align*}
\]
One soliton

Parameters: \( x \in [-12, 7], \quad \eta \in \{1, 3, 6\}, \quad \xi = -\frac{1}{3}, \quad x_0 = 1, \quad \phi = \frac{\pi}{4}. \)

**IST** \((h \approx 0.075)\)

**FSS** \((\Delta t \approx 2.6 \cdot 10^{-4})\)
One soliton

Parameters: \( x \in [-12, 7], \quad \eta \in \{1, 3, 6\}, \quad \xi = -\frac{1}{3}, \quad x_0 = 1, \quad \phi = \frac{\pi}{4}. \)

**IST** \((h \simeq 0.075)\)

- \(|u|\)
- \(x \simeq 10^{-12}\)
- \(|e|\)
- \(\text{Re } u\)
- \(\text{Im } u\)

**FSS** \((\Delta t \simeq 2.6 \cdot 10^{-4})\)

- \(|u|\)
- \(|u - u_{\text{true}}|\)
- \(\text{Re } u\)
- \(\text{Im } u\)
Truncated one-soliton

We introduce a discontinuity in $\mu$:

\[
q_\mu(x) = \begin{cases} 
0, & x < \mu \\
q(x; 0), & x \geq \mu
\end{cases}
\]  

(1)

- all the scattering data can be computed analytically
- the reflection coefficients do not vanish
- they are discontinuous
- parameters: $\eta = 1$, $\xi = 0$, $x_0 = -\ln(2)$, $\phi = \frac{\pi}{2}$
- there is one bound state if $\mu \leq -\frac{1}{2} \ln(2)$; none if $\mu > -\frac{1}{2} \ln(2)$
- numerical experiments with $\mu \in \{-1, +1\}$
Truncated one-soliton

We introduce a discontinuity in $\mu = -1$

$$q_\mu(x) = \begin{cases} 
0, & x < \mu \\
q(x;0), & x \geq \mu
\end{cases}$$ (1)

IST & FSS ($h \simeq 0.075$)

IST ($t = 0$)
Truncated one-soliton

We introduce a discontinuity in $\mu = +1$

$$q_\mu(x) = \begin{cases} 
0, & x < \mu \\
q(x;0), & x \geq \mu
\end{cases} \tag{1}$$

IST & FSS ($h \simeq 0.075$)

IST ($t = 0$)

Numerical solution to the cubic NLS - AIP09 · Vienna
References

Gardner, Greene, Kruskal, Miura

*Method for Solving the Korteweg-deVries Equation*


Zakharov, Shabat

*Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*


Taha, Ablowitz

*Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, nonlinear Schrödinger equation*


van der Mee

*Direct and inverse scattering for skewselfadjoint Hamiltonian systems*


Ablowitz, Prinari, Trubatch

*Discrete and continuous nonlinear Schrödinger systems*