# Numerical Inverse Scattering Transform for Solving the Nonlinear Schrödinger Equation

A. Aricò, G. Rodriguez, S. Seatzu

Department of Mathematics and Computer Science
University of Cagliari · Italy

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# Outline


Nonlinear Schrödinger equation and Inverse Scattering Transform
- Nonlinear Schrödinger equation (NLS)
- Numerical method

Issues related to the application of the IST to the NLS

Numerical solution to the inverse scattering problem
- Time evolution of the scattering data
- Solution to the Marchenko systems

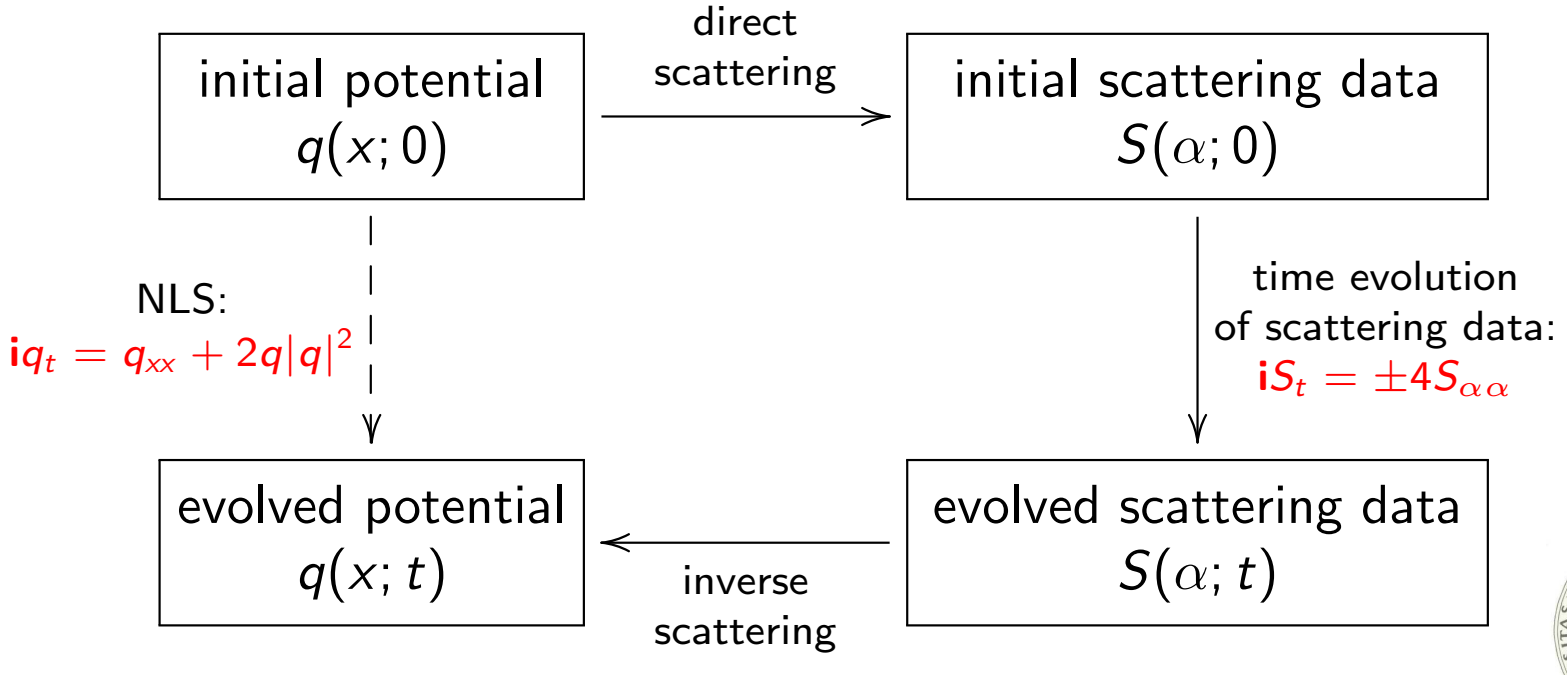
Numerical experiments
- One soliton
- Truncated one soliton

# Nonlinear Schrödinger equation (NLS)

We are interested in the IVP for the cubic Schrödinger equation (NLS)

$$\begin{cases} \mathbf{i}q_t = q_{xx} + 2q|q|^2, & x \in \mathbb{R},\ t > 0, \\ q(x;0) \quad \text{given} & x \in \mathbb{R}, \end{cases}$$

where $q = q(x;t)$; we follow the path of the *Inverse Scattering Transform* (IST):



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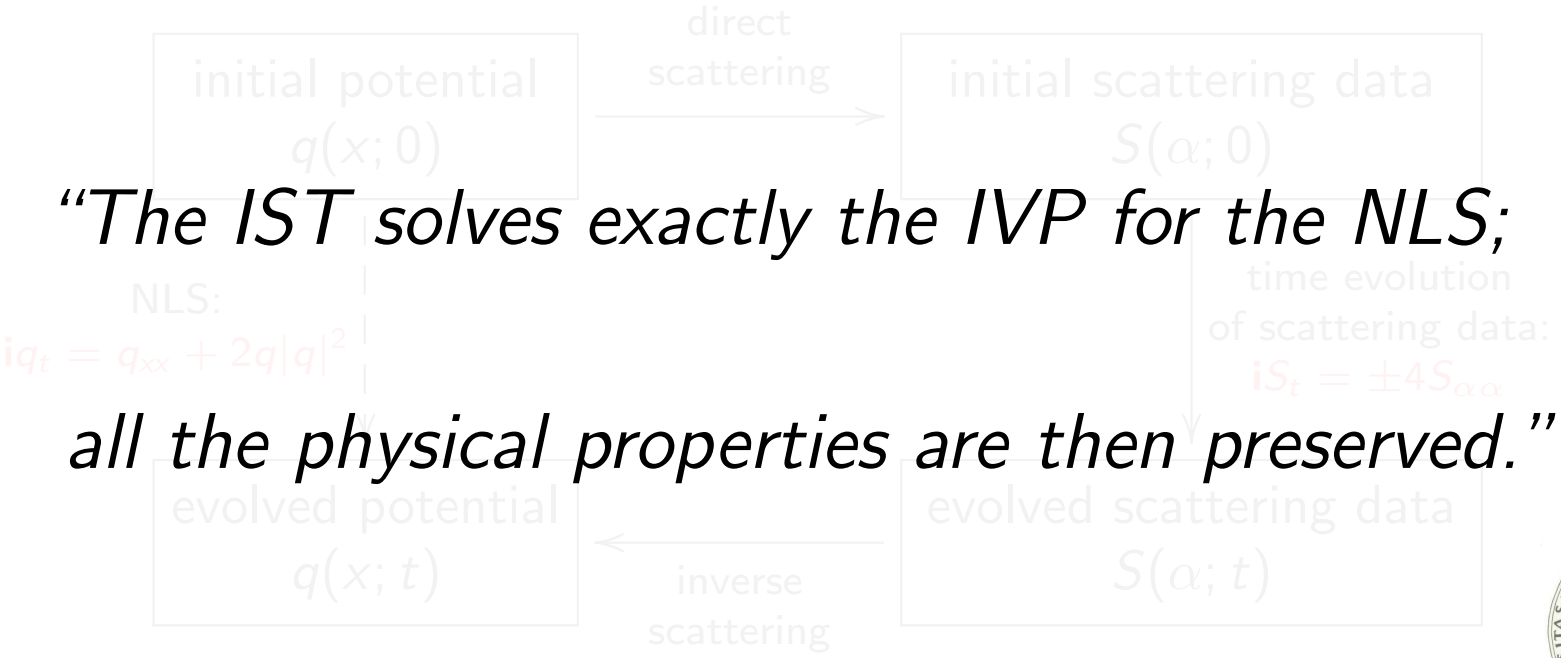


*"The IST solves exactly the IVP for the NLS;*

*all the physical properties are then preserved."*

# Numerical method

Here we validate the last two step of the procedure.

A further work will be the implementation of the direct scattering, thanks to a result due to van der Mee.

## Proposed algorithm

1. assume that we know $S(\lambda; 0)$; choose $\{t_1, \ldots, t_k\}$
2. for each $t_i$:
   - choose a discretization step $h$ and sample $S(\lambda; 0)$ with step $h$: $S_h(0)$
   - apply a numerical method to $S_h(0)$ and compute $S_h(t_i)$
   - apply a numerical method to $S_h(t_i)$ and compute [twice] $q(0; t_i)$
   - check if $q(0; t_i)$ is accurate:

     YES compute $q(x_k; t_i)$ for all $x_k$

     NO halve $h$ and restart the computation for the same $t_i$

# Issues related to the application of the IST to the NLS

- eigenvalue system $+$ FT $+$ nonlinear least square problem;
- linear evolution problem;
- two systems of two Marchenko integral equations

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## Scattering data

- "*reflection coefficients*" from the left and from the right: $L(\lambda)$, $R(\lambda)$;
- "*transmission coefficients*": $T(\lambda)$;
- "*bound states*" $\lambda_j \in \mathbb{C}^+$, and their "*norming constants*" $\Gamma_{lj}$, $\Gamma_{rj}$.

In the numerical experiments for the Inverse Scattering, we assume to know the initial scattering data and use them to compute the potential.

# Issues related to the application of the IST to the NLS

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## Time evolution of the relevant scattering data

$$
\begin{array}{ccccc}
L(\lambda) & R(\lambda) & \lambda_j & \Gamma_{lj} & \Gamma_{rj} \\
\Big\downarrow \mathrm{e}^{-4\mathbf{i}\lambda^2 t} & \Big\downarrow \mathrm{e}^{+4\mathbf{i}\lambda^2 t} & \Big\downarrow 1 & \Big\downarrow \mathrm{e}^{+4\mathbf{i}\lambda^2 t} & \Big\downarrow \mathrm{e}^{-4\mathbf{i}\lambda^2 t} \\
L(\lambda;t) & R(\lambda;t) & \lambda_j(t) & \Gamma_{lj}(t) & \Gamma_{rj}(t)
\end{array}
$$

The scattering data determines the Marchenko integral kernels, i.e. the kernels of the integral system related to the inverse scattering step.

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## Kernels of the Marchenko integral systems

Marchenko kernel from the left

$$\Omega_l(\alpha) = \hat{R}(\alpha) + \sum_j \Gamma_{lj} e^{\mathbf{i}\lambda_j \alpha}$$

$$\Big\downarrow \; \mathbf{i}(\Omega_l)_t = 4(\Omega_l)_{\alpha\alpha}$$

$$\Omega_r(\alpha; t) = \hat{R}(\alpha; t) + \sum_j \Gamma_{rj}(t) e^{\mathbf{i}\lambda_j \alpha}$$

Marchenko kernel from the right

$$\Omega_r(\alpha) = \hat{L}(\alpha) + \sum_j \Gamma_{rj} e^{\mathbf{i}\lambda_j \alpha}$$

$$\Big\downarrow \; \mathbf{i}(\Omega_r)_t = -4(\Omega_r)_{\alpha\alpha}$$

$$\Omega_r(\alpha; t) = \hat{L}(\alpha; t) + \sum_j \Gamma_{rj}(t) e^{\mathbf{i}\lambda_j \alpha}$$

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## Marchenko integral systems:

For a fixed value $t \geqslant 0$, and for any value of $x \in \mathbb{R}$, solve: $\quad (B(\mathbb{R}, \mathbb{R}^+; t))$

$$\text{left} \begin{cases} B_{l1}(x,\gamma;t) - \displaystyle\int_0^{+\infty} \overline{\Omega_l(2x+\gamma+\beta;t)} B_{l2}(x,\beta;t)\,\mathrm{d}\beta = 0 \\ \displaystyle\int_0^{+\infty} \Omega_l(2x+\gamma+\beta;t) B_{l1}(x,\beta;t)\,\mathrm{d}\beta + B_{l2}(x,\gamma;t) = -\Omega_l(2x+\gamma;t) \end{cases}$$

and/or

$$\text{right} \begin{cases} B_{r1}(x,\gamma;t) + \displaystyle\int_0^{+\infty} \Omega_r(-2x+\gamma+\beta;t) B_{r2}(x,\beta;t)\,\mathrm{d}\beta = 0, \\ -\displaystyle\int_0^{+\infty} \overline{\Omega_r(-2x+\gamma+\beta;t)} B_{r1}(x,\beta;t)\,\mathrm{d}\beta + B_{r2}(x,\gamma;t) = \overline{\Omega_r(-2x+\gamma;t)}, \end{cases}$$

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## The Marchenko system returns the solution:

Potential

$$q(x;t) = \begin{cases} 2B_{l2}(x,0^+;t), & x \geqslant 0, \\ -2B_{r2}(x,0^+;t), & x \leqslant 0; \end{cases}$$

Density energy:

$$\begin{cases} B_{l1}(x,0^+;t) = -\dfrac{1}{2}\displaystyle\int_x^{+\infty} |q(y;t)|^2 \,\mathrm{d}y, \\ B_{r1}(x,0^+;t) = -\dfrac{1}{2}\displaystyle\int_{-\infty}^{x} |q(y;t)|^2 \,\mathrm{d}y, \end{cases}$$

# Time evolution of the scattering data

- The time evolution of each bound state term is straightforward:

$$\Gamma_{lj} e^{\mathbf{i}\lambda_j \alpha} \longrightarrow \left(\Gamma_{lj} e^{\mathbf{i}\lambda_j^2 t_i}\right) e^{\mathbf{i}\lambda_j \alpha}, \qquad \Gamma_{lj} e^{\mathbf{i}\lambda_j \alpha} \longrightarrow \left(\Gamma_{lj} e^{\mathbf{i}\lambda_j^2 t_i}\right) e^{\mathbf{i}\lambda_j \alpha}.$$

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- About the time evolution of the reflection coefficients $L(\lambda)$, $R(\lambda)$:

$$\begin{tikzcd} L(\lambda) \arrow[d, "e^{-4\mathbf{i}\lambda^2 t_i}"'] & \\ L(\lambda; t_i) \arrow[r, "\mathcal{F}"] & \hat{L}(\alpha; t_i) \end{tikzcd} \qquad \begin{tikzcd} R(\lambda) \arrow[d, "e^{4\mathbf{i}\lambda^2 t_i}"'] & \\ R(\lambda; t_i) \arrow[r, "\mathcal{F}"] & \hat{R}(\alpha; t_i) \end{tikzcd}$$

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R(\lambda; t_i) \arrow[r, "\mathcal{F}"] & \hat{R}(\alpha; t_i)
\end{tikzcd}$$

1. $[\alpha_0, \alpha_N]$ support of $\hat{R}(\alpha)$, i.e. at time $t = 0$;
2. approximate $\hat{R}(\alpha; t_i)$ on $[\alpha_0, \alpha_N]$:
   apply *IFFT*; integrate (multiply by $e^{4\mathbf{i}\lambda^2 t_i}$); apply *FFT*;
3. check if $\hat{R}(\alpha; t_i)$ is negligible at the boundary;

(4.) enlarge $[\alpha_0, \alpha_N]$ and recompute $\hat{R}(\alpha; t_i)$ with the same stepsize.

# Solution to the Marchenko systems

- one routine to solve both the Marchenko systems
- discretization of the "left" system by the Nyström method
- numerical method to solve the linear system

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$$\text{left}\begin{cases} B_{l1}(x,\gamma;t) - \displaystyle\int_0^{+\infty} \overline{\Omega_l(2x+\gamma+\beta;t)} B_{l2}(x,\beta;t)\,\mathrm{d}\beta = 0 \\ \displaystyle\int_0^{+\infty} \Omega_l(2x+\gamma+\beta;t) B_{l1}(x,\beta;t)\,\mathrm{d}\beta + B_{l2}(x,\gamma;t) = -\Omega_l(2x+\gamma;t) \end{cases}$$

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1) conjugate both equations;
2) change of sign in the first integral and in the second equation;
3) replace $x$ by $-x$

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$$\text{right}\begin{cases} \overline{B_{r1}(-x,\gamma;t)} - \displaystyle\int_0^{+\infty} \overline{\Omega_r(2x+\gamma+\beta;t)}\,\left(-\overline{B_{r2}(-x,\beta;t)}\right)\mathrm{d}\beta = 0, \\ \displaystyle\int_0^{+\infty} \Omega_r(2x+\gamma+\beta;t)\,\overline{B_{r1}(-x,\beta;t)}\,\mathrm{d}\beta + \left(-\overline{B_{r2}(-x,\gamma;t)}\right) = -\Omega_r(2x+\gamma;t), \end{cases}$$

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Given $x_k$ and $t$, we collocate the left Marchenko system in each point of the grid $\{\beta_i = ih,\ i = 0,\dots,N\}$, and approximate the integrals by appliyng the repeated Simpson's quadrature rule in the same points.

- $h$ is the stepsize used in the time evolution routine
- $N$ is the number of points where $\hat{R}(\alpha;t_i)$ has been approximated
- $x_k$ belongs to the same interval $[\frac{\alpha_0}{2}, \frac{\alpha_N}{2}]$, with *half* stepsize

$$x_k = \frac{\alpha_k}{2} = \frac{\alpha_0}{2} + \frac{h}{2}k \qquad \Rightarrow \qquad 2x_k + \gamma_i + \beta_j = \alpha_{k+i+j}$$

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$$\text{left}\begin{cases} B_{l1}(x_k,\beta_i;t) - \displaystyle\sum_{j=0}^{N} \overline{\Omega_l(\alpha_{k+i+j};t)} d_j B_{l2}(x_k,\beta_j;t) = 0 \\ \displaystyle\sum_{j=0}^{N} \Omega_l(\alpha_{k+i+j};t) d_j B_{l1}(x_k,\beta_i;t) \ + B_{l2}(x_k,\beta_i;t) = -\Omega(\alpha_{k+i};t) \end{cases}$$

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The support of $\Omega_l(\alpha; t)$ is essentially $(-\infty, \alpha_N]$, so we put $\Omega_l(\alpha; t) = 0$ whenever $\alpha > \alpha_N^t$. In matrix notation, the previous sistem takes the form

$$\begin{bmatrix} I & -H_l^* D \\ H_l D & I \end{bmatrix} \begin{bmatrix} \mathbf{b}_{l1} \\ \mathbf{b}_{l2} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\omega} \end{bmatrix}$$

where, for $i, j = 0, \dots, N$,

- $D = \frac{h}{3}\,\mathrm{diag}(1, 4, 2, 4, 2, \dots)$
- $(H_l)_{ij} = \Omega_l(\alpha_{k+i+j}; t)$ (upper-left triangular + low rank)
- $(\mathbf{b}_{l1})_i = B_{1l}(x_k, \beta_i; t)$
- $(\mathbf{b}_{l2})_i = B_{2l}(x_k, \beta_i; t)$

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If the reflection coefficient vanishes ($L(\lambda) = R(\lambda) = 0$), the system matrix is "Identity + Low Rank". Low = twice the number of bound states.

- we have an exact formula for the inverse
- we can compute the limit of the solution, wrt $h$, and get the *analytical* solution of the Marchenko systems
- this has been implemented, and improves the overall results (precision, time complexity)

If the reflection coefficients do not vanish, we apply the CG to the Schur system, i.e. $(I + HDH^* D)\mathbf{b}_{l2} = -\boldsymbol{\omega}$, but in a symmetryzed version:

$$(D + DHDH^* D)\mathbf{b}_{l2} = -D\boldsymbol{\omega}.$$

Since the $B(x, \gamma; t)$ are (...) regular, we can also reduce the number of required iterations.

# One soliton

- parameters:

$$\eta,\ \xi,\ x_0,\ \phi\ \in \mathbb{R}, \quad \eta > 0$$

- initial potential:

$$q(x;0) = -2\mathbf{i}\eta e^{-\mathbf{i}(2\xi x+\phi)}\,\mathrm{sech}(x_0 - 2\eta x)$$

- scattering data:

$$\lambda_1 = \mathbf{i}\eta - \xi, \qquad \Gamma_{l1} = 2\mathbf{i}\eta e^{x_0 - \mathbf{i}\phi} \qquad \Gamma_{r1} = -2\mathbf{i}\eta e^{-x_0+\mathbf{i}\phi}$$

$$T(\lambda) = \frac{\lambda+\lambda_1}{\lambda-\lambda_1}, \qquad L(\lambda) = 0 \qquad R(\lambda) = 0,$$

- solution to the NLS:

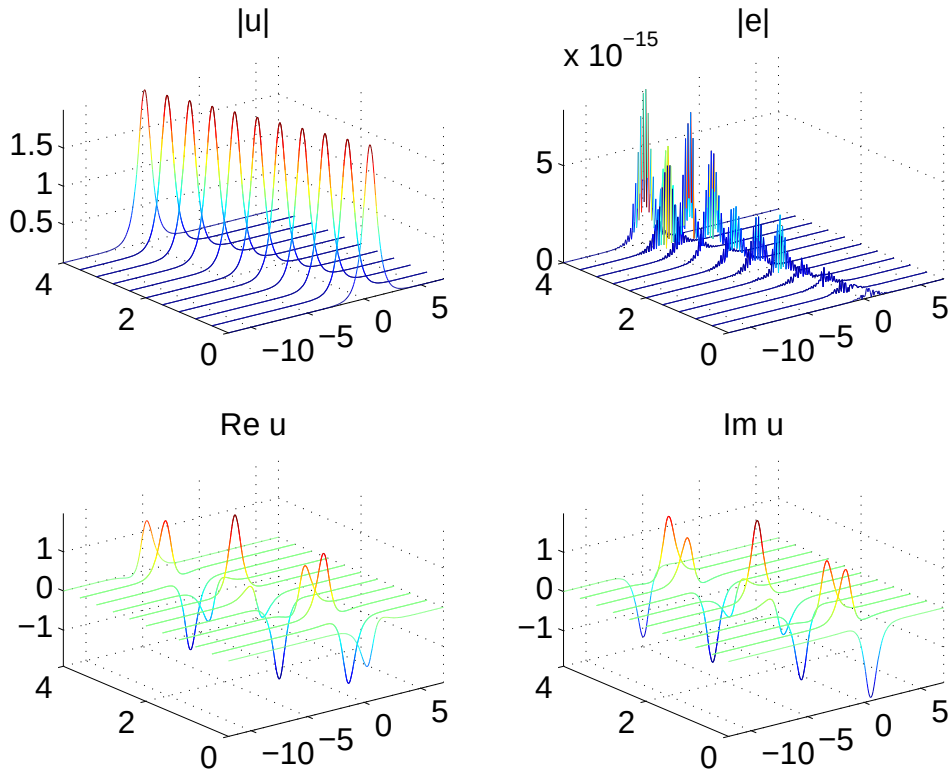
$$q(x;t) = -2\mathbf{i}\eta e^{-\mathbf{i}(2\xi x+4(\eta^2-\xi^2)t+\phi)}\,\mathrm{sech}(x_0 - 2\eta x + 8\eta\xi t)$$

The amplitude is $2\eta$ and the peak point at time $t$ is $\frac{x_0}{2\eta} + 4\xi t$.

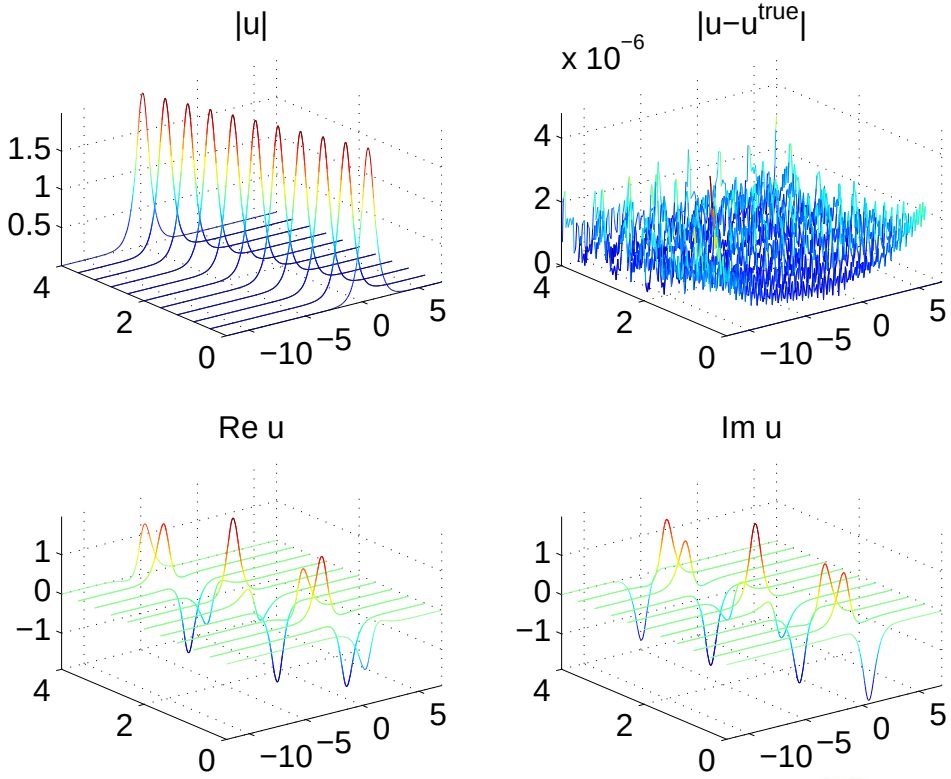
# One soliton

Parameters: $x \in [-12, 7]$, $\eta \in \{1, 3, 6\}$, $\xi = -\frac{1}{3}$, $x_0 = 1$, $\phi = \frac{\pi}{4}$.

IST $(h \simeq 0.075)$




FSS $(\Delta t \simeq 2.6 \cdot 10^{-4})$



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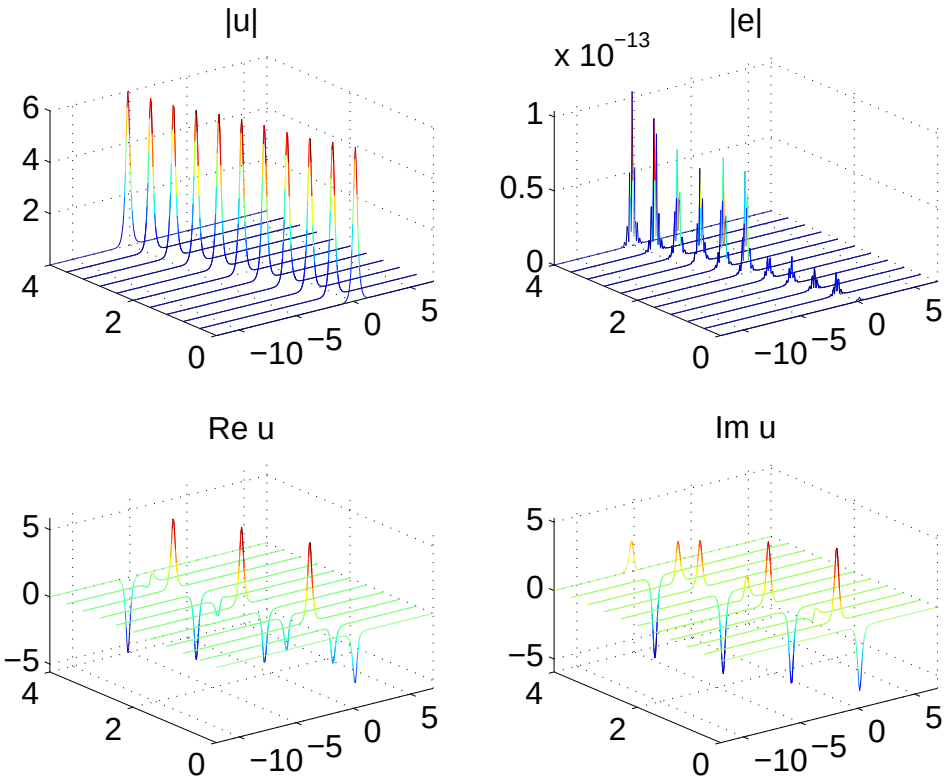


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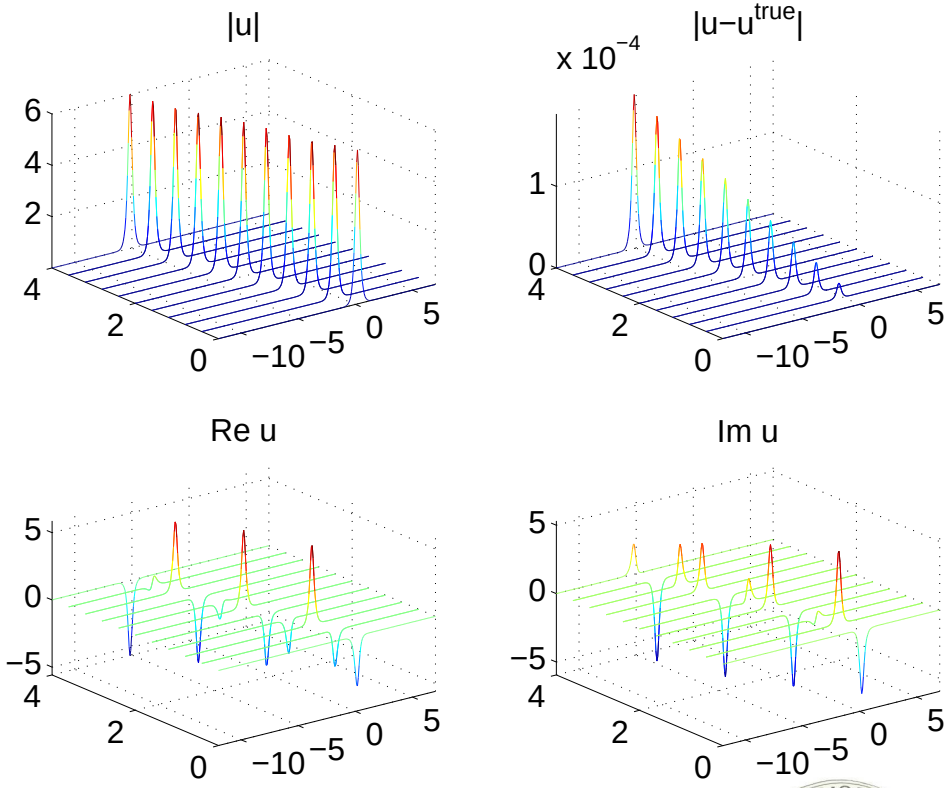

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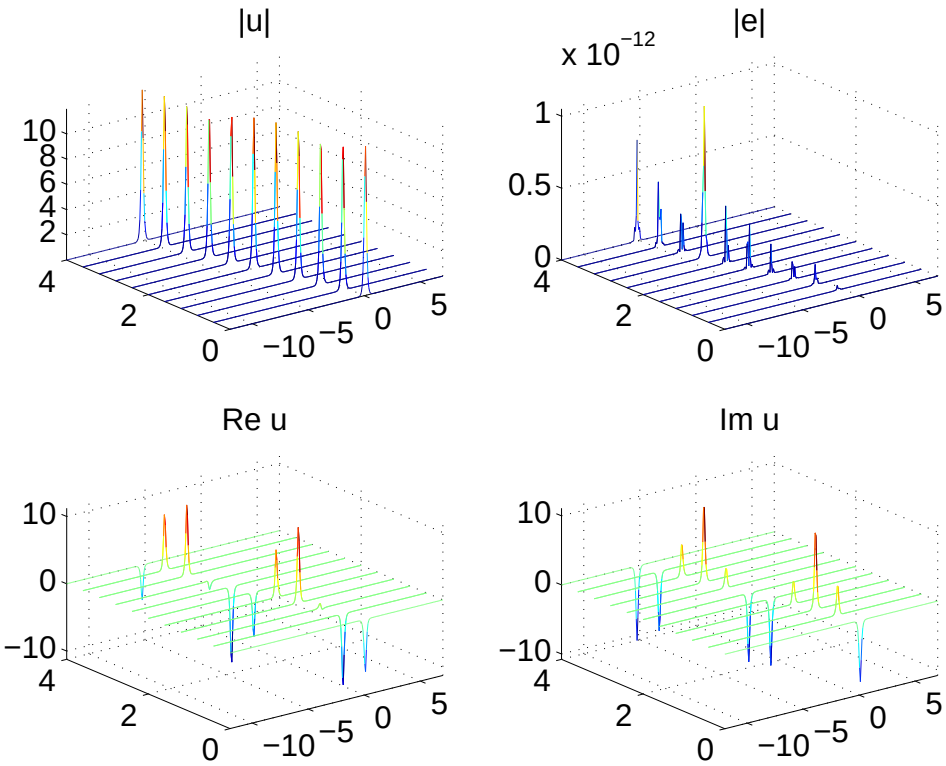


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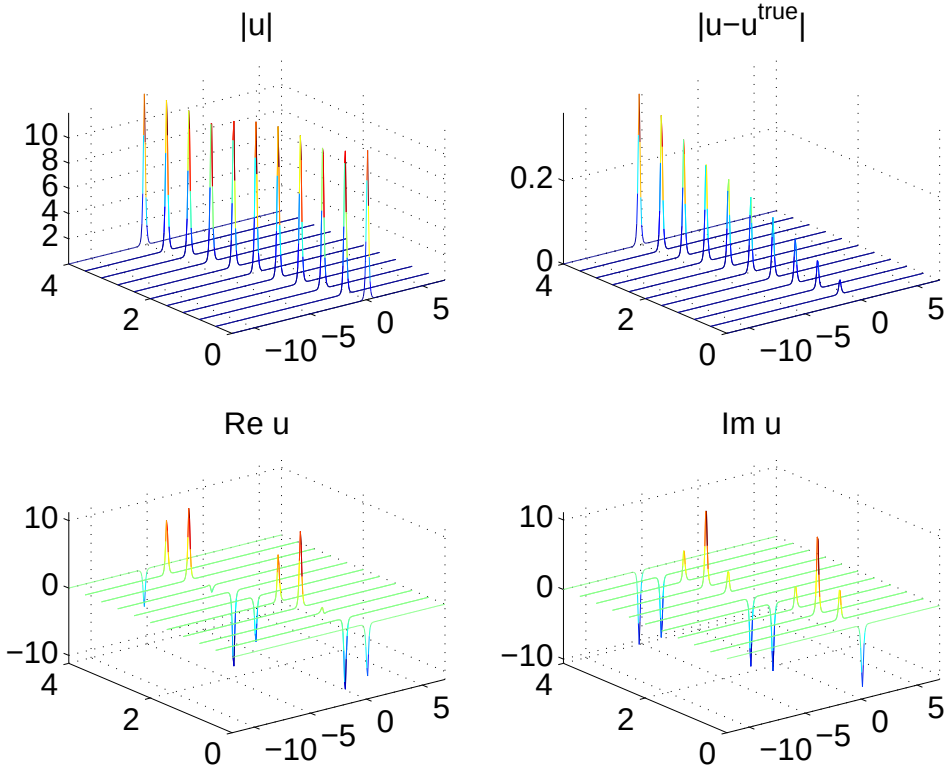

# Truncated one-soliton

We introduce a discontinuity in $\mu$:

$$q_\mu(x) = \begin{cases} 0, & x < \mu \\ q(x;0), & x \geqslant \mu \end{cases} \tag{1}$$

- all the scattering data can be computed analitically
- the reflection coefficients do not vanish
- they are discontinuous
- parameters: $\eta = 1, \quad \xi = 0, \quad x_0 = -\ln(2), \quad \phi = \frac{\pi}{2}$
- there is one bound state if $\mu \leqslant -\frac{1}{2}\ln(2)$; none if $\mu > -\frac{1}{2}\ln(2)$
- numerical experiments with $\mu \in \{-1, +1\}$

# Truncated one-soliton

We introduce a discontinuity in $\mu$=-1

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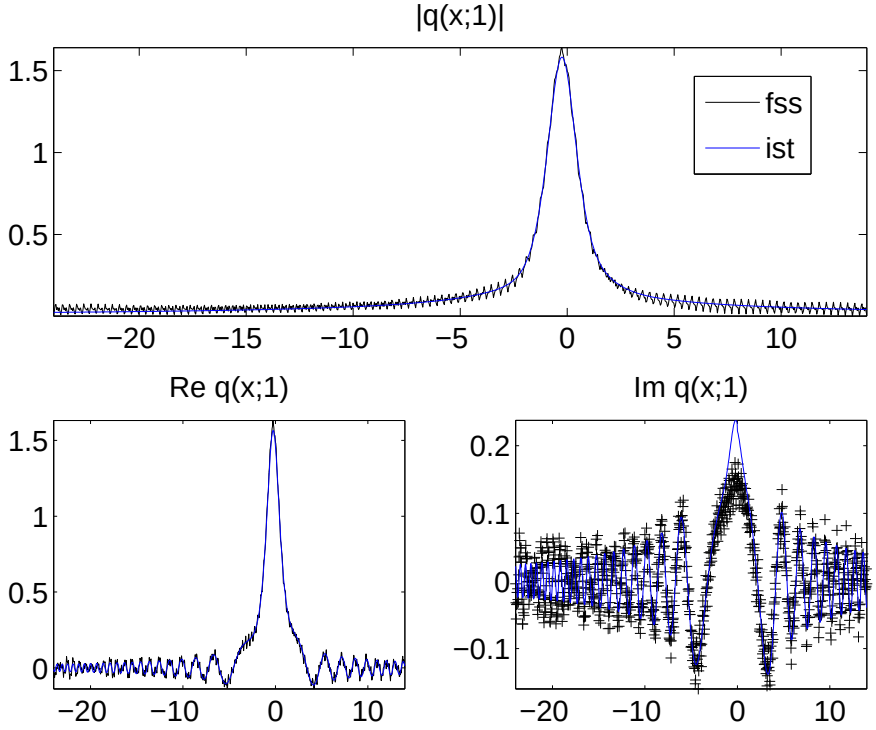
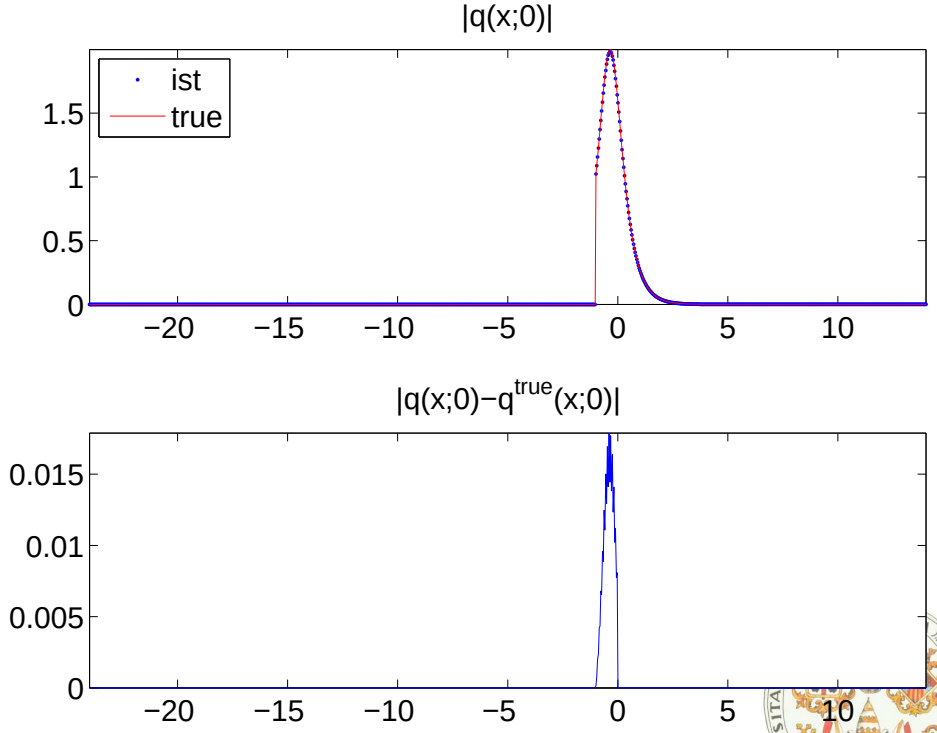
IST & FSS $(h \simeq 0.075)$




IST $(t = 0)$



# Truncated one-soliton

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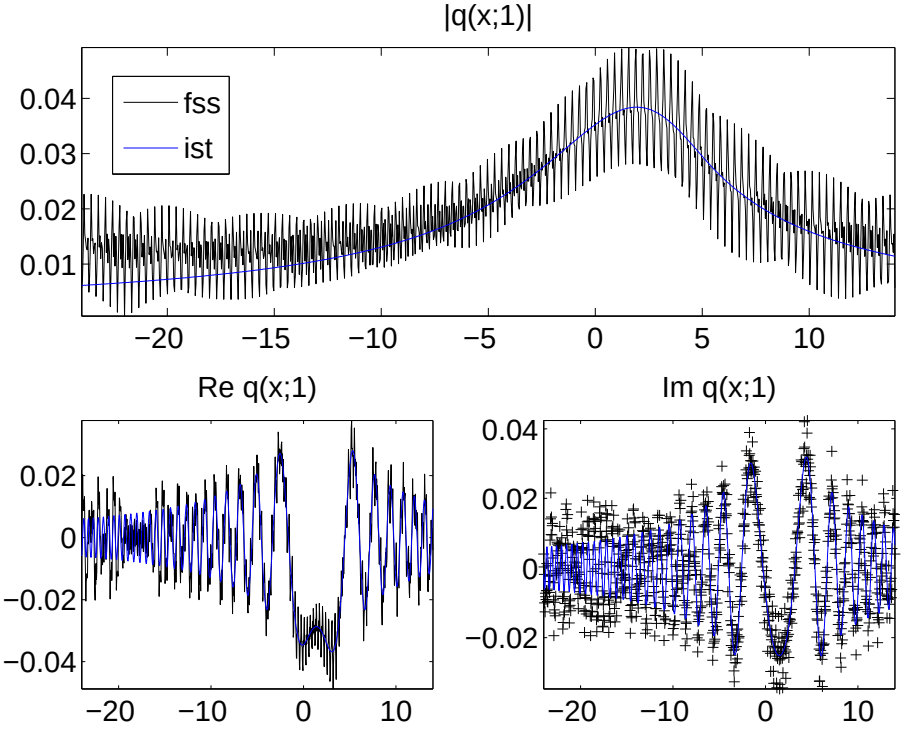
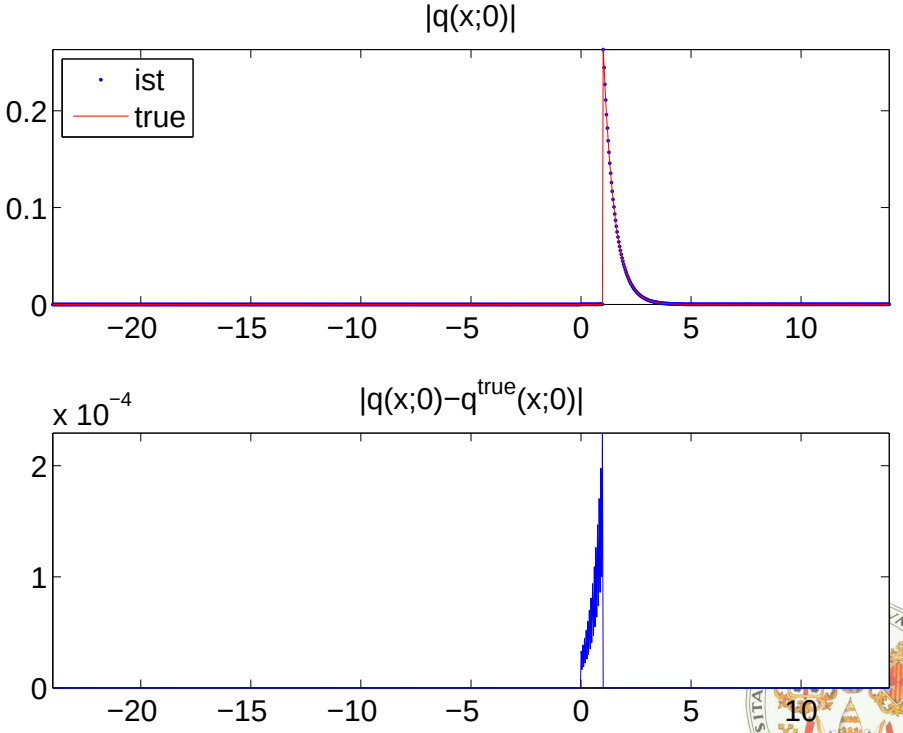
IST & FSS ($h \simeq 0.075$)




IST ($t = 0$)



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