Inverse Problems for Reconstructing the Medium Discontinuities

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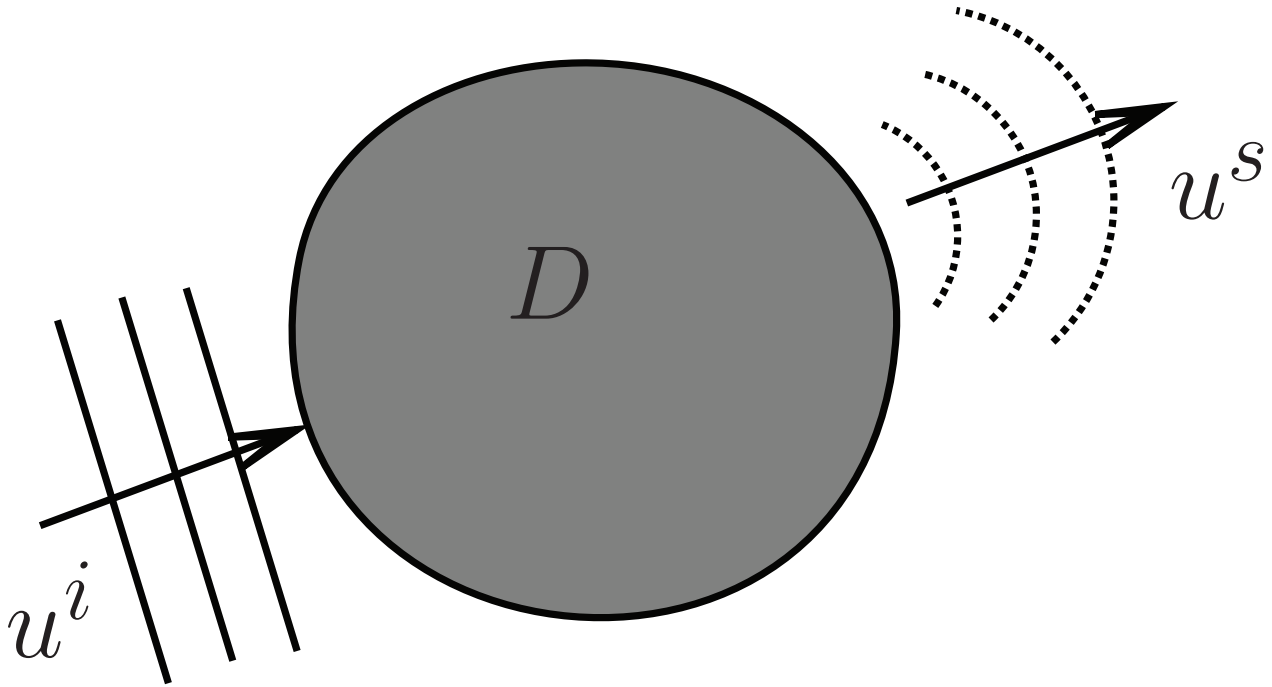
Inverse scattering problem



Joint work with **N.Honda, R.Potthast and M.Sini**

(Ann. Mat. Pura Appl., 2008)

Set up

$D \subset \mathbb{R}^3$: a bounded domain (sound soft obstacle for simplicity) with C^2 boundary ∂D, such that $\mathbb{R}^3 \setminus \overline{D}$ is connected

$u^i(x,d) := e^{i\kappa d\cdot x}$ (incident field or wave) with

$d \in \mathbb{S} := \{d \in \mathbb{R}^3 \,:\, |d| = 1\}$ (incident direction),

$\kappa > 0$ (wave number).

acoustical scattering problem by D for $u \in H^1_{\mathsf{loc}}(\mathbb{R}^3 \setminus \overline{D})$ (total field)

Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \tag{1}$$

Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D \tag{2}$$

Scattered field

Sommerfeld radiation condition

$u^s := u - u^i$ (scattered field or wave) satisfies

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - i\kappa u^s\right) = 0, \tag{3}$$

where $r = |x|$, $\theta := \frac{x}{|x|}$.

asymptotic of u^s

$$u^s(x) = u^s(x,d) = \frac{e^{i\kappa r}}{r} u^\infty(\theta, d) + O(r^{-2}) \quad (r \to \infty), \tag{4}$$

where $u^\infty(\cdot, d)$: far field pattern associated to $u^i(\cdot, d)$

Shape reconstruction problem

Inverse scattering problem

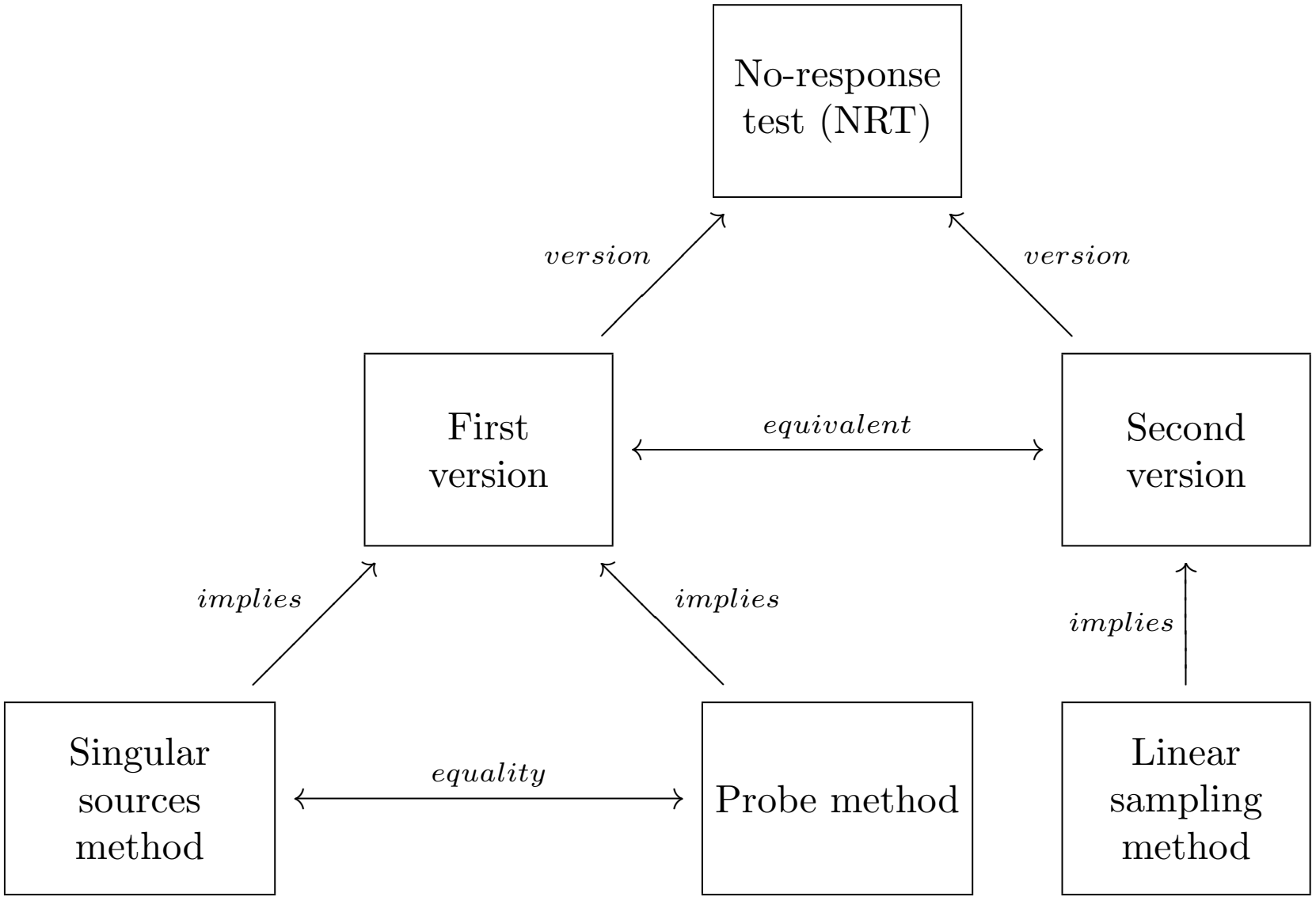
Given $u^\infty(\cdot,\cdot)$ on $\mathbb{S}\times\mathbb{S}$ for the scattering problem (1) - (3) find (reconstruct) the obstacle D.

Known non-iterative reconstruction methods

For example, they are

- *linear sampling method*: D. Colton and A. Kirsch (1996)
- *factorization method*: A. Kirsch (1998)
- *probe method*: M. Ikehata (1998)
- *singular sources method*: R. Potthast (1999)
- *no-response test*: D-R. Luke and R. Potthast (2003)
- *range test*: R. Potthast, J. Sylvester and S. Kusiak (2003)

No-response test view



Notations and terminologies

$$\Phi(x,y) := \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}, x \neq y, x, y \in \mathbb{R}^3$$

$$(\text{fundamental solution of } \Delta + \kappa^2 \text{ in } \mathbb{R}^3)$$

$\Phi^s(\cdot, z)$: scattered field of $\Phi(\cdot, z)$, $z \notin \overline{D}$

$\Phi^\infty(\cdot, z)$: far-field pattern of $\Phi^s(\cdot, z)$

Definition 1

We call a bounded domain B, with C^2 regular boundary, such that $\mathbb{R}^3 \setminus \overline{B}$ is connected a non-vibrating domain if κ^2 is not a Dirichlet eigenvalue for $-\Delta$ in B.

Notations and terminologies continued

For $g \in L^2(\mathbb{S})$,

$$(Hg)(x) = v_g(x) := \int_{\mathbb{S}} e^{i\kappa x \cdot d} g(d)\, ds(d) \;\; (x \in \mathbb{R}^3) \tag{5}$$

(Herglotz wave function)

$$v_g^s(x) := \int_{\mathbb{S}} u^s(x,d) g(d)\, ds(d) \;\; (x \in \mathbb{R}^3 \setminus D) \tag{6}$$

(scattered field associated with Herglotz incident field $v_g^i := v_g$)

$$v_g^\infty(\theta) := \int_{\mathbb{S}} u^\infty(\theta,d) g(d) ds(d) \;\; (\theta \in \mathbb{S}) \tag{7}$$

(far-field pattern of v_g^s)

Representation

$$u^\infty(\theta,d) = \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial u^s(y,d)}{\partial\nu}e^{-i\kappa\theta\cdot y} - \frac{\partial e^{-i\kappa\theta\cdot y}}{\partial\nu}u^s(y,d)\right\}ds(y) \quad (8)$$

where the unit normal ν of ∂D is directed into inside D.

$$\begin{aligned}
&\int_{\mathbb{S}}\int_{\mathbb{S}} u^\infty(-\theta,d)f(\theta)g(d)\,ds(\theta)ds(d)\\
=\;& \frac{1}{4\pi}\int_{\partial D}\left\{\int_{\mathbb{S}}\frac{\partial u^s(y,d)}{\partial\nu}g(d)\,ds(d)\cdot\int_{\mathbb{S}}e^{i\kappa\theta\cdot y}f(\theta)\,ds(\theta)\right.\\
&\left.-\int_{\mathbb{S}}\frac{\partial e^{i\kappa\theta\cdot y}}{\partial\nu}f(\theta)\,ds(\theta)\cdot\int_{\mathbb{S}}\underbrace{u^s(y,d)}_{=-u^i(y,d)}g(d)\,ds(d)\right\}ds(y)\\
=\;& \frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial v_g^s}{\partial\nu}(y)v_f^i(y)+\frac{\partial v_f^i}{\partial\nu}(y)v_g^i(y)\right\}ds(y). \qquad (9)
\end{aligned}$$

1 st version of NRT

Definition 2 (1 st version of the no response method)

For any non-vibrating domain B, *we define the* indicator function *by*

$$I_1(B) := \lim_{\epsilon \to 0} \sup\{I_{1,\epsilon}(f,g) : \ ||v_f||_{L^2(\partial B)} < \epsilon, ||v_g||_{L^2(\partial B)} < \epsilon\} \quad (10)$$

with

$$I_{1,\epsilon}(f,g) := \Big| \int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d) f(\theta) g(d) \ ds(\theta) ds(d) \Big|. \quad (11)$$

For the set $\mathcal{G}$ *of non-vabriting domains* B *no response test calculates the indicator function* $I_1(B)$ *and builds the intersection*

$$D_{rec,1} := \bigcap_{B \in \mathbf{B}_1} B, \quad (12)$$

where

$$\mathbf{B}_1 := \{B \in \mathcal{G} : \ I_1(B) = 0\}. \quad (13)$$

Convergence result NRT

Theorem 3 (Convergence of the first version)

Let $\mathcal{G}$ as in Definition 2. We have

1. *if $\overline{D} \subset B$ then $I_1(B) = 0$,*
2. *if $\overline{D} \not\subset B$ then $I_1(B) = \infty$.*

Thus the unknown obstacle D is given by the intersection of all test domains B for which $I_1(B)$ is zero, i.e

$$\overline{D} = D_{rec,1}.$$

Proof for the case $\overline{D} \subset B$

This is just to see the idea behind.

$||v_g||_{L^2(\partial B)} = ||v_g^i||_{L^2(\partial B)} < \epsilon \Longrightarrow$

$||v_g||_{L^2(B)} < c\epsilon \Longrightarrow ||v_g||_{H^2(D)} < c'\epsilon \Longrightarrow$

$$||v_g^s||_{L^2(\partial D)} < c_1\epsilon, \qquad \left|\left|\frac{\partial v_g^s}{\partial \nu}\right|\right|_{L^2(\partial D)} < c_1\epsilon \tag{14}$$

together with $||v_f||_{H^2(D)} = ||v_f^i||_{H^2(D)} < \tilde{c}\epsilon$

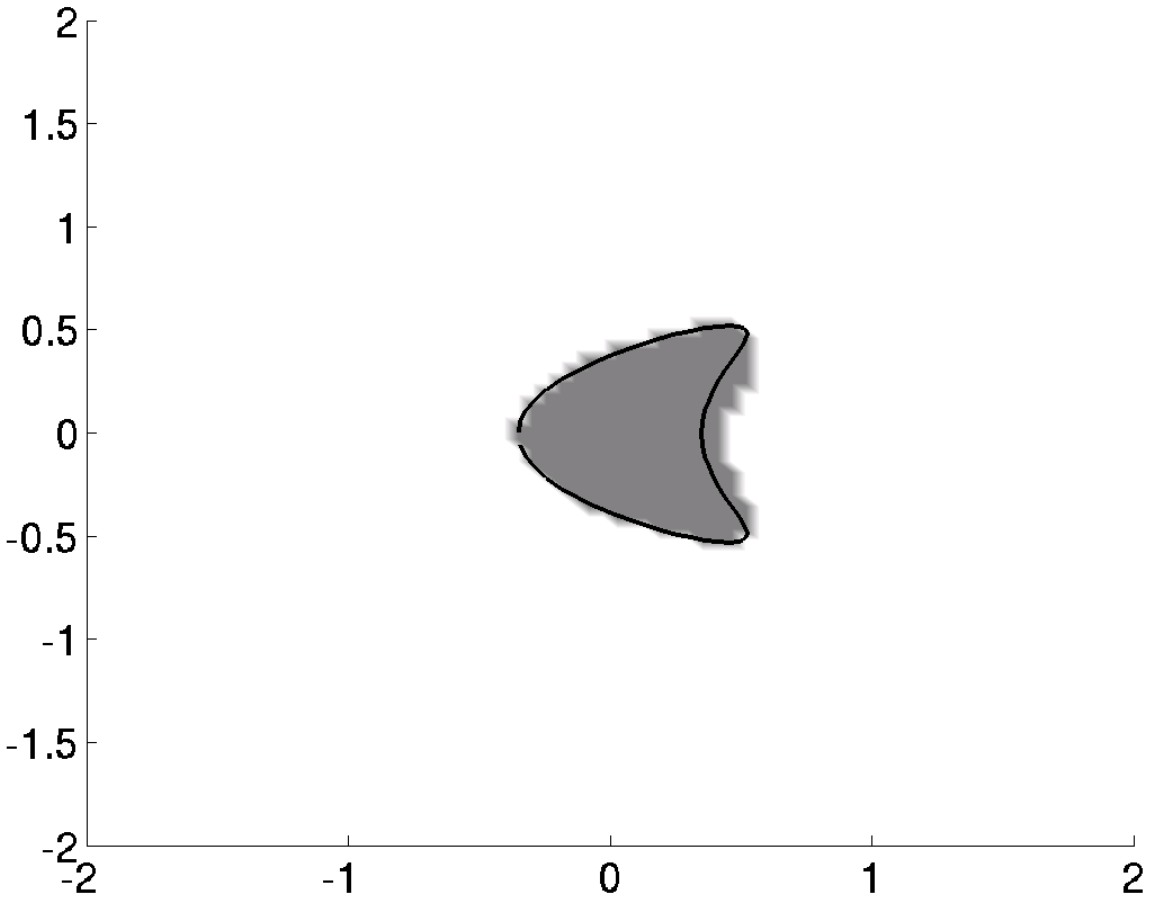
$\Longrightarrow$

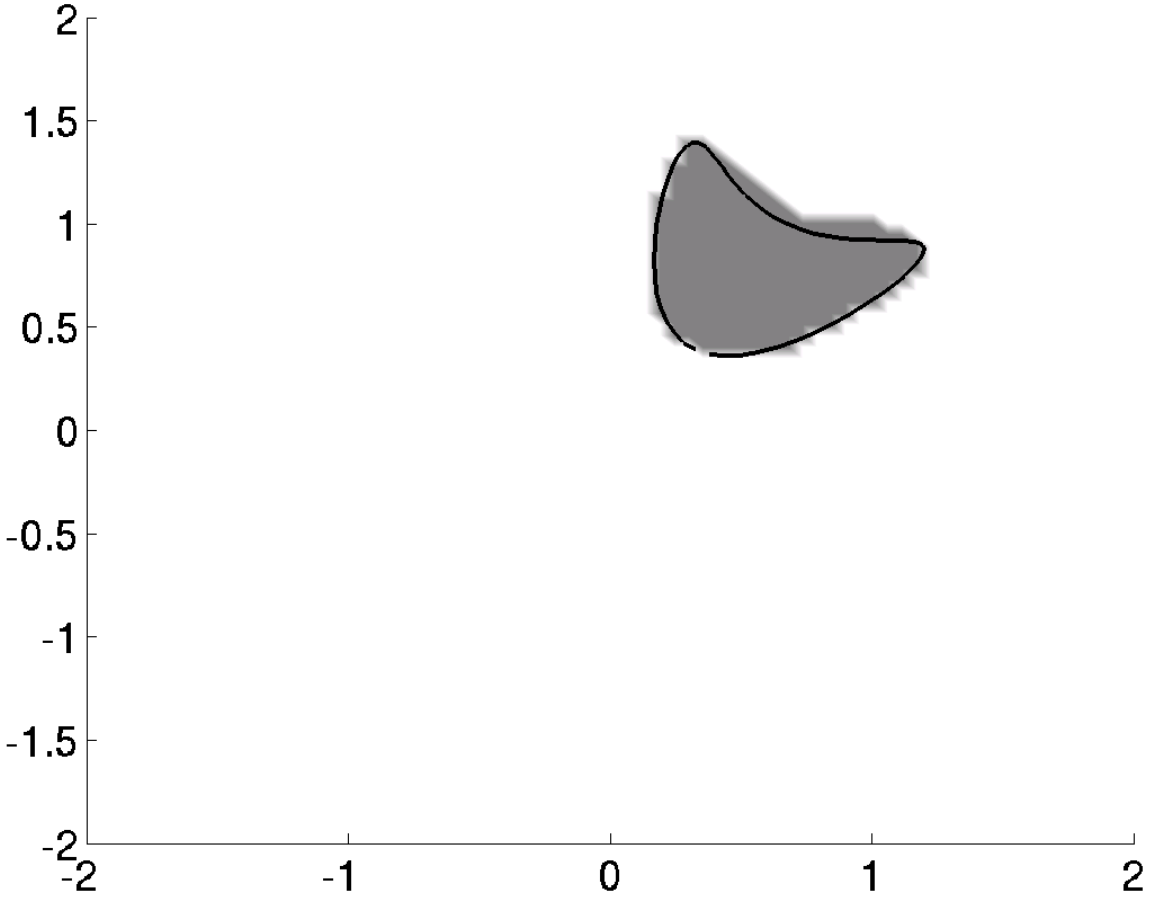
$$I_{1,\epsilon}(f,g) = |\frac{1}{4\pi}\int_{\partial D}\left\{\frac{\partial v_g^s}{\partial \nu}(y)v_f^i(y) + \frac{\partial v_f^i}{\partial \nu}(y)v_g^i(y)\right\} ds(y)| \leq C\epsilon^2 \tag{15}$$

$\Longrightarrow$

$$I_1(B) = \lim_{\epsilon\to 0}\sup\{I_{1,\epsilon}(f,g) : \ ||v_g||_{L^2(\partial B)} < \epsilon, ||v_f||_{L^2(\partial B)} < \epsilon\} = 0. \tag{16}$$

Numerical results



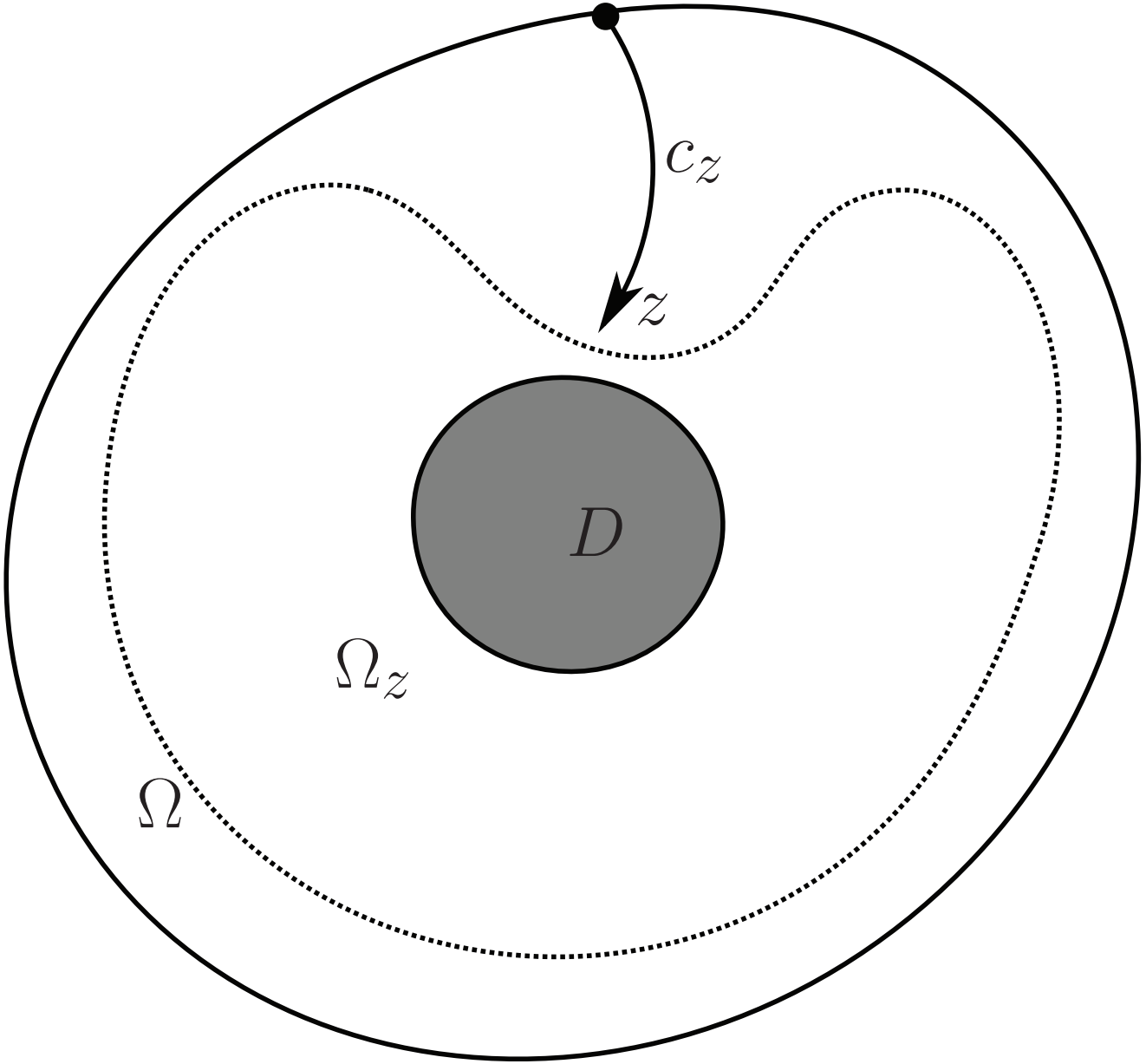

2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1
0
1
2

Singular sources method (SSM)

Suppose $z \in \mathbb{R}^3 \setminus \overline{D}$ and let B be a non-vibrating domain such that $D \Subset B$ and $z \notin \overline{B}$.

$$\left|\left| v_{g_n^z} - \Phi(\cdot, z) \right|\right|_{L^2(\partial B)} \to 0 \ (n \to \infty) \tag{17}$$

$$\begin{aligned} & 4\pi \int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d) g_n^z(\theta) g_n^z(d) \, ds(\theta) ds(d) \\ = & \int_{\partial D} \left\{ \frac{\partial v^s_{g_n^z}}{\partial \nu}(y) v^i_{g_n^z}(y) + \frac{\partial v^i_{g_n^z}}{\partial \nu}(y) v^i_{g_n^z}(y) \right\} ds(y) \qquad (18) \\ \to & \int_{\partial D} \frac{\partial(\Phi^s + \Phi)(y, z)}{\partial \nu(y)} \Phi(x, y) \, ds(y) = \Phi^s(z, z) \ (n \to \infty) \qquad (19) \end{aligned}$$


c_z
z
D
Ω_z
Ω

Indicator function of SSM

Definition 4 (indicator function)

$z \in \Omega \ni D$: *bounded domain*
$\mathcal{C}_z$: *set of continuous curves* c_z (*needles*) *joining* z *to* $\partial\Omega$.
For any curve $c_z \in \mathcal{C}_z$, $z' \in c_z$, *let* $\Omega_{z'} \Subset \Omega \setminus c_{z'}$ *be a non-vibrating domain.*

$\{Hg_n^{z'} = v_{g_n^{z'}}\}$: *Runge's approximation of* $\Phi(\cdot, z')$ *in* $\Omega_{z'}$,

$$I_{ssm}(z, z', c_{z'}, \Omega_{z'}) := 4\pi \limsup_{n\to\infty} \left| \int_{\mathbb{S}} \int_{\mathbb{S}} u^\infty(-\theta, d) g_n^{z'}(\theta) g_n^{z'}(d) ds(\theta) ds(d) \right|$$

$$(= |\Phi^s(z', z')| \text{ if } c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}).$$

Then, we define the indicator function $I(z)$ *by*

$$I(z) := \inf_{c_z \in \mathcal{C}_z} \sup_{z' \in C_z} \sup_{\Omega_{z'}} I_{ssm}(z, z', c_{z'}, \Omega_{z'}).$$

Convergence result for SSM

Theorem 5

The indicator function $I(\cdot)$ satisfies the two properties:
1) If $z \in \Omega \setminus \overline{D}$, then

$$I(z) < \infty.$$

2) If $z \in \overline{D}$, then

$$I(z) = \infty.$$

As a conclusion, the obstacle D is characterized by the indicator function $I(\cdot)$ as follows:

$$\overline{D} = \{z \in \Omega\,;\, I(z) = \infty\}.$$

Probe method

$\Omega \Supset D$: non-vibrating domain

Dirichlet-Neumann map:

$$\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega),$$

where

$\Lambda_D f := \frac{\partial u^f}{\partial \nu} |_{\partial\Omega}$, ν : unit normal of $\partial\Omega$ directed outside of Ω,

$u^f \in H^1(\Omega \setminus \overline{D})$: solution to

$$\begin{cases} \Delta u^f + k^2 u^f = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ u^f = f \quad \text{on} \quad \partial\Omega, \\ u^f = 0 \quad \text{on} \quad \partial D. \end{cases} \tag{20}$$

$\Lambda_\emptyset$: Dirichlet-Neumann map Λ_D when $D = \emptyset$

Indicator function of probe method

Let $z \in \Omega \setminus \overline{D}$, $z' \in c_z$, $c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}$, $\Omega_{z'} \Subset \Omega \setminus c_{z'}$.

Assume z' always keep some distance to $\partial\Omega$.

Approximate $\Phi(\cdot, z')$ on $\partial\Omega_{z'}$ by a sequence of Herglotz waves $v_{g^n_{z'}}$.

$$I_{pb}(z, z', c_{z'}, \Omega_{z'}) := \lim_{n\to\infty} \int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) v_{g^{z'}_n}(x) \cdot v_{g^{z'}_n}(x) ds(x)$$

$$= -\lim_{n\to\infty} \int_{\partial D} \left\{ \frac{\partial}{\partial\nu} v^s_{g^{z'}_n}(x) v_{g^{z'}_n}(x) - \frac{\partial}{\partial\nu} v_{g^{z'}_n}(x) v^s_{g^{z'}_n}(x) \right\} ds(x)$$

$$= -\int_{\partial D} \left\{ \frac{\partial \Phi^s_\Omega}{\partial\nu}(x, z') \Phi(x, z') - \frac{\partial \Phi}{\partial\nu}(x, z') \Phi^s_\Omega(x, z') \right\} ds(x)$$

$$= -\Phi^s_\Omega(z', z') + \int_{\partial\Omega} \frac{\partial \Phi^s_\Omega}{\partial\nu}(x, z') \Phi(x, z') ds(x),$$

where

$v^s_{g^n_{z'}}$, Φ_Ω : solutions of (20) replacing f by $v_{g^n_{z'}}|_{\partial\Omega}$, $\Phi(\cdot, z')|_{\partial\Omega}$,

$\Phi^s_\Omega(x, z') := \Phi_\Omega(x, z') - \Phi(x, z')$ (reflected solution) : solution to

$$\begin{cases} \Delta\Phi^s_\Omega + k^2\Phi^s_\Omega = 0 \quad \text{in} \quad \Omega \setminus \overline{D}, \\ \Phi^s_\Omega(\cdot, z') = 0 \quad \text{on} \quad \partial\Omega, \\ \Phi^s_\Omega(\cdot, z') = -\Phi(\cdot, z') \quad \text{on} \quad \partial D. \end{cases} \tag{21}$$

Hence

$$|I_{pb}(z, z', c_{z'}, \Omega_{z'}) + \Phi^s_\Omega(z', z')| = O(1)$$

By

$$|\Phi^s_\Omega(z', z') + \Phi^s(z', z')| = O(1),$$

we have

Proposition 6

$$|I_{pb}(z, z', c_{z'}, \Omega_{z'}) - I_{ssm}(z, z', c_{z'}, \Omega_{z'})| = O(1),$$

for z *in* $\Omega \setminus \overline{D}$ *near* ∂D, $z' \in c_z$ some distance away from $\partial\Omega$, $c_{z'} \subset \mathbb{R}^3 \setminus \overline{D}$ *and* $\Omega_{z'} \Subset \Omega \setminus c_{z'}$.

Equivalence of SSM and probe method

Theorem 7

The probe method and the singular sources method are identical.

The probe method has the common version as that of the singular sources method.

2nd version of NRT

$B \subset \mathbb{R}^3$: non-vibration domain

The basic idea of the multi-wave range test is to test B by the solvability of the equation

$$S^\infty \psi_g(\theta) := \frac{1}{4\pi} \int_{\partial B} e^{-i\kappa\theta\cdot x} \psi(x) ds(x) = v_g^\infty(\theta, d) \quad (\theta \in \mathbb{S})\ d \in \mathbb{S}. \tag{22}$$

with $g \in L^2(\mathbb{S})$, $||v_g||_{L^2(\partial B)} < \epsilon$.

($S^\infty : L^2(\partial B) \to L^2(\mathbb{S})$ *far field operator of single layer potential*)

We use *Tikhonov regularization scheme* for the regularization of (22):

$$\psi_g^\alpha := (\alpha + S^{\infty,*} S^\infty)^{-1} S^{\infty,*} v_g^\infty \tag{23}$$

with $\alpha > 0$ (*regularization parameter*),

Definition 8 (2nd version of NRT)

For any non-vibrating domain B we define the indicator function

$$I_2(B) := \lim_{\epsilon\to 0}\sup\Big\{\lim_{\alpha\to 0}||\psi_g^\alpha||_{L^2(\partial B)} : \ \psi_g^\alpha \text{ is the regularized solution of } S^\infty\psi_g = v_g^\infty \text{ with } g\in L^2(\mathbb{S}) \text{ and } ||v_g||_{L^2(\partial B)} < \epsilon\Big\} \tag{24}$$

For the set $\mathcal{G}$ of non-vibrating domains B, the 2nd version of NRT calculates the indicator function $I_2(B)$ and builds the intersection

$$D_{rec,2} := \bigcap_{B\in\mathbf{B}_2} B, \tag{25}$$

where

$$\mathbf{B}_2 := \{B\in\mathcal{G} : \ I_2(B) = 0\}. \tag{26}$$

Theorem 9 (Convergence of 2nd NRT)

Let $\mathcal{G}$ be as in Definition 8. We have

1. *if $\overline{D} \subset B$ then $I_2(B) = 0$,*
2. *if $\overline{D} \not\subset B$ then $I_2(B) = \infty$.*

Thus, the obstacle D can be characterized by

$$\overline{D} = D_{rec,2}.$$

Proof of Theorem 9

Just to see the idea behind we only consider the case $\overline{D} \subset B$.

$g \in L^2(\mathbb{S}) : ||v_g||_{L^2(\partial B)} < \epsilon \Longrightarrow ||v_g^s||_{H^1(\partial B)} < C\epsilon$.

Since the operator $S : L^2(\partial B) \to H^1(\partial B)$ defined by

$$(S\psi)(x) := \int_{\partial B} \Phi(x,y)\psi(y)ds(y) \ \ (x \in \partial B) \tag{27}$$

is an isomorphism due to the fact that B is non-vibrating,

$$S\psi_g = v_g^s \text{ on } \partial B \tag{28}$$

has a solution $\psi_g \in L^2(\partial B)$,

$$||\psi_g||_{L^2(\partial B)} < \tilde{c}\epsilon. \tag{29}$$

Hence, ψ_g is the solution of $S^\infty \psi_g = v_g^\infty$, which implies $I_2(B) = 0$.

Convergence of the linear sampling method implies the convergence of the no response method

The linear sampling method

For $z \in \mathbb{R}^3$, observe that the far-field of $\Phi(\cdot, z)$ is given by

$$\Phi_\infty(\theta, z) = \frac{1}{4\pi} e^{-i\kappa\theta\cdot z} \ (\theta \in \mathbb{S}).$$

Consider the far field equation

$$Fg_z = \Phi_\infty(\cdot, z), \tag{30}$$

where the far-field operator $F : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ is given by

$$Fg(\theta) := \int_{\mathbb{S}} u^\infty(\theta, d) g(d) ds(d) \ (\theta \in \mathbb{S}).$$

The idea of the linear sampling method is to approximately solve this equation and look at the behavior of the norms of g_z.

Using the density of the linear sampling method for the 2nd version of NRT

Consider $z \in D$. The linear sampling gives :

$^{\forall}\epsilon > 0$, $^{\exists}g^{\epsilon}(\cdot, z) \in L^2(\mathbb{S})$ such that

$$\|Fg^{\epsilon}(\cdot, z) - \Phi^{\infty}(\cdot, z)\|_{L^2(\mathbb{S})} < \epsilon,$$

$$\lim_{z \to \partial D} \|g^{\epsilon}(\cdot, z)\|_{L^2(\mathbb{S})} = \lim_{z \to \partial D} \|v_{g^{\epsilon}}(\cdot, z)\|_{H^1(D)} = \infty.$$

If $v_{g^{\epsilon}(\cdot,z)} \approx -\Phi(\cdot, z)$ $(z \to \partial D)$ in $H^{1/2}(\partial D)$, we have

Theorem 10

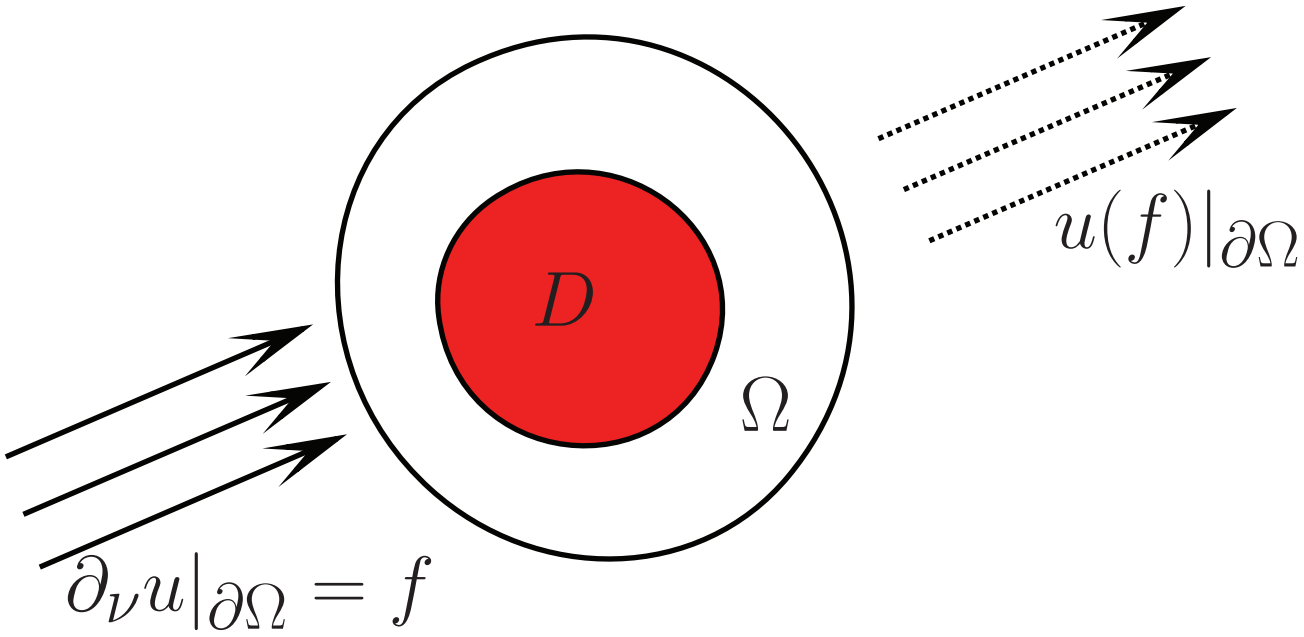
There is a density $\tilde{g}(\cdot, z)$ such that the H^2 norm of $v_{\tilde{g}(\cdot,z)}$ is bounded in a neighborhood of z as $z \to \partial D$ and the density $g_{NRT}(\cdot, z) := g^{\epsilon}(\cdot, z) + \tilde{g}(\cdot, z)$ leads to a blow-up of the functional I_2 of the 2nd version of NRT.

Equivalence of the two versions

Theorem 11

The convergence of the first version implies the convergence of the second version and vice versa.

Dynamical inverse boundary value problem



Mixed problem (set up)

$\Omega \subset \mathbb{R}^n$ $(1 \leq n \leq 3)$: bounded domain, $\partial\Omega$: C^2 $(n = 2, 3)$

$D \Subset \Omega$: open set (**inclusion**), ∂D : C^2, $\Omega \setminus \overline{D}$: connected

heat conductivity:

$$\gamma(x) = 1 + (k-1)\chi_D,$$

where $0 < k(\neq 1)$ is a constant (for simplicity).

$H^p(\partial\Omega), H^{p,q}(\Omega \times (0,T))$: usual Sobolev spaces

$L^2((0,T); H^p(\partial\Omega)) := \{f \ ; \ \int_0^T ||f(\cdot,t)||^2_{H^p(\partial\Omega)} dt < \infty\}$

Mixed problem (forward problem)

Given $f \in L^2((0,T) : (H^{1/2}(\partial\Omega))^*)$,

(?) $\exists!$ weak solution
$u = u(f) \in W(\Omega_T) := \{u \in H^{1,0}(\Omega_T),\ \partial_t u \in L^2((0,T); H^1(\Omega)^*)\}$ such that

$$\begin{cases} \mathcal{P}_D u(x,t) := \partial_t u(x,t) - \mathrm{div}_x(\gamma(x)\nabla_x u(x,t)) = 0 \quad \text{in } \Omega_T \\ \\ \partial_\nu u(x,t) := \nu \cdot \nabla u(x,t) = f(x,t) \quad \text{in } \partial\Omega_T, \quad u(x,0) = 0 \quad \text{for } x \in \Omega, \end{cases}$$

where
$\Omega_T = \Omega_{(0,T)} := \Omega \times (0,T)$, $\partial\Omega_T = \partial\Omega_{(0,T)} := \partial\Omega \times (0,T)$.
(cylindrical sets)

This is a well-posed problem.

Measured data

Neumann-to-Dirichlet map Λ_D:

$$\begin{aligned}\Lambda_D : L^2((0,T);(H^{1/2}(\partial\Omega))^*) &\to L^2((0,T);H^{1/2}(\partial\Omega)) \\ f &\mapsto u(f)|_{\partial\Omega_T}.\end{aligned}$$

Inverse boundary value problem

Identify the unknown inclusion D from Λ_D. Particularly interesting in the reconstruction.

Known results

* H. Bellout (1992): Local uniqueness and stability.

* A. Elayyan and V. Isakov (1997): Global uniqueness using the localized *Neumann-to-Dirichlet map*.

* M. Di Cristo and S. Vessella (2008)(ArXiv preprint and oral communication): Optimal stability estimate (i.e. log type stability estimate)

* Y. Daido, H. Kang and N (2007) (Inverse Problems) : Reconstruction scheme using probing type method (1D).

* Y. Daido, Y. Lai, J. Liu and N (2009) (to appear in Applied Mathematics and Computation) : Numerics for 1D.

* Y. Lei, K. Kim and N (2009) (to appear in Journal of Computational Mathematics) : Numerics for 2D.

* V. Isakov, K. Kim and N (submitted) : Reconstruction scheme using probing type method (3D).

Probe method (fundamental solutions)

For $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ with $(y,s) \neq (y',s')$, let $\Gamma(x,t;y,s)$ be

$$\Gamma(x,t;y,s) = \begin{cases} \dfrac{1}{[4\pi(t-s)]^{n/2}} \exp\Big[-\dfrac{|x-y|^2}{4(t-s)}\Big], & t > s, \\ 0, \quad t \leq s, \end{cases}$$

and $\Gamma^*(x,t;y',s') = \Gamma(x,s';y',t)$ for $(x,t) \in \Omega_T$.

Then,

$$\mathcal{P}_\emptyset \Gamma(x,t;y,s) = 0 \quad \text{if } (x,t) \neq (y,s)$$

$$\mathcal{P}_\emptyset^* \Gamma^*(x,t;y',s') = 0 \quad \text{if } (x,t) \neq (y',s'),$$

where

$$\mathcal{P}_\emptyset = \partial_t - \triangle_x, \quad \mathcal{P}_\emptyset^* = -\partial_t - \triangle_x.$$

Probe method (Runge's approximation functions)

$\exists\ \{v^j_{(y,s)}\},\ \{\varphi^j_{(y',s')}\} \in H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ for $\forall\varepsilon > 0$ s.t.

$$\begin{cases} \mathcal{P}_\emptyset v^j_{(y,s)} = 0 & \text{in} \quad \Omega_{(-\varepsilon,T+\varepsilon)}, \\ v^j_{(y,s)}(x,t) = 0 & \text{if} \quad -\varepsilon < t \le 0, \\ v^j_{(y,s)} \to \Gamma(\cdot,\cdot;y,s) & \text{in} \quad H^{2,1}(U) \quad \text{as} \quad j \to \infty, \end{cases}$$

and

$$\begin{cases} \mathcal{P}^*_\emptyset \varphi^j_{(y',s')} = 0 & \text{in} \quad \Omega_{(-\varepsilon,T+\varepsilon)}, \\ \varphi^j_{(y',s')}(x,t) = 0 & \text{if} \quad T \le t < T+\varepsilon, \\ \varphi^j_{(y',s')} \to \Gamma^*(\cdot,\cdot;y',s') & \text{in} \quad H^{2,1}(U) \quad \text{as} \quad j \to \infty \end{cases}$$

for $\forall U \subset \Omega_{(-\varepsilon,T+\varepsilon)}$: open s.t.

$\overline{U} \subset \Omega_{(-\varepsilon,T+\varepsilon)}$, $(\Omega_{(-\varepsilon,T+\varepsilon)} \setminus \overline{U}) \cap \{t = t_0\}$: connected for each

$t_0 \in (-\varepsilon, T+\varepsilon)$, ∂U : Lipschitz, $\overline{U} \not\ni (y,s),(y',s')$.

$\{v^j_{(y,s)}\}, \{\varphi^j_{(y',s')}\}$: *Runge's approximation functions*

Pre-indicator function

Definition 12

$(y,s),(y',s') \in \Omega_T$ *with* $(y,s) \neq (y',s')$

$\{v^j_{(y,s)}\},\{\varphi^j_{(y',s')}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$: *Runge's approximation functions*

pre-indicator function :

$$I(y',s';y,s)$$
$$= \lim_{j\to\infty} \int_{\partial\Omega_T} \left[\partial_\nu v^j_{(y,s)}|_{\partial\Omega_T}\ \varphi^j_{(y',s')}|_{\partial\Omega_T} - \Lambda_D(\partial_\nu v^j_{(y,s)})|_{\partial\Omega_T}\ \partial_\nu \varphi^j_{(y's')}|_{\partial\Omega_T} \right]$$

whenever the limit exists.

Reflected solution

Lemma 13

$(y,s) \notin \overline{D}$, $\{v^j_{(y,s)}\} \subset H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$: *Runge's approximation functions,*

$u^j_{(y,s)} := u(\partial_\nu v^j_{(y,s)}|_{\partial\Omega_T})$, $w^j_{(y,s)} := u^j_{(y,s)} - v^j_{(y,s)}$

Then, $w^j_{(y,s)}$ *has a limit* $w_{(y,s)} \in W(\Omega_T)$ *satisfying*

$$\begin{cases} \mathcal{P}_D w_{(y,s)} = (k-1)\mathrm{div}_x(\chi_{D(t)}\nabla_x\Gamma(\cdot,\cdot;y,s)) & \text{in } \Omega_T, \\ \\ \partial_\nu w_{(y,s)} = 0 \quad \text{on } \partial\Omega_T, \quad w_{(y,s)}(x,0) = 0 & \text{for } x \in \Omega. \end{cases}$$

$w_{(y,s)}$: reflected solution

Representation formula

Theorem 14

For $(y,s),(y',s') \in \Omega_T \setminus \overline{D}$ *such that* $(y,s) \neq (y',s')$, *the pre-indicator function* $I(y',s';y,s)$ *has the representation formula in terms of the reflected solution* $w_{(y,x)}$ *:*

$$I(y',s';y,s) = -w_{(y,s)}(y',s') - \int_{\partial\Omega_T} w_{(y,s)} \partial_\nu \Gamma^*(\cdot,\cdot;y',s') d\sigma dt$$

Main result (indicator function)

Definition 15

$c := \{c(\lambda)\,;\, 0 \le \lambda \le 1\}$ *: non-selfintersecting* C^1 *curve in* $\overline{\Omega}$*,*

$c(0), c(1) \in \partial\Omega$ *(We call this* c *a needle.)*

Then, for each $c(\lambda) \in \Omega$ *and each fixed* $s \in (0, T)$*, we define an*

indicator function (mathematical testing machine)

$$J(c(\lambda), s) := \lim_{\epsilon \downarrow 0} \limsup_{\delta \downarrow 0} |I(c(\lambda - \delta), s + \epsilon^2; c(\lambda - \delta), s)|$$

whenever the limit exists.

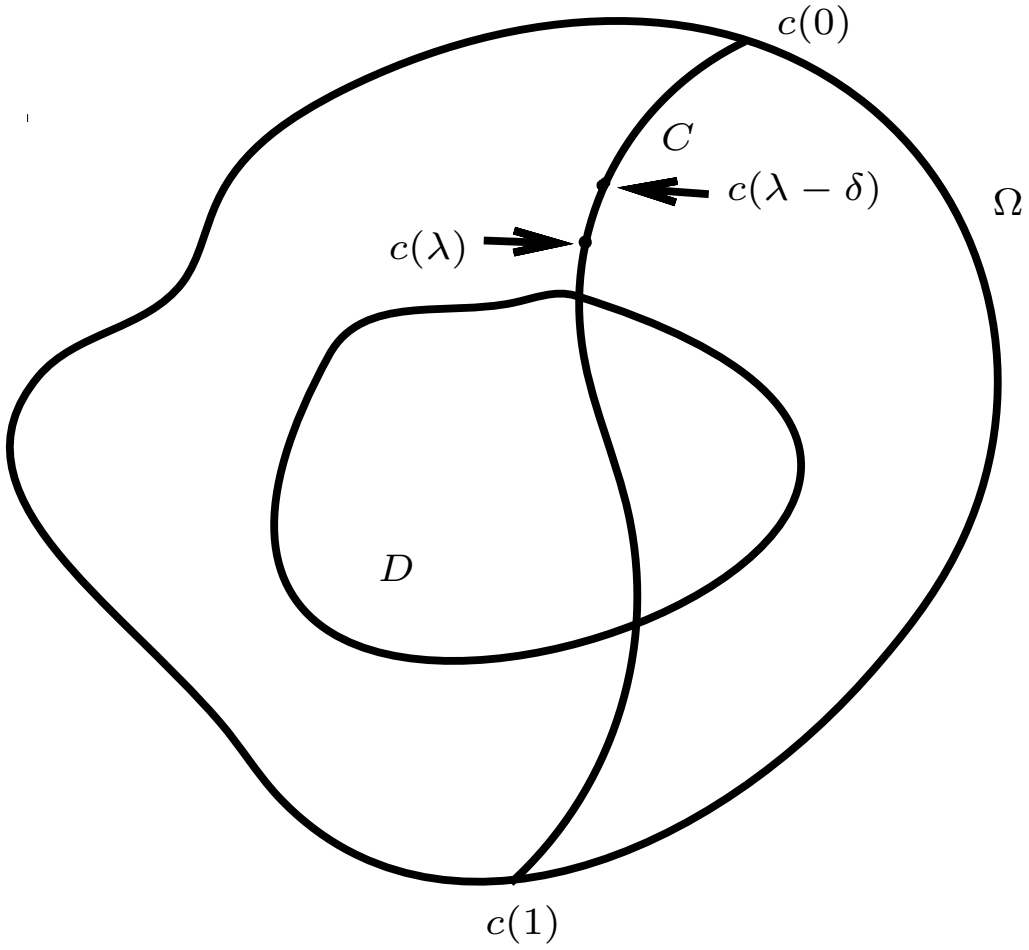


Figure: Domains Ω, D, and a curve C

Main result (theorem)

Theorem 16

c, $c(\lambda)$: *as in Definition above*

Fix $s \in (0, T)$.

(i) $c \subset \Omega \setminus \overline{D}$ *except* $c(0)$ *and* $c(1)$

$\Longrightarrow J(c(\lambda), s) < \infty$ for all λ, $0 \leq \lambda \leq 1$

(ii) $c \cap \overline{D} \neq \emptyset$

λ_s $(0 < \lambda_s < 1)$ *s.t.* $c(\lambda_s) \in \partial D$, $c(\lambda) \in \Omega \setminus \overline{D}$ $(0 < \lambda < \lambda_s)$

$\Longrightarrow$

$$\lambda_s = \sup\{\, 0 < \lambda < 1\,;\, J(c(\lambda'), s) < \infty \quad \text{for any} \quad 0 < \lambda' < \lambda \,\}.$$

Remark :

(i) The theorem is true even for the mixed boundary condition case.

(ii) We also achieved a numerical realization of this reconstruction scheme for one and two space dimensions.

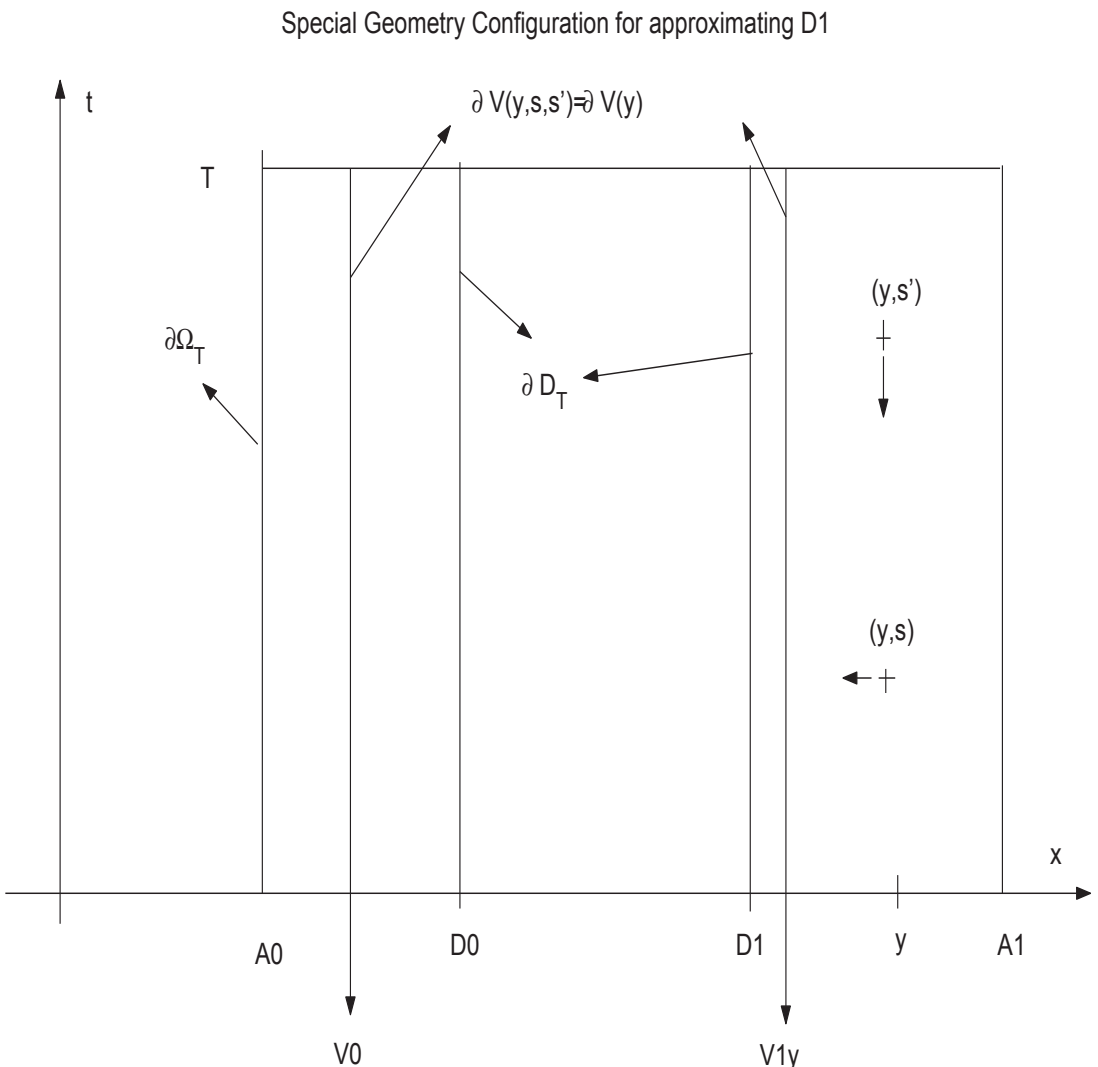
(iii) The theorem is also true for time dependent conductors and anisotropic conductors.

(The only remaining case is for the time dependent anisotropic conductors.)

(iv) By the representation of pre-indicator function in terms of the reflected solution $w_{(y,s)}(x,t)$, the proof of the main theorem reduce to study the behavior of $w_{(y,s)}(x,t)$ as $t \to s$, $y \to \partial D$. This can be achieved by obtaining an upper estimate of the gradient of the fundamental solution of $\mathcal{P}_D$ and extracting the dominant part of the reflected solution.

Numerical realization

1D case



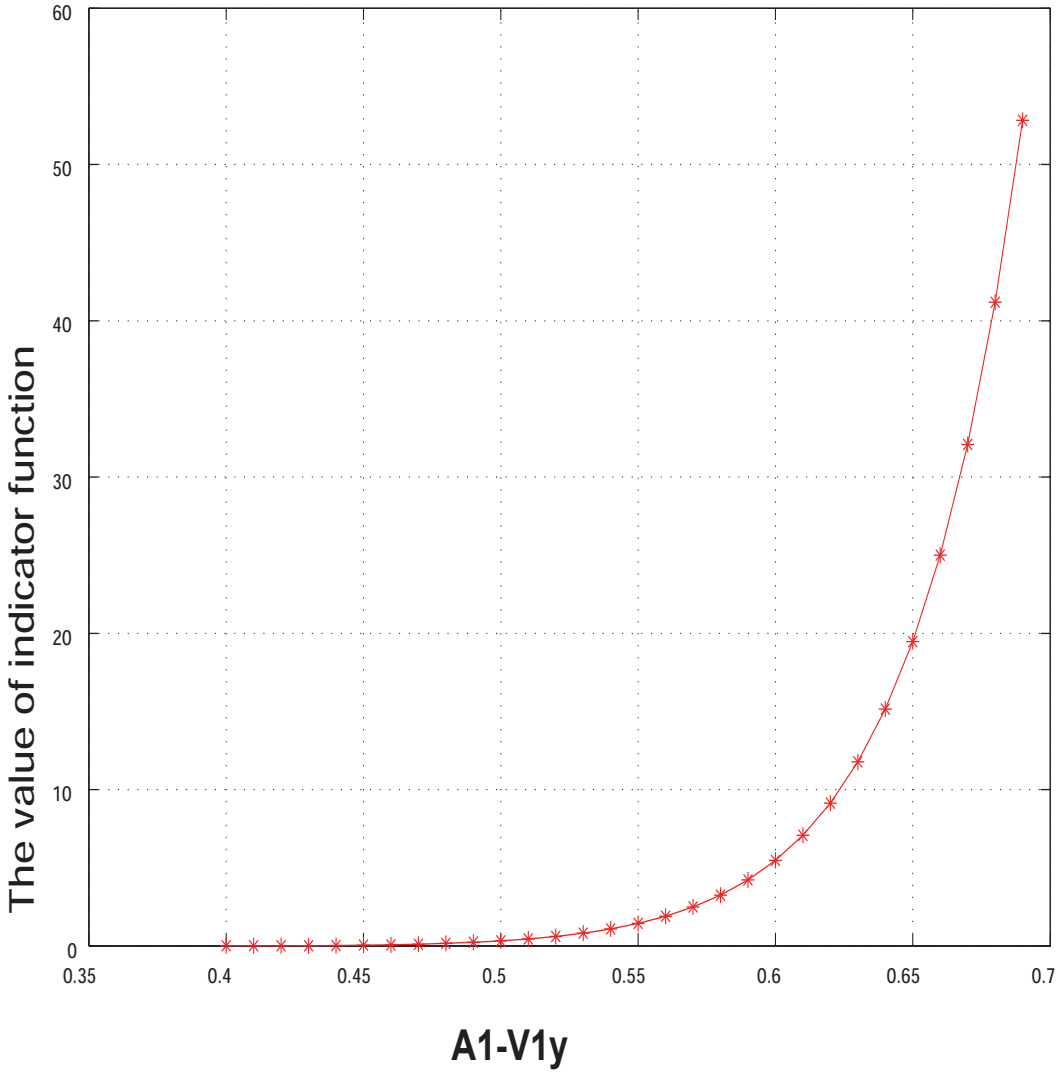


Figure: 1D case ($k = 3,\ D_1 = 0.3,\ N = 40,\ S = 0.3,\ \delta = 0.1$)

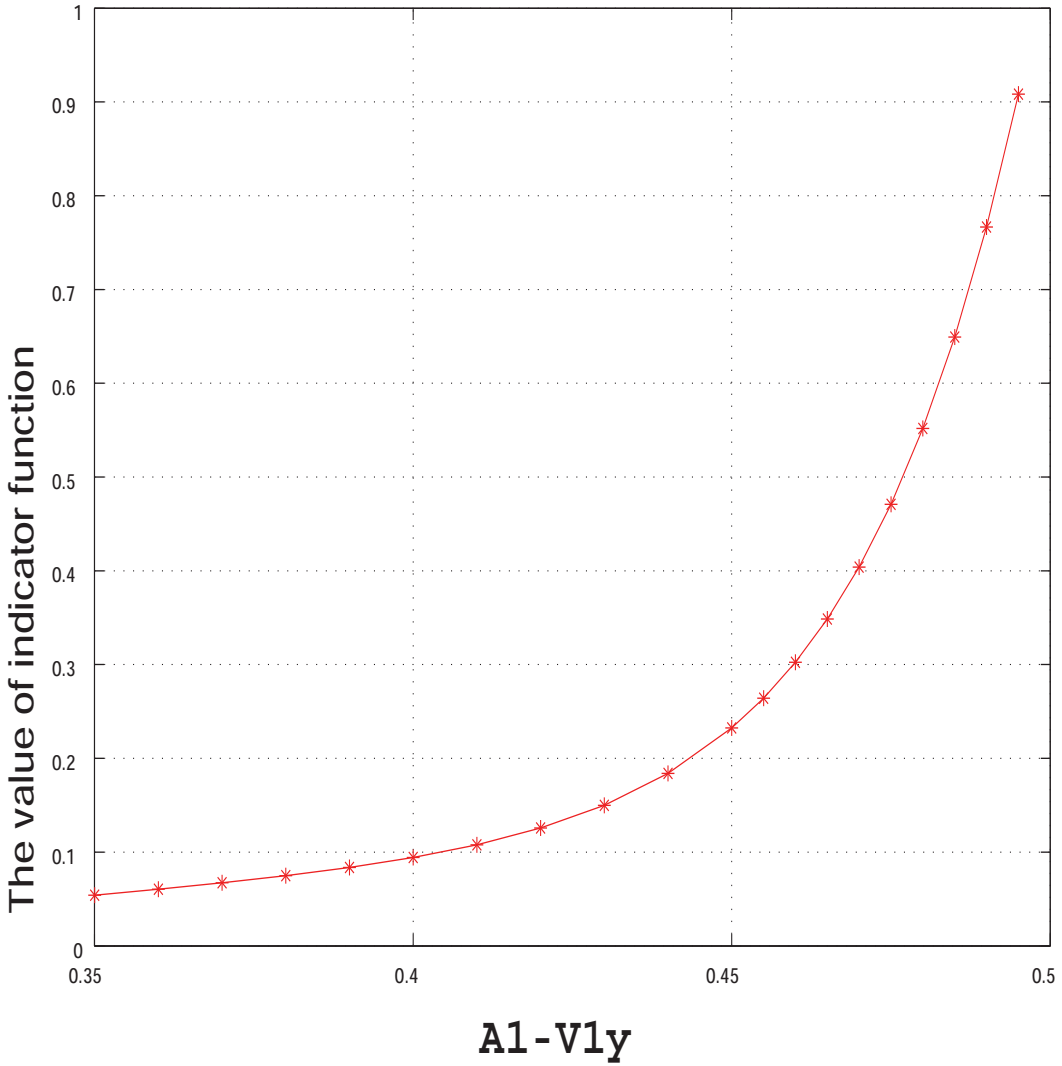


Figure: 1D case ($k = 3,\ D_1 = 0.5,\ N = 40,\ S = 0.3,\ \delta = 0.1$)

2D case

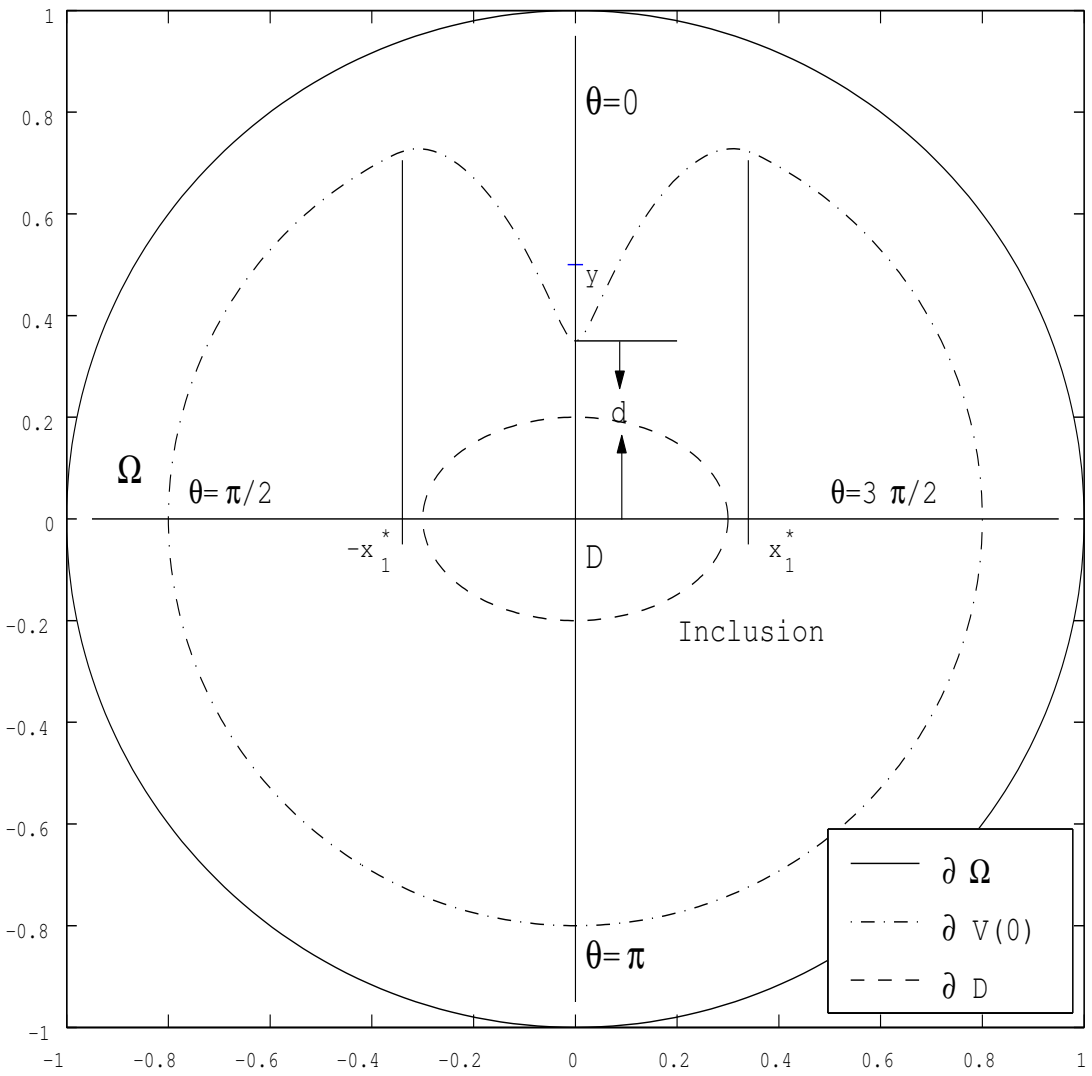


Figure: Approximated domain $V(0)$

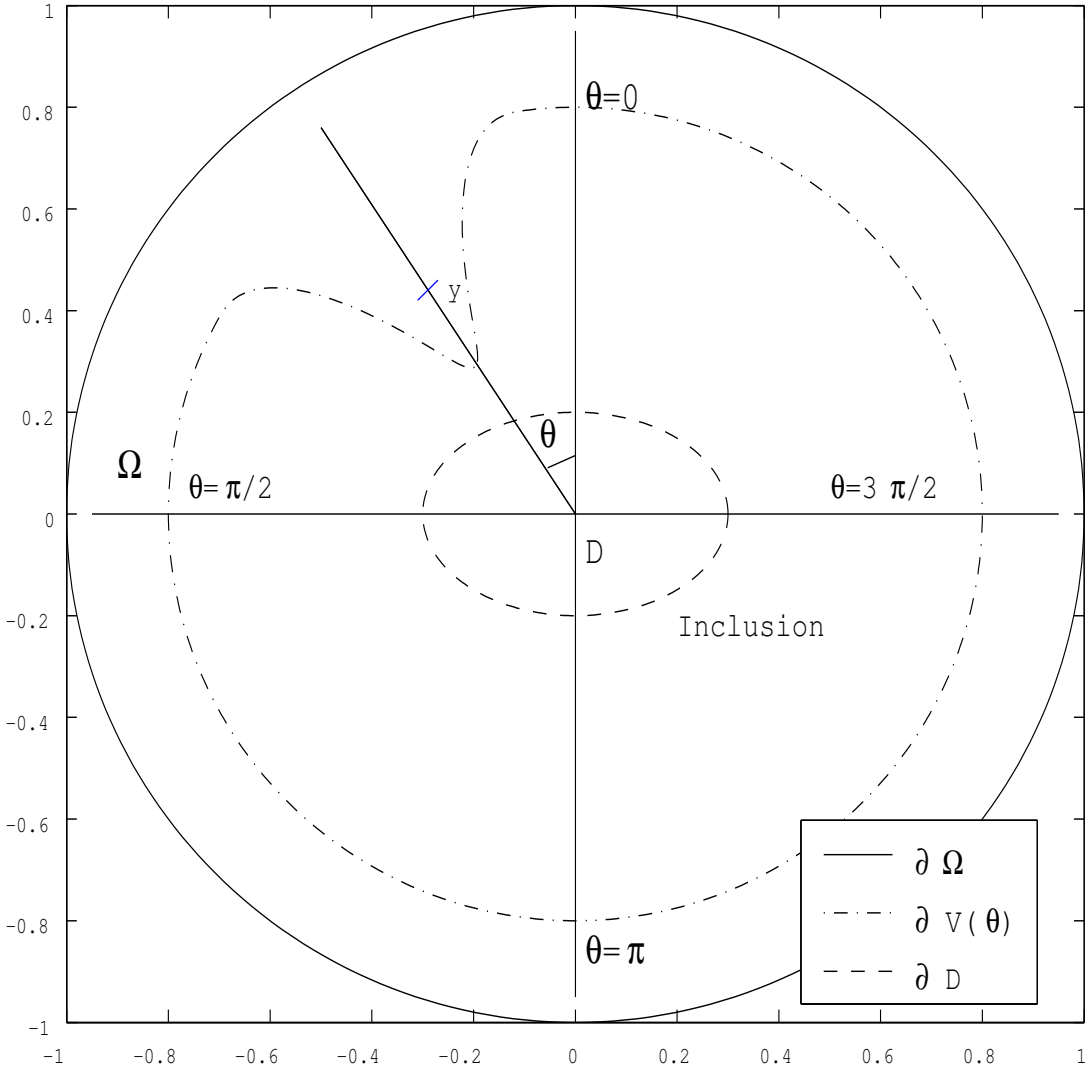


Figure: Approximated domain $V(\theta)$ rotating from $V(0)$

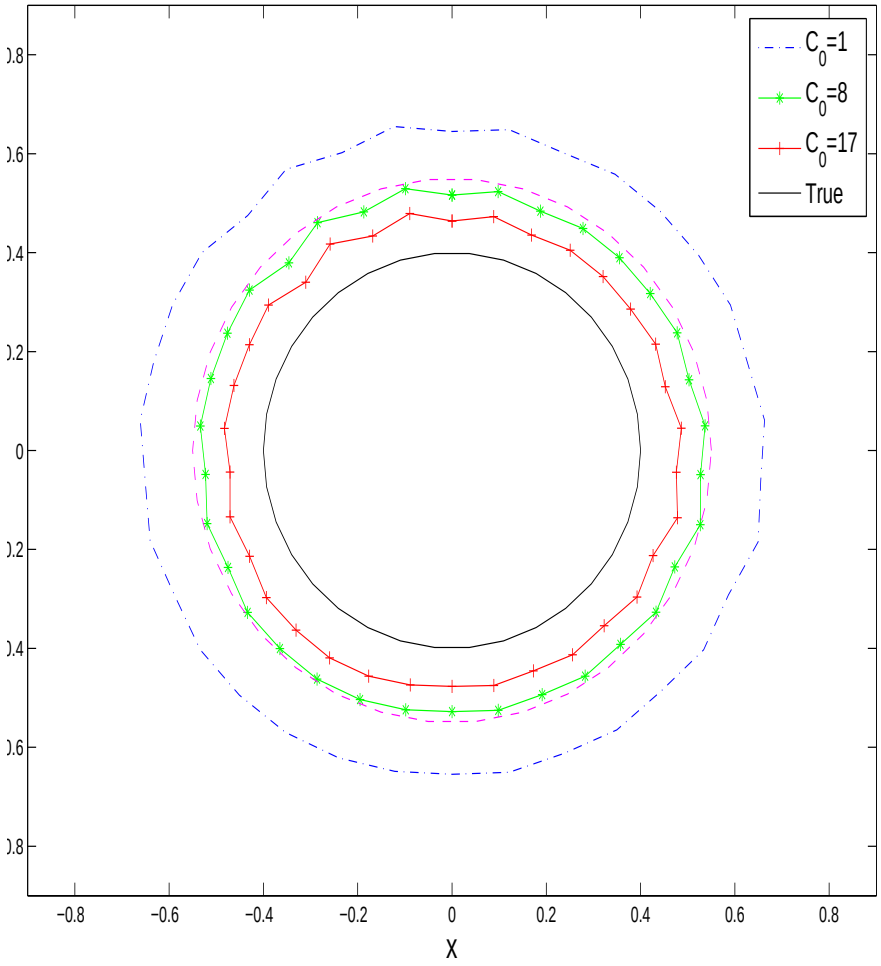


Figure: Reconstruction of D for Example 1

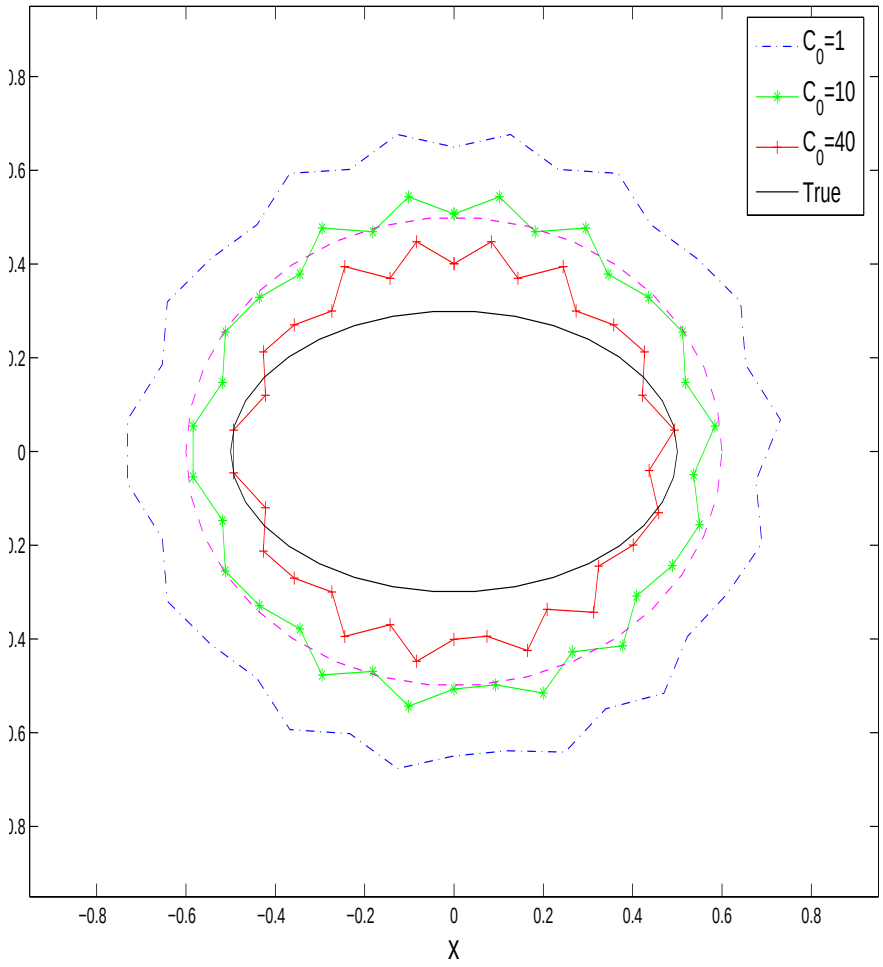


Figure: Reconstruction of D for Example 2

Thank you for your attention.