# Generalized Local Regularization for Ill-Posed Problems

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# Basic problem

**Problem:** Solve

$$\mathcal{A}u = f$$

for $\bar{u} \in \mathrm{dom}(\mathcal{A}) \subset X$, where the "ideal" data $f \in \mathcal{R}(\mathcal{A}) \subset X$ is only approximately given.

We assume

- the operator $\mathcal{A}$ may be linear or nonlinear;
- $X$ is a Hilbert or reflexive Banach space;
- Existence/uniqueness of solution $\bar{u}$ for $f \in \mathcal{R}(\mathcal{A})$, which does not depend continuously on $f$;
- The data is only approximately given by

$$f^\delta \in X, \quad \text{with} \quad \|f - f^\delta\|_X < \delta.$$

(Precise conditions on $\mathcal{A}$ to be given later.)

# Examples of (related) regularization methods

- **Equation for 0th order Tikhonov regularization:**
  - Let $\alpha > 0$ and find $u_\alpha^\delta \in \mathrm{dom}(\mathcal{A}) \subset X$ minimizing
  $$\|\mathcal{A}u - f^\delta\|_X^2 + \alpha\|u\|_X^2.$$
  - If $\mathcal{A}$ <u>bounded linear</u>, the regularized solution $u_\alpha^\delta$ satisfies
  $$\boxed{\mathcal{A}^\star\mathcal{A}u + \alpha u = \mathcal{A}^\star f^\delta.} \qquad (1)$$

<u>**Features**</u> (for generic linear case):

**Pros:**

- Solution of (1) easily handled for $f^\delta \in X$
- "All-purpose" regularizer
- Easy to implement

**Cons :**

- Stability of $(\mathcal{A}^\star\mathcal{A})^{-1}$ vs $\mathcal{A}^{-1}$
- $\mathcal{A}$ nonlinear $\Longrightarrow$ modify or discard (1)
- Ignores special structure of $\mathcal{A}$
- Can be oversmoothing

# Examples of (related) regularization methods

- **Equation for Lavrent'ev (or simplified) regularization:**

In contrast to the Tikhonov regularization equation for linear $\mathcal{A}$,

$$\mathcal{A}^\star \mathcal{A} u + \alpha u = A^\star f^\delta,$$

the method of Lavrent'ev finds $u_\alpha^\delta \in \mathrm{dom}(\mathcal{A})$ satisfying

$$\boxed{\mathcal{A} u + \alpha u = f^\delta.}$$

**Features:**

**Pros:**

- Underlying operator is still $\mathcal{A}$, so preserves structure of equation;
- **Method the same for both linear and nonlinear problems**
- Easy to implement

**Cons:**

- Solvability guaranteed only for certain $\mathcal{A}$
- Numerical results not always satisfactory

# "Diagonal" quantities

**Comparisons**:

Both **Tikhonov** (linear case) and **Lavrent'ev** methods stabilize using the "diagonal quantity" $\alpha u$.

But **local regularization** also involves a "diagonal quantity":

- **Tikhonov** and **Lavrent'ev regularization:**
  - $\alpha u$ **added** to the underlying operator, which makes a small shift of singular values of the operator away from zero.
- **Local regularization:**
  - We first **subtract** a diagonal quantity from the operator ... use this to construct a diagonal $\sim \alpha u$ to be **added** back in. $\Longrightarrow$ $\alpha u$ more closely linked to $\mathcal{A}$ $\Longrightarrow$ can preserve structure
  - The **subtraction** effectively splits the underlying operator (As we'll see later, the splitting should be related to the **sensitivity** of the data to local changes in the solution.)

# Local regularization

- **The equation for <u>local regularization</u>:**

For each $\alpha \in (0, \bar{\alpha}]$ and $X_\alpha \subset X$, let $u_\alpha^\delta \in \mathrm{dom}(\mathcal{A}_\alpha) \subset X_\alpha$ satisfy

$$\boxed{\mathcal{A}_\alpha u + a_\alpha u \;=\; f_\alpha^\delta}$$

for given

- (nonlinear) operator $\mathcal{A}_\alpha : \mathrm{dom}(\mathcal{A}_\alpha) \subset X_\alpha \mapsto X$,
- parameter (function/operator) $a_\alpha(\cdot)$, and for
- $f_\alpha^\delta \in X_\alpha$ defined via

$$f_\alpha^\delta = T_\alpha f^\delta,$$

for a family $\{T_\alpha\}$ of bounded linear operators, $T_\alpha : X \mapsto X_\alpha$,

$$\| T_\alpha f \|_{X_\alpha} \leq \| f \|_X, \quad \text{uniformly in } f \in X.$$

*[E.g., $T_\alpha = I$, or $T_\alpha =$ a (normalized) mollification operator, or a number of other choices.]*

# Compare Tikhonov, Lavrent'ev, and local regularization

Comments about the **local regularization equation**,

$$\mathcal{A}_\alpha u + a_\alpha u \;=\; T_\alpha f^\delta$$

- Tikhonov and Lavrent'ev are special cases, with parameter-independent $\mathcal{A}_\alpha$ and $X_\alpha = X$ for each:

| **Method** | $\mathcal{A}_\alpha$ | $a_\alpha$ | $T_\alpha$ |
|---|---|---|---|
| **Tikhonov:** | $\mathcal{A}^\star\mathcal{A}$ | $\alpha$ *(scalar)* | $\mathcal{A}^\star$ |
| **Lavrent'ev:** | $\mathcal{A}$ | $\alpha$ *(scalar)* | $I$ |

- In **local regularization**, $T_\alpha$ plays the role of "consolidating" a <u>local</u> piece of the function $f^\delta$:

**Tikhonov:** $T_\alpha = \mathcal{A}^\star$ *(not local at all)*

**Lavrent'ev:** $T_\alpha = I$ *(may be <u>too</u> local)*

# Operators/parameters for local regularization

**Selection of $\mathcal{A}_\alpha$, $\mathbf{a}_\alpha$, and $\mathbf{T}_\alpha$ for local regularization method:**

**Motivation:** Look at what's needed to show *stability* and *convergence* to $\bar{u}$ of $u_\alpha^\delta$, the solution of the (generic) equation

$$\mathcal{A}_\alpha u + a_\alpha u = T_\alpha f^\delta. \tag{2}$$

- **To simplify the motivation:**
  - Take operators $\mathcal{A}$, $\mathcal{A}_\alpha$ bounded linear, $\alpha > 0$.
  - Take $X_\alpha = X$, $\alpha > 0$.
- **Will make comparisons with:**
  - *stability*/*convergence* results for Tikhonov/Lavrent'ev (since these methods also take the form of equation (2) ).

# Stability of the three methods

**(1) Stability for methods based on** $\mathcal{A}_\alpha \mathbf{u} + \mathbf{a}_\alpha \mathbf{u} = \mathbf{T}_\alpha \mathbf{f}^\delta$:

Want: solution $u_\alpha^\delta$ unique + depends continuously on data $f^\delta$.

$\Longrightarrow$ need $(\mathcal{A}_\alpha + a_\alpha I)$ **invertible with bounded inverse**.

- This is *easy* for Tikhonov regularization:

  $\mathcal{A}_\alpha = \mathcal{A}^\star \mathcal{A}$ *automatically* nonnegative self-adjoint.

- *Not* automatic for Lavrent'ev or local regularization:

  Need added conditions on $\mathcal{A}_\alpha$.

  E.g., *assume* $\mathcal{A}_\alpha$ nonnegative self-adjoint.

  **Note difference:**

  $\star$ For Lavrent'ev $\Longrightarrow$ assumption is on $\mathcal{A}_\alpha = \mathcal{A}$;

  $\star$ For local reg. $\Longrightarrow$ assumption is on $\mathcal{A}_\alpha$ (we construct).

# Convergence for the three methods

**So assume we already have stability**, or that

$$\|(\mathcal{A}_\alpha + a_\alpha I)^{-1}\| = \mathcal{O}\left(\frac{1}{c(\alpha)}\right)$$

where $c(\alpha) \to 0$ as $\alpha \to 0$; here $\|\cdot\|$ denotes the operator norm.

**(2) Convergence for methods based on $\boldsymbol{\mathcal{A}_\alpha \mathbf{u} + \mathbf{a}_\alpha \mathbf{u} = \mathbf{T}_\alpha \mathbf{f}^\delta}$:**

*Must show:*

For each $f \in \mathcal{R}(A)$ there is a choice of $\alpha = \alpha(\delta, f^\delta)$ for which

$$\alpha(\delta, f^\delta) \to 0 \quad \textit{as} \;\; \delta \to 0,$$

and for which

$$u^\delta_{\alpha(\delta, f^\delta)} \to \bar{u} \quad \textit{as} \;\; \delta \to 0,$$

for all $f^\delta \in X$ satisfying $\|f - f^\delta\|_X \le \delta$.

# Convergence for the three methods, cont'd.

Verification of **convergence** of $u_\alpha^\delta$ to $\bar{u}$, where

$$\mathcal{A}_\alpha u_\alpha^\delta + a_\alpha u_\alpha^\delta = T_\alpha f^\delta. \tag{3}$$

with $\mathcal{A}_\alpha$, $a_\alpha$, $T_\alpha$ given:

- <u>First</u> note $\bar{u} \in X$ uniquely satisfies

$$\begin{aligned} \mathcal{A}\bar{u} &= f, \\ \Longrightarrow \quad T_\alpha \mathcal{A}\bar{u} &= T_\alpha f, \\ \Longrightarrow \quad \mathcal{A}_\alpha \bar{u} + a_\alpha \bar{u} &= T_\alpha f + (\mathcal{A}_\alpha \bar{u} + a_\alpha \bar{u} - T_\alpha \mathcal{A}\bar{u}). \end{aligned} \tag{4}$$

- <u>So subtract</u>: equation (3)−(4) to obtain

$$(\mathcal{A}_\alpha + a_\alpha I)(u_\alpha^\delta - \bar{u}) = T_\alpha\, d^\delta \; - \; (\mathcal{A}_\alpha \bar{u} + a_\alpha \bar{u} - T_\alpha \mathcal{A}\bar{u}),$$

where $d^\delta \equiv (f^\delta - f) \in X$ satisfies $\|d^\delta\|_X \le \delta$.

# Convergence for the three methods, cont'd.

We then have

$$\begin{aligned}(u_\alpha^\delta - \bar{u}) &= (\mathcal{A}_\alpha + a_\alpha I)^{-1}\, T_\alpha d^\delta \\ &\quad + (\mathcal{A}_\alpha + a_\alpha I)^{-1}\left[(T_\alpha \mathcal{A}\bar{u} - \mathcal{A}_\alpha \bar{u}) - a_\alpha \bar{u}\right],\end{aligned}$$

where

$$\left\|(\mathcal{A}_\alpha + a_\alpha I)^{-1}\, T_\alpha d^\delta\right\|_X = \mathcal{O}\left(\frac{\delta}{c(\alpha)}\right).$$

**Goal:** select $\alpha = \alpha(\delta, f^\delta)$ so that $\alpha(\delta, f^\delta) \to 0$ as $\delta \to 0$, and so that both terms above in $(u_\alpha^\delta - \bar{u})$ go to zero as $\delta \to 0$. That is,

- **The term due to data error goes to zero**, i.e.,

$$\frac{\delta}{c\,(\,\alpha(\delta, f^\delta)\,)} \to 0 \quad as\ \ \delta \to 0.$$

- **The term due to method error goes to zero:**

$$(\mathcal{A}_\alpha + a_\alpha I)^{-1}\,(D_\alpha \bar{u} - a_\alpha \bar{u}) \to 0 \quad as\ \ \delta \to 0.$$

Here we have defined $D_\alpha = T_\alpha \mathcal{A} - \mathcal{A}_\alpha$.

# Convergence for the three methods, cont'd.

**Note:**

• There is generally little difficulty in making an *a priori* choice of $\alpha = \alpha(\delta, f^\delta)$ so that $\alpha(\delta, f^\delta) \to 0$ and

$$\frac{\delta}{c\,(\,\alpha(\delta, f^\delta)\,)} \to 0 \quad \textit{as} \;\; \delta \to 0.$$

• The problem then is to show that for a selection of $\alpha$-values with $\alpha \to 0$, the operator $(\mathcal{A}_\alpha + a_\alpha I)$ is continuously invertible, and

$$(\mathcal{A}_\alpha + a_\alpha I)^{-1}\,(D_\alpha \bar{u} - a_\alpha \bar{u}) \to 0, \quad \textit{as} \quad \alpha \to 0.$$

It is well-known that these last statements are true (for **linear** $\mathcal{A}$) in the case of:

- **Tikhonov regularization**, for $\mathcal{A}$ bounded;
- **Lavrent'ev regularization**, for a narrower class of operators $\mathcal{A}$ (e.g. $\mathcal{A}$ nonnegative self-adjoint).

# Local regularization: choice of $\mathcal{A}_\alpha$ and $a_\alpha$

For **local regularization**, we'll use the desired convergence condition,

$$(\mathcal{A}_\alpha + a_\alpha I)^{-1}(D_\alpha \bar{u} - a_\alpha \bar{u}) \to 0 \quad \textit{as} \quad \alpha \to 0, \tag{5}$$

to <u>motivate the selection of</u> $\mathcal{A}_\alpha$ <u>and</u> $a_\alpha$. (Recall $D_\alpha \equiv T_\alpha \mathcal{A} - \mathcal{A}_\alpha$.)

- That is, $\bar{u}$ satisfies

$$\begin{aligned} \mathcal{A}\bar{u} &= f \\ \Longrightarrow \quad T_\alpha \mathcal{A}\bar{u} &= T_\alpha f, \end{aligned}$$

  where the operator $T_\alpha \mathcal{A}$ may be <u>split</u> as follows:

$$\begin{aligned} T_\alpha \mathcal{A} &= \mathcal{A}_\alpha + (T_\alpha \mathcal{A} - \mathcal{A}_\alpha) \\ \Longrightarrow \quad T_\alpha \mathcal{A} &= \mathcal{A}_\alpha + D_\alpha. \end{aligned}$$

- Using (5), this splitting of $T_\alpha \mathcal{A}$ needs to be such that (roughly)

$$D_\alpha \bar{u} \approx a_\alpha \bar{u} \qquad \textit{for} \quad \alpha \approx 0,$$

  for some choice of $a_\alpha$, i.e., $D_\alpha \approx$ "diagonal" when applied to $\bar{u}$.

# Local regularization: choice of $\mathcal{A}_\alpha$ and $a_\alpha$

So in **local regularization**:

- We **first select** $T_\alpha$ (selection criteria to be discussed later).
- We **then select** $\mathcal{A}_\alpha$ and $a_\alpha$ (and define $D_\alpha \equiv T_\alpha \mathcal{A} - \mathcal{A}_\alpha$) so that for $\alpha > 0$ we have:
  1. $(\mathcal{A}_\alpha + a_\alpha I)$ continuously invertible, and
  2. the operator $(D_\alpha - a_\alpha I)$ satisfies

$$\left\| (\mathcal{A}_\alpha + a_\alpha I)^{-1} (D_\alpha - a_\alpha I)\, \bar{u} \right\|_X \to 0 \quad as \quad \alpha \to 0.$$

Roughly, we take the modified equation $T_\alpha \mathcal{A} u = T_\alpha f$, and split its leading operator via $T_\alpha \mathcal{A} = \mathcal{A}_\alpha + D_\alpha$, where

- $\mathcal{A}_\alpha =$ the "bigger, nearly continuously invertible" part of $T_\alpha \mathcal{A}$,
- $D_\alpha =$ the "smaller, diagonal-like" part of $T_\alpha \mathcal{A}$.

# Statement of convergence

The above statement of convergence, namely,

$$\left\| (\mathcal{A}_\alpha + a_\alpha I)^{-1} (D_\alpha - a_\alpha I)\, \bar{u} \right\|_X \to 0 \quad as \quad \alpha \to 0. \tag{6}$$

is obtained for **Tikhonov** & **Lavrent'ev**, <u>without</u> splitting $T_\alpha \mathcal{A}$:

- **Tikhonov:** $T_\alpha \mathcal{A} = \mathcal{A}_\alpha = \mathcal{A}^\star \mathcal{A}, \quad D_\alpha = 0, \quad a_\alpha = \alpha.$
- **Lavrent'ev:** $T_\alpha \mathcal{A} = \mathcal{A}_\alpha = \mathcal{A}, \quad D_\alpha = 0, \quad a_\alpha = \alpha.$

  In these cases the convergence as $\alpha \to 0$ in (6) takes the form (respectively) of

$$\left\| (\mathcal{A}^\star A + \alpha I)^{-1} (-\alpha \bar{u}) \right\|_X \to 0, \quad \left\| (\mathcal{A} + \alpha I)^{-1} (-\alpha \bar{u}) \right\|_X \to 0.$$

**<u>In contrast</u>: With** *local regularization* **we can obtain a <u>stronger</u> statement of convergence (than (6)) for many problems.**

# Local regularization: choice of $\mathcal{A}_\alpha$ and $a_\alpha$

**Stronger statement of convergence:** A stronger result than

$$\left\| (\mathcal{A}_\alpha + a_\alpha I)^{-1} (D_\alpha \bar{u} - a_\alpha \bar{u}) \right\|_X \to 0 \quad \text{as} \quad \alpha \to 0,$$

is obtained if the method satisfies

$$\|D_\alpha \bar{u} - a_\alpha \bar{u}\|_X = o\left(c(\alpha)\right) \quad \text{as} \quad \alpha \to 0 \tag{7}$$

(recall that $\|(\mathcal{A}_\alpha + a_\alpha I)^{-1}\| = \mathcal{O}(\frac{1}{c(\alpha)})$ ).

- **The stronger result (7) <u>cannot</u> hold for Tikhonov or Lavrent'ev** (in general). For these methods the usual case is

$$\left\| (\mathcal{A}_\alpha + \alpha I)^{-1} \right\| = \mathcal{O}\left(\tfrac{1}{\alpha}\right) \quad \text{as} \quad \alpha \to 0,$$

i.e., $c(\alpha) = \alpha$, but $\|D_\alpha \bar{u} - a_\alpha \bar{u}\|_X = \|-\alpha \bar{u}\|_X = \mathcal{O}(\alpha)$

$$\Longrightarrow \quad \|D_\alpha \bar{u} - a_\alpha \bar{u}\|_X \neq o\left(c(\alpha)\right) \quad \text{as} \quad \alpha \to 0.$$

# Local regularization: the selection of $T_\alpha$

**How to select $T_\alpha$ in the <u>method of local regularization</u> ?**

- **We will select $T_\alpha$ in order to:**
  - Facilitate a regularization method which is "localized" in an appropriate sense.
  - Work toward the goal of finding a method which is easier/faster than classical regularization methods.
  - Effectively utilize information about the sensitivity of the data $f$ to (local) changes in $u$ (for $f$, $u$ satisfying the $\mathcal{A}u = f$).

We will demonstrate the selection of $T_\alpha$, as well as of $\mathcal{A}_\alpha$ and $a_\alpha$, through some examples.

# Examples of local regularization

**Three motivating examples** ... from where local regularization theory is the most complete, i.e., for *Volterra operators*.

1. **Linear problem** on $X = L^p(0,1)$, $1 < p \leq \infty$:

$$\int_0^t k(t,s)u(s)\,ds = f(t), \quad a.e. \;\; t \in [0,1].$$

2. **Nonlinear Hammerstein problem** on $X = C[0,1]$:

$$\int_0^t k(t,s)g(s,u(s))\,ds = f(t), \quad a.e. \;\; t \in [0,1],$$

under (standard) conditions on $g$.

3. **Nonlinear autoconvolution problem**, $X = L^2(0,1)$, $C[0,1]$:

$$\int_0^t u(t-s)u(s)\,ds = f(t), \quad a.e. \;\; t \in [0,1].$$

# Linear Volterra example

- First consider the **linear** Volterra problem

$$\mathcal{A}u(t) = \int_0^t k(t,s)u(s)\,ds = f(t), \quad a.e. \;\; t \in [0,1],$$

in the special case of $X = L^2(0,1)$, and for $k \in L^2([0,1]\times[0,1])$.

- To determine an appropriate choice for $T_\alpha$, we first look at **sensitivities** of the data $f = \mathcal{A}u$ to **local parts** of $u$:

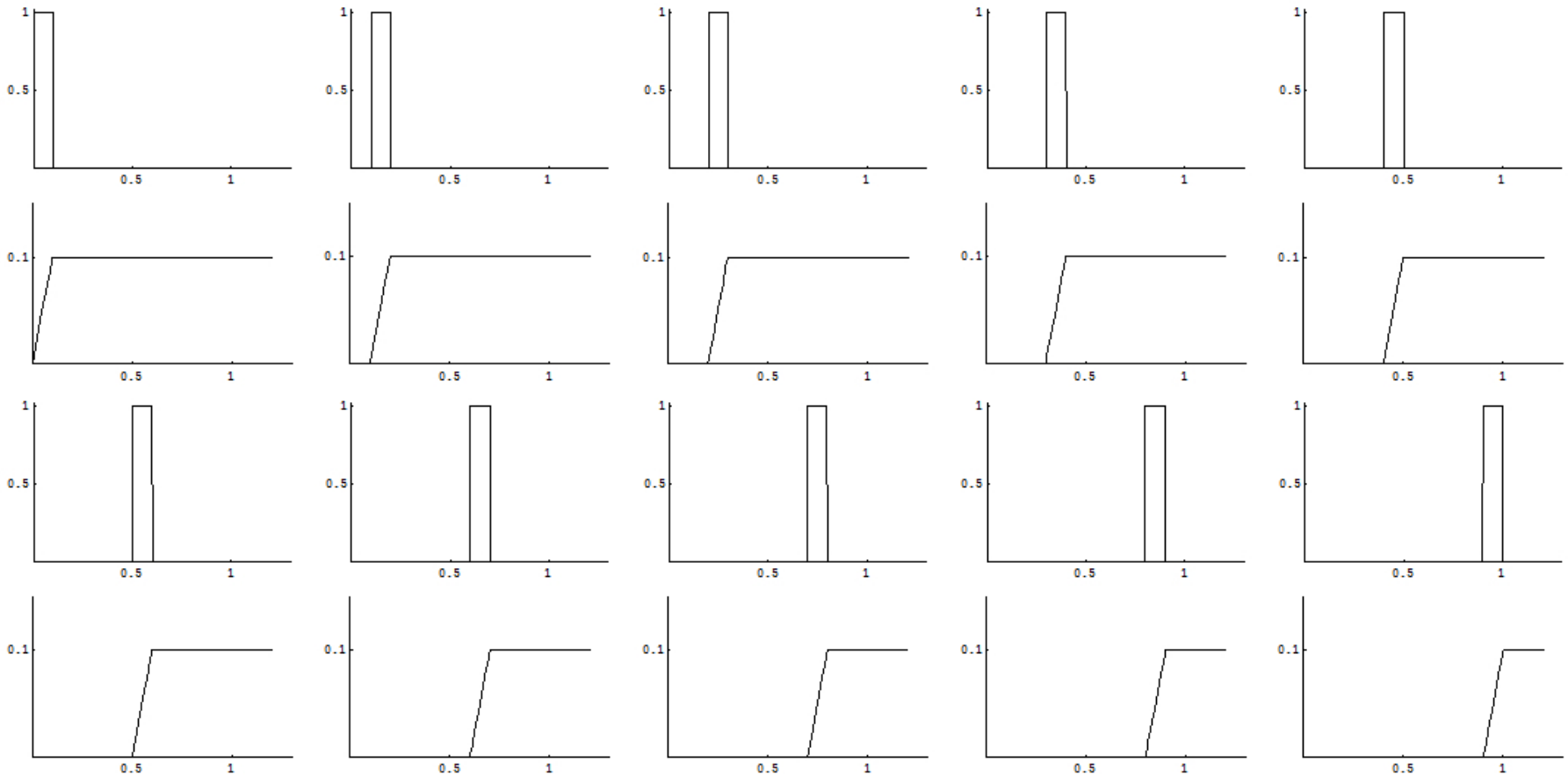
$\mathcal{A}$ is bounded linear on $X \implies$ the Frechet differential of $\mathcal{A}$ at $u \in X$ with increment $\Delta u \in X$ is given by

$$\mathcal{D}\mathcal{A}(u;\Delta u) = \mathcal{A}\Delta u,$$

i.e.,

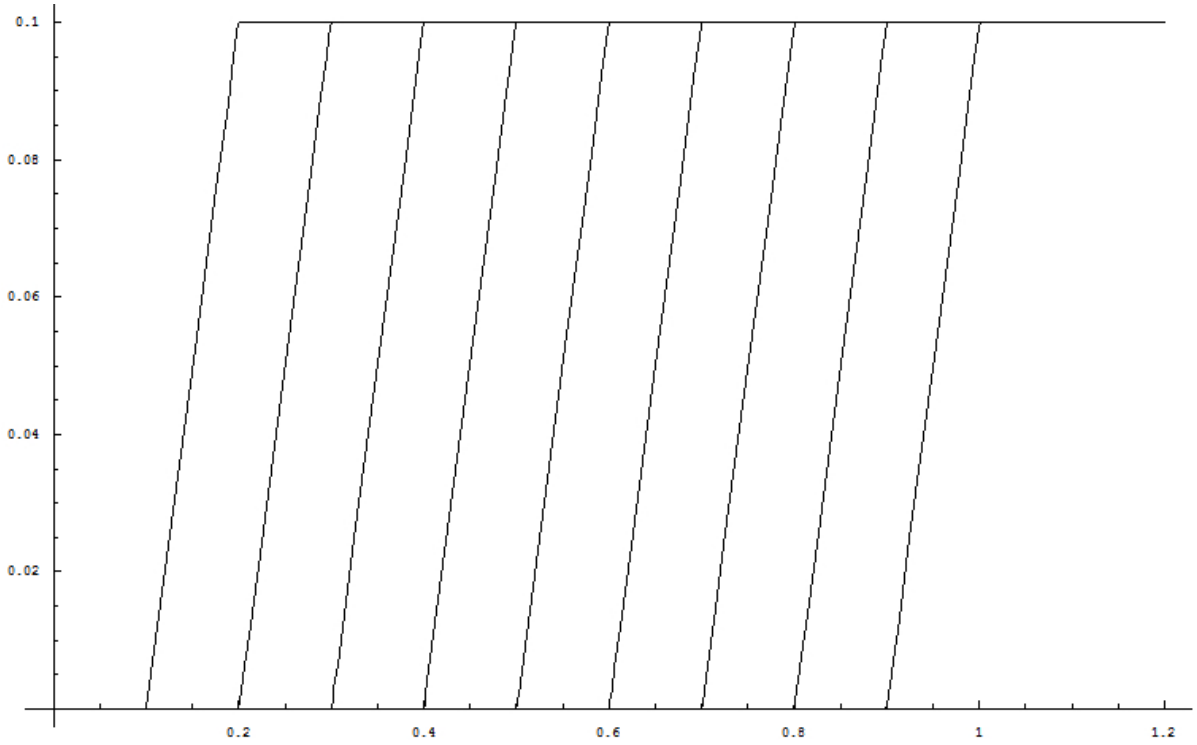
$$\mathcal{D}\mathcal{A}(u;\Delta u)(t) = \int_0^t k(t,s)\Delta u(s)\,ds, \quad a.e. \;\; t \in [0,1].$$

# Sensitivities of data for linear Volterra problem



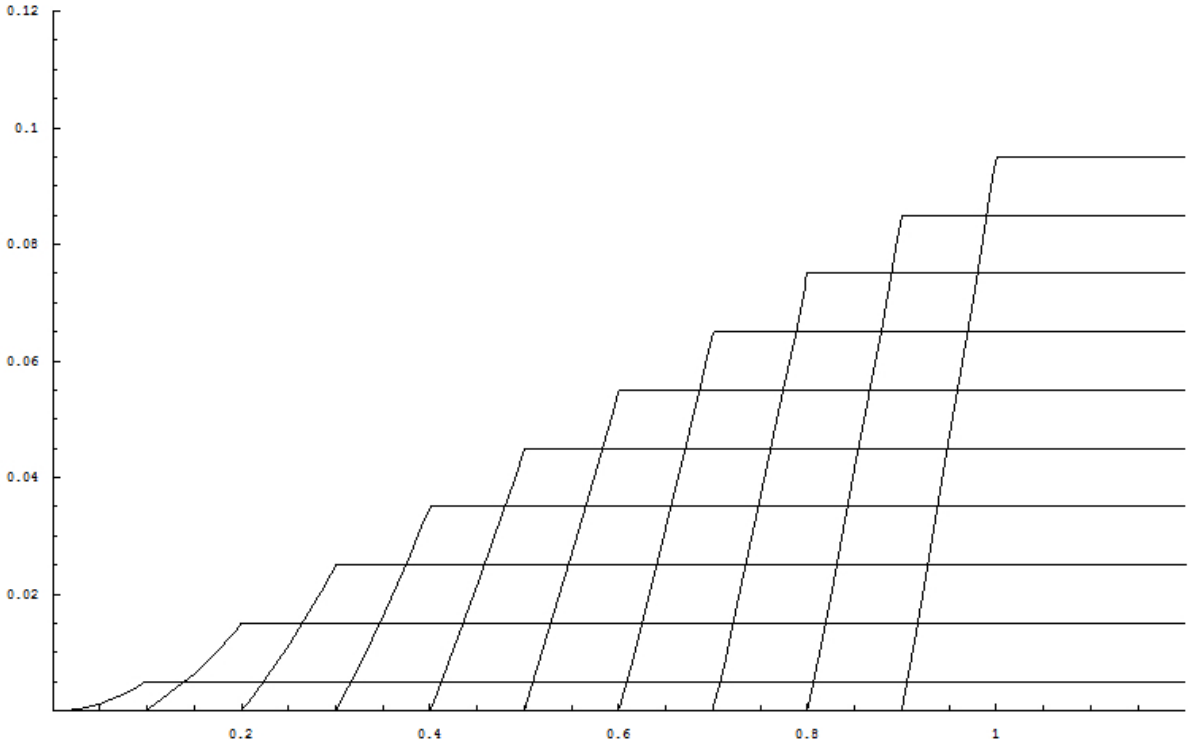

Graphs of: (1) $\Delta u =$ "unit impulse" of width .1, (1st & 3rd rows of graphs above), and (2) the response $\mathcal{D}\mathcal{A}(u; \Delta u)$ (2nd & 4th rows), immediately below each $\Delta u$. (Here the kernel for $\mathcal{A}$ is $k(t, s) = 1$.)

# Sensitivities of data for linear Volterra problem



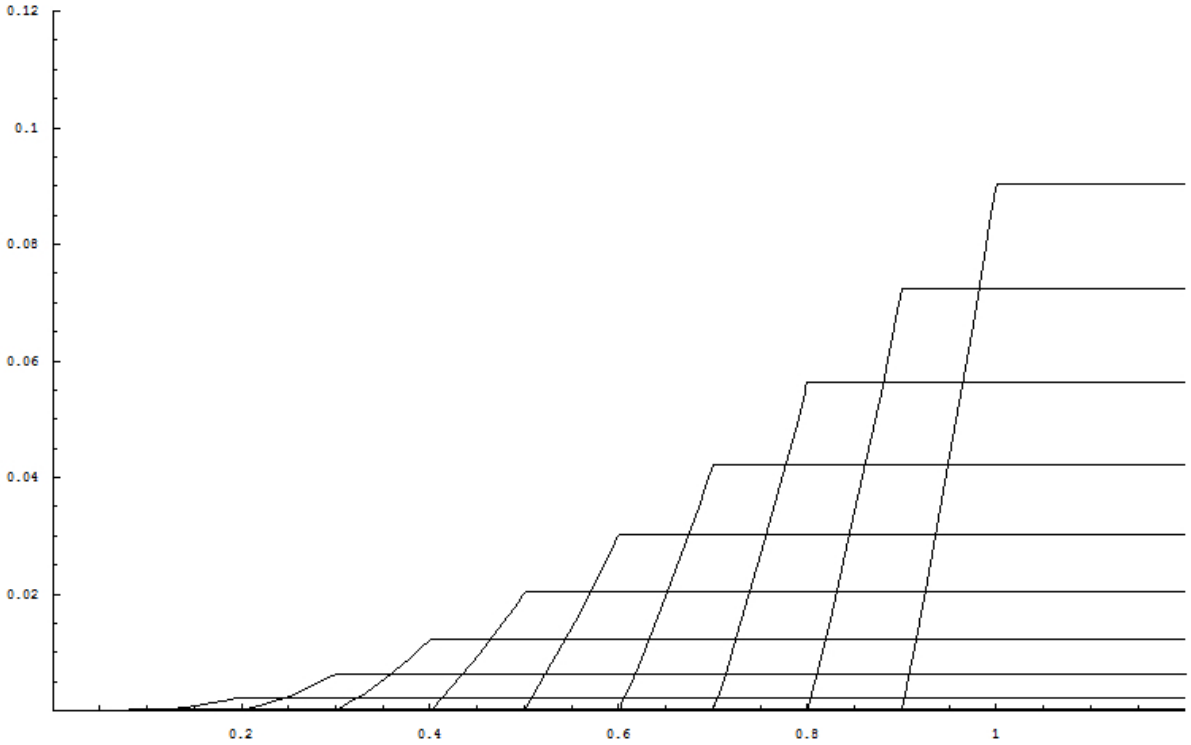

Combined graphs of $\mathcal{D}\mathcal{A}(u;\Delta u)$ for each "unit impulse" $\Delta u$ of width .1 (As before, the kernel for $\mathcal{A}$ is $k(t,s)=1$.)

# Sensitivities of data for linear Volterra problem



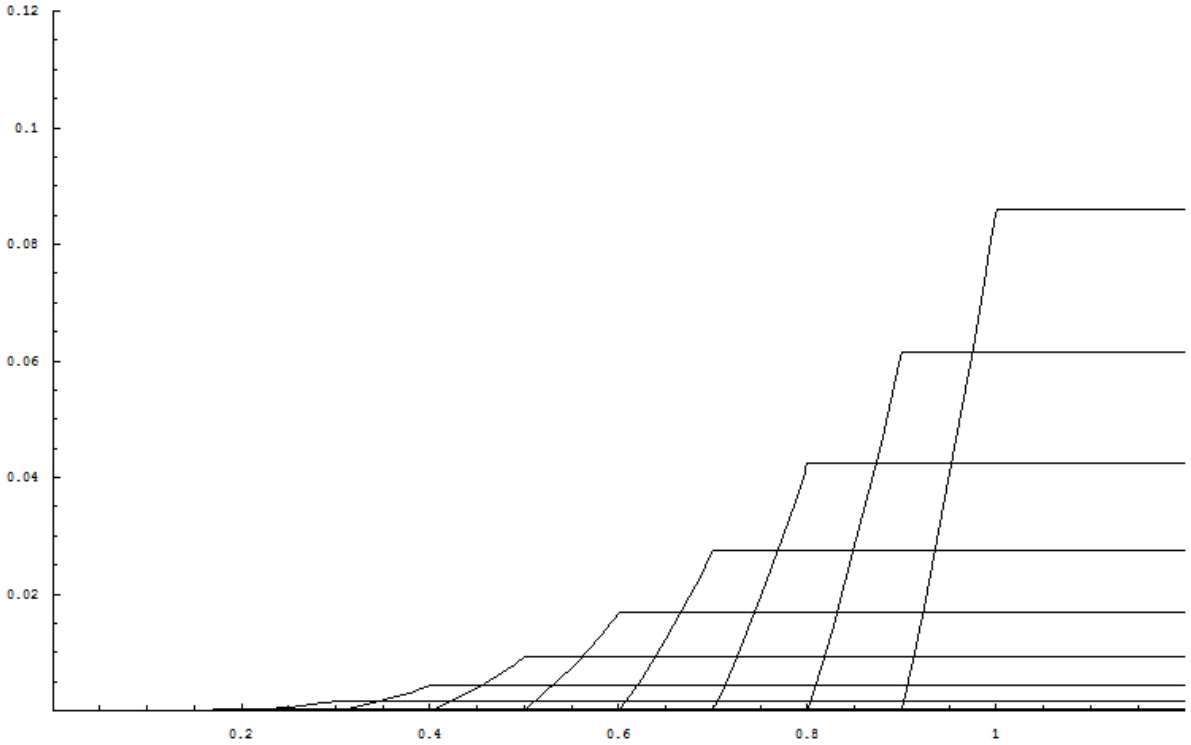

Combined graphs of $\mathcal{D}\mathcal{A}(u;\Delta u)$ for each "unit impulse" $\Delta u$ of width .1 (Here the kernel for $\mathcal{A}$ is $k(t,s) = t - s$.)

# Sensitivities of data for linear Volterra problem




Combined graphs of $\mathcal{D}\mathcal{A}(u;\Delta u)$ for each "unit impulse" $\Delta u$ of width .1 (Here the kernel for $\mathcal{A}$ is $k(t,s) = (t-s)^2$ .)

# Sensitivities of data for linear Volterra problem




Combined graphs of $\mathcal{DA}(u;\Delta u)$ for each "unit impulse" $\Delta u$ of width .1 (Here the kernel for $\mathcal{A}$ is $k(t,s) = (t-s)^3$ .)

# Selection of $T_\alpha$ for linear Volterra problem

$\Longrightarrow$ for the linear Volterra problem, **the solution** $u$ **at** $t$ **most influences data** $f$ on the interval $[t, t+\alpha]$, for some $\alpha > 0$ small.

The length $\alpha$ of the interval is the **regularization parameter**.

(Alternatively, use two parameters – one for sensitivity region and another for regularization. E.g., see [PKL & Dai, 2005] )

- One possible choice of $T_\alpha$:

$$T_\alpha f(t) = \frac{1}{\alpha} \int_0^\alpha f(t+\rho)\, d\rho, \quad a.e. \;\; t \in [0, 1-\alpha],$$

(averaging function on $[t, t+\alpha]$).

- More generally,

$$T_\alpha f(t) = \frac{\int_0^\alpha f(t+\rho)\, d\eta_\alpha(\rho)}{\int_0^\alpha d\eta_\alpha(\rho)}, \quad a.e. \;\; t \in [0, 1-\alpha],$$

for suitable measure $\eta_\alpha$ on $[0, \alpha]$.

# Selection of $\mathcal{A}_\alpha$, $a_\alpha$ for linear Volterra problem.

For example, we will apply the simple averaging $T_\alpha$ to the original linear Volterra equation. Assume $\alpha \in (0, \bar{\alpha}]$, for $\bar{\alpha} > 0$ small:

- Start with the original equation $\mathcal{A}u = f$, or

$$\int_0^t k(t,s)u(s)\,ds = f(t), \quad a.e.\ \ t \in [0,1].$$

- Next compute $T_\alpha \mathcal{A}u = T_\alpha f$, for a.e. $t \in [0, 1-\alpha]$,

$$\frac{1}{\alpha}\int_0^\alpha \int_0^{t+\rho} k(t+\rho,s)u(s)\,ds\,d\rho = \frac{1}{\alpha}\int_0^\alpha f(t+\rho)\,d\rho.$$

- **Now split:** $T_\alpha \mathcal{A} = \mathcal{A}_\alpha + D_\alpha$, where $D_\alpha$ is nearly "diagonal":

$$\begin{aligned} T_\alpha \mathcal{A}u(t) &= \frac{1}{\alpha}\int_0^\alpha \int_0^t k(t+\rho,s)u(s)\,ds\,d\rho \\ &\quad + \frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho,s)u(s)\,ds\,d\rho. \end{aligned}$$

# Selection of $\mathcal{A}_\alpha$, $a_\alpha$ for linear Volterra problem.

So the splitting of $T_\alpha \mathcal{A}$ is thus given by

$$T_\alpha \mathcal{A} = \mathcal{A}_\alpha + D_\alpha,$$

where, for a.e. $t \in [0, 1-\alpha]$,

$$\begin{aligned} \mathcal{A}_\alpha u(t) &= \int_0^t k_\alpha(t,s) u(s)\, ds, \\ D_\alpha u(t) &= \frac{1}{\alpha} \int_0^\alpha \int_t^{t+\rho} k(t+\rho, s) u(s)\, ds\, d\rho, \end{aligned}$$

and $k_\alpha(t,s) \equiv \frac{1}{\alpha} \int_0^\alpha k(t+\rho, s)\, d\rho$.

$\Longrightarrow$ $\mathcal{A}_\alpha$ **is of the same form as** $\mathcal{A}$**.**

- **Finally: select** $a_\alpha = a_\alpha(t)$ **so that** $D_\alpha \bar{u} \approx a_\alpha \bar{u}$ **for** $\alpha$ **small.**

# Selection of $a_\alpha$ for linear Volterra problem.

- Need $a_\alpha = a_\alpha(t)$ so that, for a.e. $t \in [0, 1-\alpha]$,

$$D_\alpha u(t) = \frac{1}{\alpha} \int_0^\alpha \int_t^{t+\rho} k(t+\rho, s) u(s)\, ds\, d\rho \quad \approx \quad a_\alpha(t) u(t),$$

when $u = \bar{u}$.

- If we approximate $u$ on the local interval $[t, t+\alpha]$ by the constant $u(t)$, then we have (formally)

$$D_\alpha u(t) \approx \left( \frac{1}{\alpha} \int_0^\alpha \int_t^{t+\rho} k(t+\rho, s) ds\, d\rho \right) u(t) = a_\alpha(t) u(t),$$

for

$$a_\alpha(t) = \frac{1}{\alpha} \int_0^\alpha \int_t^{t+\rho} k(t+\rho, s) ds\, d\rho, \quad a.e. \;\; t \in [0, 1-\alpha].$$

# Local regularization for the linear Volterra problem

**To summarize**: The particular construction we have chosen leads to a local regularization equation for the linear Volterra problem given, for noisy data $f^\delta$, by

$$\mathcal{A}_\alpha u + a_\alpha u = f_\alpha^\delta \quad \text{on} \quad X_\alpha = L^2(0, 1-\alpha),$$

where for a.e. $t \in [0, 1-\alpha]$,

$$\begin{aligned} f_\alpha^\delta(t) = T_\alpha f^\delta(t) &= \frac{1}{\alpha}\int_0^\alpha f^\delta(t+\rho)\, d\rho, \\ \mathcal{A}_\alpha u(t) &= \int_0^t k_\alpha(t,s) u(s)\, ds \\ &= \int_0^t \left(\frac{1}{\alpha}\int_0^\alpha k(t+\rho, s)\, d\rho\right) u(s)\, ds, \\ a_\alpha(t) &= \frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho, s) ds\, d\rho. \end{aligned}$$

# Local regularization for the linear Volterra problem

**Result:** For a large class of kernels $k$ and for $\alpha$ sufficiently small,

1. The quantity $a_\alpha$,

$$a_\alpha(t) = \frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho, s)ds\, d\rho,$$

satisfies

$$0 < \; c_1\, c(\alpha) \; \leq \; a_\alpha(t) \; \leq \; c_2\, c(\alpha),$$

a.e. $t \in [0, 1-\alpha]$, for some $c(\alpha) > 0$ with the property that

$$c(\alpha) \to 0 \quad \text{as} \;\; \alpha \to 0; \;\; \text{and},$$

2. $(\mathcal{A}_\alpha + a_\alpha I)$ is invertible with

$$\left\|(\mathcal{A}_\alpha + a_\alpha I)^{-1}\right\| = \mathcal{O}\left(\frac{1}{c(\alpha)}\right) \;\; \text{as} \;\; \alpha \to 0.$$

# Stronger convergence result

**Question:** So why does the local regularization method satisfy the **stronger convergence result**?

Recall that in this case we need to verify

$$\|(D_\alpha \bar{u} - a_\alpha \bar{u})\|_{X_\alpha} = o\left(c(\alpha)\right) \quad as \quad \alpha \to 0.$$

- In contrast to **Tikhonov** and **Lavrent'ev**, where $X_\alpha = X$ and

$$\|(D_\alpha \bar{u} - \alpha \bar{u})\|_{X_\alpha} = \|0 - \alpha \bar{u}\|_{X_\alpha} \neq o(\alpha),$$

- the method of **local regularization** gives

$$D_\alpha \bar{u}(t) - a_\alpha(t)\bar{u}(t) =$$

$$= \frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho, s)\bar{u}(s)\, ds\, d\rho - \left(\frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho, s)\, ds\, d\rho\right)\bar{u}(t).$$

# Stronger convergence result, cont'd

So for **local regularization** we have

$$D_\alpha \bar{u}(t) - a_\alpha(t)\bar{u}(t) \;=\; \frac{1}{\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho, s)\,(\bar{u}(s) - \bar{u}(t))\; ds\, d\rho,$$

a.e. $t \in [0, 1-\alpha)$, and thus as $\alpha \to 0$,

$$\begin{aligned} \|D_\alpha \bar{u} - a_\alpha \bar{u}\|_{X_\alpha} &= \mathcal{O}\left(\|a_\alpha(\,\cdot\,)\|_{L^\infty}\right) \cdot \|S_\alpha \bar{u} - \bar{u}\|_{X_\alpha} \\ &= \mathcal{O}\left(c(\alpha)\right) \cdot \|S_\alpha \bar{u} - \bar{u}\|_{X_\alpha} \end{aligned}$$

where

$$S_\alpha \bar{u}(t) = \frac{1}{\alpha}\int_0^\alpha \bar{u}(t+s)\, ds.$$

Note that $S_\alpha \bar{u}(t)$ is a special case of the Hardy-Littlewood maximal function for $\bar{u}$ at $t$.

# Stronger convergence result, cont'd

So we have

$$\|D_\alpha \bar{u} - a_\alpha \bar{u}\|_{X_\alpha} = \mathcal{O}\left(c(\alpha)\right) \cdot \|S_\alpha \bar{u} - \bar{u}\|_{X_\alpha},$$

and it follows from the Lebesgue differentiation theorem that

$$\|S_\alpha \bar{u} - \bar{u}\|_{X_\alpha} \to 0 \quad as \ \alpha \to 0,$$

for any $\bar{u} \in X$. Thus

$$\|D_\alpha \bar{u} - a_\alpha \bar{u}\|_{X_\alpha} = o\left(c(\alpha)\right) \quad as \ \alpha \to 0,$$

and **the stronger convergence result is obtained.**

For smoother $\bar{u}$, it's obvious that from this same estimate we can obtain a *rate* of convergence.

# General results for linear Volterra problems

**More general results for the linear Volterra problem**

( [PKL] for earlier work]; [Brooks thesis, 2007] and [Brooks & PKL, 2009] for $L^p$ results and discrepancy principle)

- $f \in \mathcal{R}(\mathcal{A}) \subset X = L^p[0,1]$, $f^\delta \in L^p[0,1]$, $1 < p \leq \infty$.
- For $\nu$-smoothing problems, with a large class of measures (depending on $\nu$) used to define suitable operators $T_\alpha$.
- Convergence as $\delta \to 0$ of $u_\alpha^\delta$ to $\bar{u}$ in $X_\alpha = L^p[0, 1-\alpha]$ for *a priori* choices of $\alpha = \alpha(\delta, f^\delta)$.
- Optimal rates of convergence for the $\nu$-smoothing problem.
- A discrepancy principle of the form

$$d(\alpha) = \tau\delta,$$

where $$d(\alpha) \equiv \left(\alpha^m \frac{\|f^\delta\|_X}{\|T_\alpha f^\delta\|_{X_\alpha}}\right) \|\mathcal{A}_\alpha u_\alpha^\delta - T_\alpha f^\delta\|_{X_\alpha}.$$

# Nonlinear Volterra problems

**Nonlinear Hammerstein problem**:

**Start with the usual local regularization equation**

$$\mathcal{A}_\alpha u + a_\alpha u = T_\alpha f^\delta,$$

except now $\mathcal{A}_\alpha$ and the term $a_\alpha u$ are both **nonlinear** in $u$.

- **How to handle the nonlinearity?**
  Use the local regularization interval to facilitate linearization of the nonlinear problem [Luo thesis '07; Brooks, PKL, Luo '09]

  i.e., for $t > 0$, let $\tau = \min\{t, \alpha\}$, and linearize $a_\alpha = a_\alpha(t, u(t))$ about the *prior* $u$-value $u(t - \tau)$.

- **Numerical implementation of the linearized problem:**

$$u = (T_\alpha f^\delta - \mathcal{A}_\alpha u)/a_\alpha(\cdot), \quad t > 0$$

  $\Longrightarrow$ solve a **linear** problem for $u(t)$ **sequentially**, since $\mathcal{A}_\alpha u$ and (the linearized) $a_\alpha$ only involve values of $u$ prior to $t$.

# Nonlinear Volterra problems, cont'd

**Nonlinear autoconvolution problem**:

For the **autoconvolution problem** (as with Hammerstein), the usual local regularization equation

$$\mathcal{A}_\alpha u + a_\alpha u = T_\alpha f^\delta \tag{8}$$

involves **nonlinearities** in both $\mathcal{A}_\alpha$ and the term $a_\alpha\, u$.

- **First: Solve** (8) **for u(t), t $\in$ [0, $\alpha$]** **($\alpha > 0$ small)**:
  - Nonlinear equation (8) not too difficult to solve on $[0, \alpha]$;
  - Need only a reasonable approximation to $u$ on $[0, \alpha]$ for convergence
- **Then: Solve** (8) **for u(t), t $\in$ ($\alpha$, 1]** using **linear** methods:
  - The (approximate) solution u on $(\alpha, 1]$ can be found **sequentially** from (8) using **linear** techniques.

- Thus, linearization of (8) not required in practice for the autoconvolution problem (but used as tool in convergence theory).

[Dai thesis, 2006; Dai & PKL, 2008; Brooks, Dai, & PKL, 2009]

# Nonlinear Volterra problems, cont'd

- **Numerical results for nonlinear Volterra problems:**

  In general,

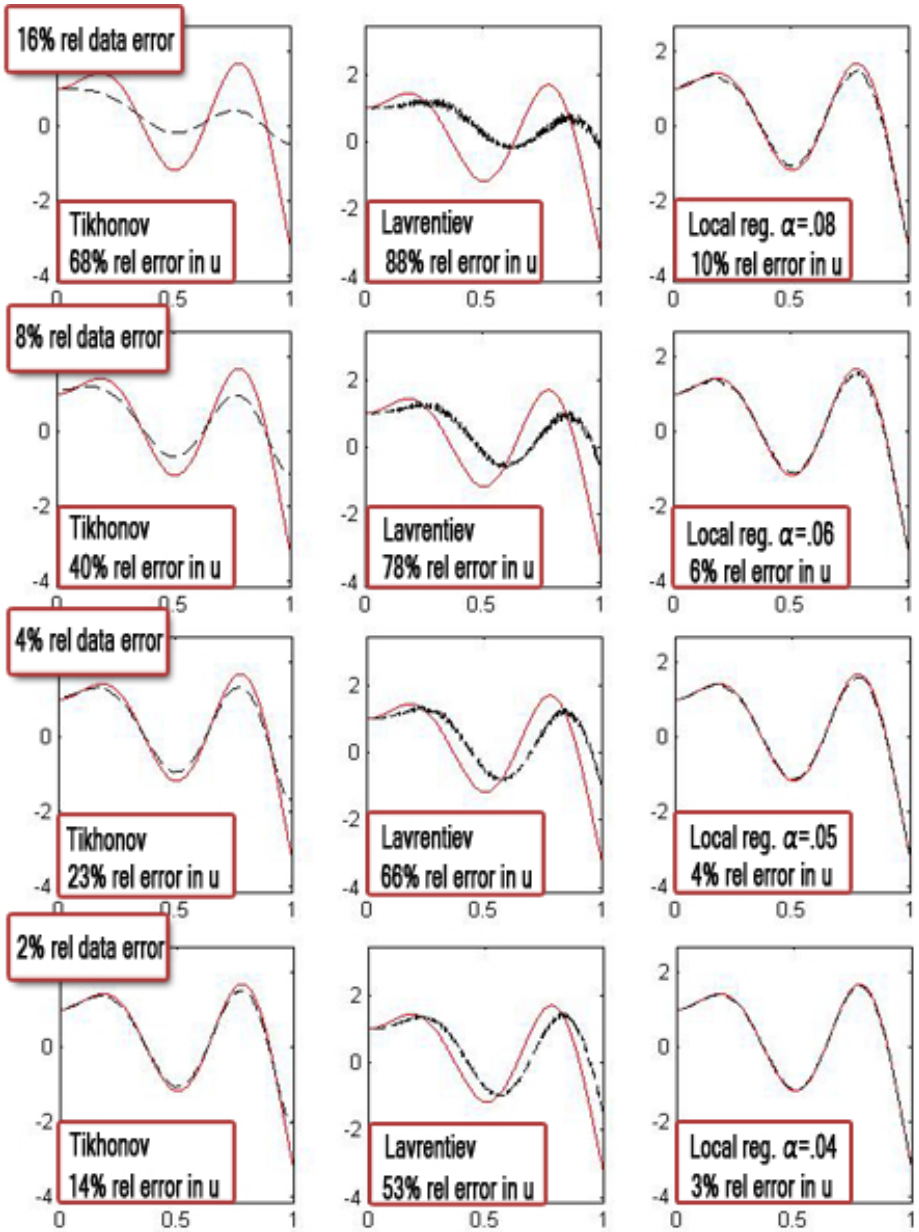
  - Qualitatively equivalent to Tikhonov solution, except not oversmoothed; considerably better than Lavrent'ev solution.
  - Many times faster than a nonlinear Tikhonov (iterative) approach in practice.

  (See above references.)

- **Other non-Volterra work:**

  - **2-D image deblurring**
    Older work: [Cui thesis], [Cui, Lamm, Scofield, '07].
    Newer results (to appear) are based on *generalized* local regularization approach $\Longrightarrow$ less computationally intensive.
  - **General $n$-dimensional problems governed by a nonlinear monotone operator** (to appear; from *generalized* theory).

# Sample numerical results



# Sample numerical results, cont'd