Algebraic Differential Equations – Rational Solutions and Classification

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Consider an algebraically closed field $\mathbb{K}$ of characteristic zero. Let

$$F(x, y_0, y_1, \ldots, y_n)$$

be an irreducible polynomial over $\mathbb{K}$. The *algebraic ordinary differential equation (AODE)* defined by F is of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

where y is an indeterminate over the differential field of rational functions $\mathbb{K}(x)$ with the usual derivation $' = \frac{d}{dx}$. So $F(x, y, y', \ldots, y^{(n)})$ is a differential polynomial in $\mathbb{K}[x]\{y\}$. Such an AODE is *autonomous* iff F does not depend on x, i.e., $F \in \mathbb{K}\{y\}$.

The solutions of $F(x, y, y', \ldots, y^{(n)}) = 0$ can be identified with the zeros of the radical differential ideal generated by F, denoted by $\{F\}$. The ideal $\{F\}$ can be decomposed as

$$\{F\} = (\{F\} : S) \cap \{F, S\},$$

where $S = \frac{\partial F}{\partial y'}$ is the separant of F, i.e., the derivative of F w.r.t. $y^{(n)}$. Here, $(\{F\} : S)$ is called the general component, whereas $\{F, S\}$ is the singular component. Since F is irreducible, the differential ideal $\{F\} : S$ is prime and therefore has a generic zero, which can be described by giving a Gröbner basis of the differential ideal $\{F\} : S$. For this approach we refer to [Hub96]. A generic zero of the prime ideal $\{F\} : S$ is called a *general solution* of the AODE $F(x, y, y', \ldots, y^{(n)}) = 0$.

A few years ago Feng and Gao, in their papers [FeG04, FeG06], have shown that rational solutions, and indeed rational general solutions, for autonomous AODEs of order 1 of the form $F(y, y') = 0$, can be effectively computed. Their algorithm is based on the parametrization of the corresponding algebraic curve $\mathcal{C} : F(u, v) = 0$, and on the knowledge of upper bounds for the rational functions in such a parametrization, as derived in [SeW01].

Recently we have generalized Feng and Gao's algorithmic method to general non-autonomous AODEs of order 1; see [NgW10, NgW11a, NgW11b]. In these papers we describe a full algorithm in the generic case for deciding the existence of a rational general solution of a parametrizable AODE of order 1, i.e., we consider an AODE of the form $F(x, y, y') = 0$, where F is a trivariate polynomial and the algebraic surface (the solution surface) $\mathcal{S} : F(u, v, w) = 0$ admits a rational parametrization $\mathcal{P}(s, t)$. Obviously a rational solution of the AODE generates a rational curve on the corresponding solution surface. Having a rational parametrization of $\mathcal{S} : F(u, v, w) = 0$ allows us to associate the differential equation $F(x, y, y') = 0$ with a planar system of autonomous differential equations of order 1 and of degree 1 in the derivatives of the two parameters s and t. If we can compute a rational general solution of this *associated system*, (indeed, we can do this in the generic case), then we obtain a rational general solution of $F(x, y, y') = 0$ via the parametrization mapping. Moreover, if the parametrization is proper, then the original AODE has a rational general solution if and only if its associated system w.r.t. $\mathcal{P}(s,t)$ has a rational general solution.

A transformation of the ambient geometric space also transforms the AODE, but might leave the associated system invariant. In this case, the original AODE and the transformed one share the same associated system, and therefore also their rational solvability. So w.r.t. certain such groups of transformations (affine, birational) one might try to determine a simple, canonical form for a given AODE, decide rational solvability for this canonical form, and then transform this knowledge back to the original AODE.

In [NSW12] we have determined such a group for the case of affine transformations. We have studied some well-known classes of equations and we have related them to our algebro-geometric approach. It turns out that being autonomous is not an intrinsic property of an AODE; certain classes contain both autonomous and non-autonomous AODEs. Currently we are working on a classification of AODEs w.r.t. birational transformations. We will report on first results in this direction.

References

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