Solving systems of linear differential equations over integro-differential algebras

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Systems of linear differential equations

Systems of first order n linear differential equations have the matrix form

$$u'(x) = M(x)u(x) + f(x),$$

where $u = (u_i), u_i \in C^1[a, b], M = (m_{ij}) \in C[a, b]^{n \times n}$ is the system matrix and $f = (f_i) \in C[a, b]^n$ the forcing function.

We want to find u in terms of symbolic f.

Initial value problems

An *initial-value problem* (IVP) is to find $u(x)$ which satisfies $u'(x) = M(x)u(x) + f(x)$ and which also satisfies an initial condition of the form $u(x_0) = y_0$, where $x_0 \in [a, b]$ and $y_0 \in \mathbb{R}^n$.

General existence-uniqueness theorem:

Theorem (Apostol [1])

Assume $M(x) \in C[a, b]^{n \times n}$ is a matrix function continuous on an open interval (a, b). If $x_0 \in [a, b]$ and $y_0 \in \mathbb{R}^n$ are given, then the system

$$u'(x) = M(x)u(x),$$
$$u(x_0) = y_0 \neq 0$$

has a unique solution $u(x)$.

Solution of IVPs

Theorem (Apostol [1])

Given $M(x) \in C[a,b]^{n\times n}$ and $f(x) \in C[a,b]^n$, both continuous on an open interval (a,b), the solution of the IVP

$$u'(x) = M(x)u(x) + f(x),$$
$$u(x_0) = y_0$$

on $[a,b]$ is given by the formula

$$u(x) = U(x)U(x_0)^{-1}y_0 + U(x)\int_{x_0}^{x} U(\xi)^{-1}f(\xi)\, d\xi,$$

where $U(x)$ is the fundamental matrix.

Duhamel's formula

If $M(x)$ is a constant matrix, then $U(x)U(x_0)^{-1} \equiv e^{M(x-x_0)}$. The precise formula for the solution of IVP is

$$u(x) = e^{M(x-x_0)}y_0 + e^{Mx}\int_{x_0}^{x} e^{-M\xi}f(\xi)\,d\xi,$$

which is called Duhamel's formula.

Two-point boundary problems

A system of linear differential equations with two-point boundary conditions is:

$$u'(x) = M(x)u(x) + f(x), \quad x \in [a, b], \tag{1}$$

$$\Gamma_a u(a) + \Gamma_b u(b) = 0, \tag{2}$$

where $\Gamma_a, \Gamma_b \in \mathbb{R}^{n \times n}$.

Note The boundary conditions (2) include initial conditions as a special case, where $\Gamma_a = I$ and $\Gamma_b = 0$.

Given the fundamental matrix $U(x)$ of the homogeneous linear system and a pair of matrices Γ_a, Γ_b given by (2), the boundary condition matrix Γ is defined by

$$\Gamma = \Gamma_a U(a) + \Gamma_b U(b).$$

Condition for uniqueness of solutions to (1)(2).

Theorem (Starr-Rokhlin [3])

If the matrix $\Gamma = \Gamma_a U(a) + \Gamma_b U(b)$ is regular, then there is a unique solution $u(x)$ to the system (1)(2). Furthermore, the solution to the homogeneous system of equations with boundary conditions defined by (2) is $u(x) \equiv 0$.

Definition (Starr-Rokhlin [3])

A continuous function $g : [a, b] \times [a, b] \to C[a, b]^{n\times n}$ is the *Green's matrix* for a boundary problem (1)(2) if

1. $\frac{\partial}{\partial x} g(x, y)$ is continuous except at $x = y$,
2. $g(x + 0, x) - g(x - 0, x) = I$ for all $x \in [a, b]$,
3. $\frac{\partial}{\partial x} g(x, y) + M(x)g(x, y) = 0$ for all $x, y \in [a, b], x \neq y$,
4. $\Gamma_a \cdot g(a, y) + \Gamma_b \cdot g(b, y) = 0$ for all $y \in [a, b]$.

Solution of two-point BPs

Theorem (Whyburn [4])

If the matrix Γ is regular, then there exists a unique Green's matrix g for the system

$$u'(x) = M(x)u(x) + f(x),$$
$$\Gamma_a u(a) + \Gamma_b u(b) = 0$$

given by the formula

$$g(x,y) = \begin{cases} U(x)\Gamma^{-1}\Gamma_a U(a)U(y)^{-1} & \text{if } y \leq x, \\ -U(x)\Gamma^{-1}\Gamma_b U(b)U(y)^{-1} & \text{if } y \geq x, \end{cases}$$

with $U(x)$, the fundamental matrix for homogeneous system, and the solution is given by

$$u(x) = \int_a^b g(x,\xi)f(\xi)\ d\xi.$$

BPs with Stieltjes boundary conditions

More general boundary conditions are known as *Stieltjes boundary conditions*. Such conditions take the form

$$\sum_{i=1}^{\infty} \Gamma_i u(z_i) + \int_a^b F(\xi)u(\xi)\ d\xi = 0,$$

where $Z = (z_1, z_2, \dots)$ is a point sequence of the first species, $\Gamma_1, \Gamma_2, \dots$ is an absolutely convergent sequence of constant matrices and $F(x)$ is square matrix of order n and the elements of $F(x)$ are of bounded variation on $[a, b]$.

For two points

$$\Gamma_a u(a) + \Gamma_b u(b) + \int_a^b F(\xi)u(\xi)\ d\xi = 0, \tag{3}$$

the boundary condition matrix given by (3) with fundamental matrix is

$$\Gamma = \Gamma_a U(a) + \Gamma_b U(b) + \int_a^b F(\xi)U(\xi)\, d\xi. \tag{4}$$

Theorem (Whyburn [4])

If the matrix Γ defined by (4) is regular, then there exists a unique Green's matrix $g(x, y)$ for the system

$$u'(x) = M(x)u(x) + f(x),$$
$$\Gamma_a u(a) + \Gamma_b u(b) + \int_a^b F(\xi)u(\xi)\, d\xi = 0$$

given by the formula

$$g(x, y) = \begin{cases} U(x)\Gamma^{-1}\left[\Gamma_a U(a) + \int_a^y F(t)U(t)dt\right] U(y)^{-1} & \text{if } y \leq x, \\ -U(x)\Gamma^{-1}\left[\Gamma_b U(b) + \int_y^b F(t)U(t)dt\right] U(y)^{-1} & \text{if } y \geq x, \end{cases}$$

where $U(x)$ is the fundamental matrix for the homogeneous system, and the solution of the system is given by

$$u(x) = \int_a^b g(x, \xi)f(\xi)\ d\xi.$$

Integro-differential algebra

Definition (Rosenkranz-Regensburger [2])

We call $(\mathcal{F}, \partial, \int)$ an *integro-differential algebra* over K if $\mathcal{F}$ is a commutative K-algebra with K-linear operators ∂ and $\int$ such that

$$(\textstyle\int f)' = f, \tag{5}$$

$$(fg)' = f'g + fg', \tag{6}$$

$$(\textstyle\int f')(\int g') + \int (fg)' = (\int f')g + f(\int g') \tag{7}$$

are satisfied, where $'$ is the usual shorthand notation for ∂.

1. ∂ and $\int$ respectively *derivation* and *integral* of $\mathcal{F}$.
2. Axiom (5) is called *section axiom*, Axiom (6) is called *Leibnitz axiom*, and Axiom (7) is called *differential Baxter axiom*.
3. The projectors $\mathtt{J} = \int \circ \partial$ and $\mathtt{E} = 1 - \int \circ \partial$ respectively called the initialization and the evaluation of $\mathcal{F}$.

We apply the action of the operators $\partial, \int$ and $\mathtt{E}$ componentwise to prove the following.

Proposition

Let $\mathcal{F}$ be an integro-differential algebra over a field K. Then the matrix ring $\mathcal{F}^{n \times n}$ is again an integro-differential algebra over K.

Proof.

Let $F = (F_{ij}), G = (G_{ij}) \in \mathcal{F}^{n \times n}$. Then the section axiom (5) clearly satisfied, since $(\int F)' = F$. For $i, j = 1, \ldots, n$ we have
$\sum_{r=1}^{n} (F_{ir} G_{rj})' = \sum_{r=1}^{n} (F'_{ir} G_{rj}) + \sum_{r=1}^{n} (F_{ir} G'_{rj})$, hence (6) is satisfied and we have also
$\sum_{r=1}^{n} (\int F'_{ir})(\int G'_{rj}) + \sum_{r=1}^{n} \int (F_{ir} G_{rj})' = \sum_{r=1}^{n} (\int F'_{ir}) G_{rj} + \sum_{r=1}^{n} F_{ir} (\int G'_{rj})$,
hence (7) is also satisfied. □

IVPs over integro-differential algebras

Theorem (Rosenkranz-Regensburger [2])

Let $(\mathcal{F}, \partial, \int)$ be an ordinary integro-differential algebra. Given a fundamental matrix $U \in \mathcal{F}^{n \times n}$, the system

$$u' = Mu + f,$$
$$\mathtt{E}u = 0$$

has the unique solution

$$u = U \int U^{-1} f,$$

for every $f \in \mathcal{F}^n$.

Proof.

We have $u' = (U \int U^{-1} f)' = U' \int U^{-1} f + UU^{-1} f$ by Leibnitz axiom and section axiom. Since $u_1, \ldots, u_n$ are solutions of $u' = Mu$, we have $MU = U'$. For checking the initial condition, note that $\mathtt{E} \int U^{-1} f$ is already the zero vector, so $\mathtt{E}(u) = 0$ for $\mathtt{E}$ is multiplicative. Hence $u \in \mathcal{F}^n$ is a solution of the first order system.

For uniqueness, If u is a solution of homogeneous system, then choose $c = (c_1, \ldots, c_n)^T \in K^n$ such that $u = c_1 u_1 + \cdots + c_n u_n$ and the initial conditions yield $\mathtt{E}(Uc) = 0$. Since $\mathtt{E}$ is linear functional , $\mathtt{E}(Uc) = (\mathtt{E}U)c$ and $\det \mathtt{E}U = \mathtt{E}(\det U)$. Since $U \in \mathcal{F}^{n \times n}$ is regular, $\mathtt{E}U \in K^{n \times n}$ is regular, so $c = (\mathtt{E}U)^{-1} 0 = 0$ and $u = 0$. □

BPs over integro-differential algebras

Restate the system (3) as the following:

$$(D - M)u \equiv u' - Mu = f, \tag{8}$$

where $\partial I = D, M \in \mathcal{F}^{n\times n}$ and $D - M \in \mathcal{F}[\partial, \int]^{n\times n}$.

Note *Stieltjes boundary conditions* take a form of a matrix whose entries are (scalar) Stieltjes conditions and these conditions form right a ideal generated by vector characters in the matrix operator ring $\mathcal{F}[\partial, \int]^{n\times n}$.

We want to find an operator G that satisfies $(D - M)G = 1$ and $\Gamma G = 0$, where Γ is a boundary condition matrix. This G is known as the *Green's operator*.

A boundary problem for a systems is given by a pair $(T, \mathcal{B})$, where $T \equiv D - M$ and $\mathcal{B} \leq (\mathcal{F}^*)^n$ is a space of boundary conditions.

Definition

A boundary problem is called *regular* if for every forcing function there exists exactly one solution that satisfies the boundary conditions.

Definition

$u \in \mathcal{F}^n$ is a solution of $(T, \mathcal{B})$ for a given $f \in \mathcal{F}^n$ if $Tu = f$ and $u \in \mathcal{B}^\perp$.

Let $\beta_1, \ldots, \beta_n$ be a basis for $\mathcal{B}$. Then the boundary problem can be as follows: Given $f \in \mathcal{F}^n$, find $u \in \mathcal{F}^n$ such that

$$\begin{aligned} Tu &= f, \\ \beta_1(u) &= \cdots = \beta_n(u) = 0. \end{aligned}$$

The boundary conditions can be any linear functionals. In particular, they can be point evaluations of derivatives or Stieltjes boundary conditions.

Algorithmic criterion of regularity

The following Lemma is a generalization of scalar case (Rosenkranz-Regensburger [2]) to vector case.

Lemma

A boundary problem $(T, \mathcal{B})$ with $dim(Ker(T)) = dim(\mathcal{B})$ is regular iff the matrix

$$\beta(u) = \begin{pmatrix} \sum_{i=1}^{n} \beta_{1i}(u_{1i}) & \cdots & \sum_{i=1}^{n} \beta_{1i}(u_{ni}) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} \beta_{ni}(u_{1i}) & \cdots & \sum_{i=1}^{n} \beta_{ni}(u_{ni}) \end{pmatrix}$$

is regular, where $\beta_i = (\beta_{ij})$ and $u_i = (u_{ij})$, $i, j = 1, \ldots, n$ are any basis of $\mathcal{B}$ and Ker(T) respectively.

Finding the Green's operator

The general algorithm for scalar is applicable to this case. The main steps to compute Green's operator are:

1. Compute the fundamental right inverse $T^\blacklozenge \in \mathcal{F}[\partial, \textstyle\int]^{n\times n}$ by
$$T^\blacklozenge = U \textstyle\int U^{-1},$$
with the help of variation-of-parameters formula.
2. Compute the projector $P \in \mathcal{F}[\partial, \textstyle\int]^{n\times n}$ onto $\mathrm{Ker}(T)$ along $\mathcal{B}^\perp$. This can be done as follows: Let $\mathrm{Im}(P) = [u_1, \ldots, u_n]$ and $\mathrm{Ker}(P) = [\beta_1, \ldots, \beta_n]^\perp$. Since $(T, \mathcal{B})$ is regular, $\beta(u)$ is invertible. Set $(\overline{\beta}_1, \ldots, \overline{\beta}_n)^T = \beta(u)^{-1}(\beta_1, \ldots, \beta_n)^T$. Then the projector P is given by
$$P(u) = \sum_{i=1}^{n} \overline{\beta}_i(u) u_i$$

Now the Green's operator for $(T, \mathcal{B})$ is $G = (1 - P)T^\blacklozenge$.

Example

Consider a system of equations

$$u_1' = u_2,$$
$$u_2' = f_2$$
$$\text{with } u_1(0) = 0, u_1(1) = 0.$$

Matrix notation: $u' = Mu + f$,
where $u = (u_1, u_2)^T, f = (0, f_2)^T$ and

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\beta_1 = (L \quad 0), \beta_2 = (R \quad 0), \text{ where}$$
$$L(u) = u_1(0), R(u) = u_1(1)$$

Fundamental Matrix:

$$U = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$$

Fundamental Right Inverse:

$$T^{\blacklozenge} = \begin{pmatrix} A & xA - Ax \\ 0 & A \end{pmatrix}$$

Projector:

$$P = \begin{pmatrix} L + xR - xL & 0 \\ -L + R & 0 \end{pmatrix}$$

Green's operator:

$$G = \begin{pmatrix} A - x\int_0^1 & xA - Ax - x^2\int_0^1 \\ -\int_0^1 & -\int_0^1 x + A \end{pmatrix}$$

Solution: u(x) = Gf

$$\Big(-x\int_0^x f_2 - \int_0^x xf_2dx - x^2\int_0^1 f_2 \ dx, -\int_0^1 x$$

Sample computation with Maple

We used the Mapple package IntDiffOp implemented by Anja Korporal [4].

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T:=MATRIXDIFFOP(M);
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$$\mathtt{T} := \begin{bmatrix} \mathtt{D} & -1 \\ 0 & \mathtt{D} \end{bmatrix}$$

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U:=FundamentalMatrix(T);
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$$\mathtt{U} := \begin{bmatrix} \mathtt{x} & 1 \\ 1 & 0 \end{bmatrix}$$

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FRIM:=FundamentalRightInverseMatrix(T);
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$$\mathtt{FRIM} := \begin{bmatrix} \mathtt{A} & \mathtt{x.A-A.x} \\ 0 & \mathtt{A} \end{bmatrix}$$

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THANK YOU