Localization and the Noncommutative Mikusiński Calculus

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Let R be a ring and S a multiplicative subset containing no zero divisors and satisfying the Ore condition. Then $S^{-1}R \hookrightarrow R$ exists.

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- Need S with zero divisors, then $\varepsilon\colon S^{-1}R \to R$ not injective.

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Definition (Rosenkranz 2003/ Rosenkranz-Regensburger 2006)

We call $(\mathcal{F}, \partial, \int)$ an **integro-differential algebra** over K if $\mathcal{F}$ is a commutative K-algebra with K-linear operations ∂ and $\int$ satisfying

$$(\textstyle\int f)' = f,$$
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In particular, $D \in \mathcal{F}[\partial]$ and $A, B \in \mathcal{F}[\int]$ for standard case $\mathcal{F} = C^\infty(\mathbb{R})$.

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Multiplication of boundary problems:

$$(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 T_2 + \mathcal{B}_2).$$

Recall example:

$$(D^2, [L, R]) = (D, [A + B]) \cdot (D, [L])$$

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A **boundary problem** is a pair $(T, \mathcal{B})$ consisting of a monic differential operator $T \in \mathcal{F}_0[\partial]$ and a boundary space $\mathcal{B} \leq \mathcal{F}^*$ of Stieltjes conditions.

Multiplication of boundary problems:

$$(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 T_2 + \mathcal{B}_2).$$

Recall example:

$$(D^2, [L, R]) = (D, [A + B]) \cdot (D, [L])$$

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} = \boxed{\begin{array}{l} u' = f \\ \int_0^1 u = 0 \end{array}} \cdot \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}}$$

Algebraic Green's Operators

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Theorem (Factorization Theorem)

Given $(T, \mathcal{B}) \in \mathcal{F}_0[\partial]_\Phi$, any factorization $T = T_1 T_2$ lifts to a problem factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ with $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \in \mathcal{F}_0[\partial]_\Phi$ and $\mathcal{B}_2 \leq \mathcal{B}$.

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Let Φ be an umbral character set for $\mathcal{F}$. Then for any $(T, \mathcal{B}) \in \mathcal{F}_0[\partial]_{(\Phi)}$ there is a $(S, \mathcal{A}) \in \mathcal{F}_0[\partial]_\Phi$ that has $(T, \mathcal{B})$ as a subproblem.

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Let M be a left R-module, and let $S \subseteq R$ be a multiplicative, right permutable and right reversible denominator set $S \subseteq R$. Then there exists a left $S^{-1}R$-module $S^{-1}M$.

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Let $(\mathcal{F},\partial,\int)$ be an integro-differential algebra with character set Φ. Then $\mathcal{F}_\Phi$ is a $K\mathcal{F}_0[\partial]_\Phi$-module.

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G = (D^2, [L, R])^{-1} &= XAX + XBX - AX - XB
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References

Jan Mikusiński.
Operational calculus, volume 8 of *International Series of Monographs on Pure and Applied Mathematics*.
Pergamon Press, New York, 1959.

Markus Rosenkranz.
The Green's Algebra: A Polynomial Approach to Boundary Value Problems.
Phd thesis, Johannes Kepler University, Research Institute for Symbolic Computation, September 2003.
Also available as RISC Technical Report 03-05, July 2003.

Markus Rosenkranz.
A new symbolic method for solving linear two-point boundary value problems on the level of operators.
J. Symbolic Comput., 39(2):171–199, 2005.

Markus Rosenkranz and Georg Regensburger.
Solving and factoring boundary problems for linear ordinary differential equations in differential algebras.
Journal of Symbolic Computation, 43(8):515–544, 2008.