Module structure of rings of partial differential operators

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The purpose of this talk is to develop constructive versions of Stafford's theorems on the module structure of Weyl algebras $A_n(k)$ (i.e., the rings of partial differential operators with polynomial coefficients) over a base field of characteristic 0 ([6]). Using the very simplicity of the ring $A_n(k)$, we show how to explicitly compute a unimodular element of a finitely generated left $A_n(k)$-module of rank at least 2. This result is used to constructively decompose any finitely generated left $A_n(k)$-module into a direct sum of a free left $A_n(k)$-module and a left $A_n(k)$-module of rank at most 1. If the latter is torsion-free, then we explicitly show that it is isomorphic to a left ideal of $A_n(k)$ which can be generated by 2 elements. Then, we give an algorithm which reduces the number of generators of a finitely presented left $A_n(k)$-module having a module of relations of rank at least 2 to 2. In particular, any finitely generated torsion left $A_n(k)$-module can always be generated by at most 2 generators and is the image of a projective ideal whose construction is explicitly given. Moreover, a non-torsion left $A_n(k)$-module of rank r, which is not free, can be generated by $r+1$ elements but no fewer. Similar results hold for right $A_n(k)$-modules which can easily be obtained from the above results. These results are implemented in the Stafford package ([3]), and their system-theoretical interpretations are given within an algebraic analysis (D-module) approach ([1, 3, 5]). Using [4], we show that the above results can be extended to the case of the ring of ordinary differential operators with either formal power series or locally convergent power series coefficients. Finally, using a result due to Caro and Levcovitz ([2]), we show that the above results also hold for the ring of partial differential operators with coefficients in the field of fractions of the ring of formal power series or in the field of fractions of the ring of locally convergent power series. For more details, see [5].

References

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