Computer algebra application to numerical solving of nonlinear KdV-like equations

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Contents

Introduction

We apply our algorithmic approach [1][Gerdt, Blinkov, Mozzhilkin'2006] to generate finite-difference approximations (FDAs) to the nonlinear Korteweg-de Vries (KdV)-type equations and to construct numerical solutions.

The main steps of the approach are:

1. For a PDE system its completion to involution (Thomas decomposition, in general).
2. Conversion into the integral form of conservation law.
3. Enlargement with integral relations between unknown functions and their partial derivatives.
4. Discretization by applying the finite volume method and numerical integration.
5. Difference elimination of the partial derivatives to obtain FDA.
6. Verification of s(trong)-consistency [3][Gerdt'2012] of FDA.
7. Addition of discrete boundary condition (BVP) or/and initial conditions (IVP) and numerical solving.

Conservation Law Form for Two Independent Variables

A wide class of PDEs can be written in the conservation law form

$$\frac{\partial}{\partial x}\mathbf{F}(\mathbf{v}) + \frac{\partial}{\partial y}\mathbf{G}(\mathbf{v}) = 0\,.$$

Here $\mathbf{v}$ is a $m-$vector function in unknown $n-$vector function $\mathbf{u}$ and its partial derivatives $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_{xx}\ldots$; $\mathbf{F}$ and $\mathbf{G}$ are functions that map R^m into R^m.

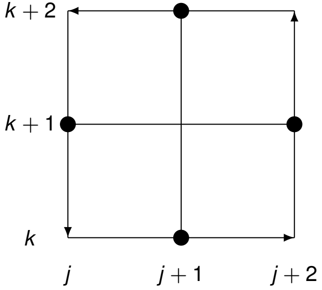
By Green's theorem (curl theorem in the plane), the above equation is equivalent to

$$\oint_\Gamma -\mathbf{G}(\mathbf{v})dx + \mathbf{F}(\mathbf{v})dy = 0\,.$$

where Γ is arbitrary closed contour.

Discretization

We set $\mathbf{u}(x,y) = \mathbf{u}(x_j, y_k) \equiv \mathbf{u}_{j\,k},\ \mathbf{u}_x(x,y) = \mathbf{u}_x(x_j, y_k) \equiv (\mathbf{u}_x)_{j\,k}, \ldots$, choose the integration contour, e.g.,



and add the relations for partial derivatives $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_{xx}, \ldots$

$$\int_{x_j}^{x_{j+2}} \mathbf{u}_x dx = \mathbf{u}(x_{j+2}, y) - \mathbf{u}(x_j, y), \quad \int_{y_k}^{y_{k+2}} \mathbf{u}_y dy = \mathbf{u}(x, y_{k+2}) - \mathbf{u}(x, y_k), \ldots .$$

Difference Elimination

Using a numerical integration method, e.g. the midpoint one, with

$$x_{j+1} - x_j = h_1 \,, \qquad y_{k+1} - y_k = h_2 \,,$$

we rewrite the equations and the relations as

$$\begin{gathered}
-(\mathbf{G}(\mathbf{v})_{j+1\,k} - \mathbf{G}(\mathbf{v})_{j+1\,k+2})h_1 + (\mathbf{F}(\mathbf{v})_{j+2\,k+1} - \mathbf{F}(\mathbf{v})_{j\,k+1})h_2 = 0 \,, \\
(\mathbf{u}_x)_{j+1\,k} \cdot 2h_1 = \mathbf{u}_{j+2\,k} - \mathbf{u}_{j\,k} \,, \\
(\mathbf{u}_y)_{j\,k+1} \cdot 2h_2 = \mathbf{u}_{j\,k+2} - \mathbf{u}_{j\,k} \,, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

A conservative finite-difference approximation to the PDEs is obtained by elimination of all partial derivatives $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_{xx}$, ...

KdV Equation

Describes waves on shallow water surfaces. It is example of an exactly solvable model.

$u = u(t, x)$

$$\frac{\partial}{\partial t} u + 6u \frac{\partial}{\partial x} u + \frac{\partial^3}{\partial x^3} u = 0\,.$$

Exact solution:

$$\theta = kx - \omega t\,, \qquad k, \omega \text{ are constants}\,,$$

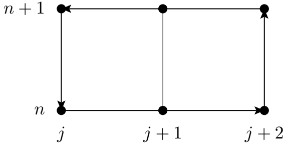
$$u = \frac{\omega}{6k} + \frac{4}{3}k^2 - 2k^2 \tanh^2(\theta)\;.$$

Hereafter, we choose the orthogonal and uniform grid with the spacings τ in t and h in x.

Discretization

$$u_t + 6uu_x + u_{xxx} = 0\,,$$

$$\oint_{\partial\Omega} -(3u^2 + u_{xx})\,dt + u\,dx = 0\,.$$



$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t, x_{j+1}) - u(t, x_j)\,, \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t, x_{j+1}) - u_x(t, x_j)\,,$$

$$\begin{cases} -(1 + \theta_t - \theta_x^2 - \theta_t\theta_x^2) \circ (3u^2 + u_{xx}) \cdot \dfrac{\tau}{2} + (\theta_x\theta_t - \theta_x) \circ u \cdot 2h = 0\,, \\ \text{midpoint rule for } x \text{ and trapezoidal rule for } t \\ (\theta_x + 1) \circ u_x \cdot \dfrac{h}{2} = (\theta_x - 1) \circ u\,, \\ \text{trapezoidal rule} \\ \theta_x \circ u_{xx} \cdot 2h = (\theta_x^2 - 1) \circ u_x\,, \\ \text{midpoint rule} \end{cases}$$

θ_x and θ_t are the right-shift operators in x and t, respectively.

Input of LDA

To eliminate partial derivatives we replace $3u^2$ by F and use Maple package LDA (Linear Difference Algebra) [2][Gerdt, Robertz'2012]

```
> restart:
> libname:=libname, "/usr/local/lib/LDA"":
> with(LDA):
> L:=[(-(F(n,j)+F(n+1,j)-F(n,j+2)-F(n+1,j+2)) -
> (uxx(n,j)+uxx(n+1,j)-uxx(n,j+2)-uxx(n+1,j+2)))*tau/2+
> (u(n+1,j+1)-u(n,j+1))*2*h,
> (ux(n,j+1)+ux(n,j))*h/2-(u(n,j+1)-u(n,j)),
> 2*uxx(n,j+1)*h-(ux(n,j+2)-ux(n,j))];
```

$$
\begin{aligned}
L := [&(\tfrac{1}{2}(-F(n,j) - F(n+1,j) + F(n,j+2) + F(n+1,j+2) \\
&- uxx(n,j) - uxx(n+1,j) + uxx(n,j+2) + uxx(n+1,j+2)))\tau \\
&+ (2(u(n+1,j+1) - u(n,j+1)))h, \\
&(\tfrac{1}{2}(ux(n,j+1) + ux(n,j)))h - u(n,j+1) + u(n,j), \\
&2uxx(n,j+1)h - ux(n,j+2) + ux(n,j)]
\end{aligned}
$$

Output of LDA

> JanetBasis(L, [n,j], [uxx,ux,u,F],2);

$$[[2\tau u(n,j+1) - \tau u(n,j) + \tau u(n+1,j+4) - 2\tau u(n+1,j+3) + 4h^3 u(n+1,j+2)$$
$$+2\tau u(n+1,j+1) - \tau u(n+1,j) + \tau u(n,j+4) - 2\tau u(n,j+3)$$
$$-4h^3 u(n,j+2) + h^2\tau F(n+1,j+3) - h^2\tau F(n+1,j+1) + h^2\tau F(n,j+3)$$
$$-h^2\tau F(n,j+1), hux(n,j+1) + hux(n,j) - 2u(n,j+1) + 2u(n,j),$$
$$h^2 uxx(n,j+1) - u(n,j+2) + 2u(n,j+1) - u(n,j), -h^2\tau uxx(n+1,j)-$$
$$h^2\tau uxx(n,j) + \tau u(n+1,j+3) - 2\tau u(n+1,j+2) + (\tau + 4h^3)u(n+1,j+1)$$
$$+\tau u(n,j+3) - 2\tau u(n,j+2) + (\tau - 4h^3)u(n,j+1) + h^2\tau F(n+1,j+2)$$
$$-h^2\tau F(n+1,j) + h^2\tau F(n,j+2) - h^2\tau F(n,j)], [n,j], [uxx, ux, u, F]]$$

Extraction of FDA

> collect(%[1,1]/(4*tau*h**3),[tau,h,sigma]);

$$\frac{\frac{1}{4}F(n+1,j+3)-\frac{1}{4}F(n+1,j+1)+\frac{1}{4}F(n,j+3)-\frac{1}{4}F(n,j+1)}{h}$$
$$+(\tfrac{1}{4}u(n+1,j+4)-\tfrac{1}{2}u(n+1,j+3)+\tfrac{1}{2}u(n,j+1)-\tfrac{1}{4}u(n,j)$$
$$-\tfrac{1}{4}u(n+1,j)+\tfrac{1}{4}u(n,j+4)-\tfrac{1}{2}u(n,j+3)+\tfrac{1}{2}u(n+1,j+1))/h^3$$
$$+\frac{u(n+1,j+2)-u(n,j+2)}{\tau}$$

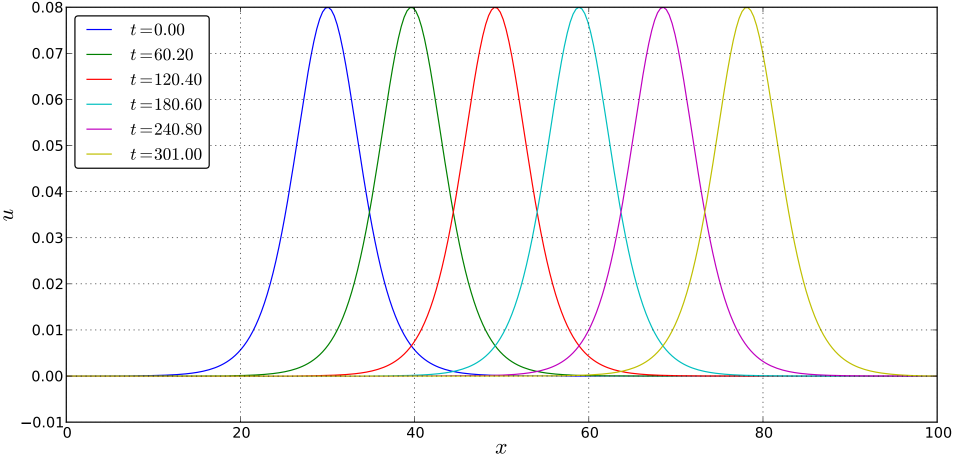
Finite Difference Approximation

Substituting back $F = 3u^2$ we obtain

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \\ + 3\,\frac{({u^2}_{j+1}^{n+1} - {u^2}_{j-1}^{n+1}) + ({u^2}_{j+1}^{n} - {u^2}_{j-1}^{n})}{4h} + \\ + \frac{(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + (u_{j+2}^{n} - 2u_{j+1}^{n} + 2u_{j-1}^{n} - u_{j-2}^{n})}{4h^3} = 0\,.$$

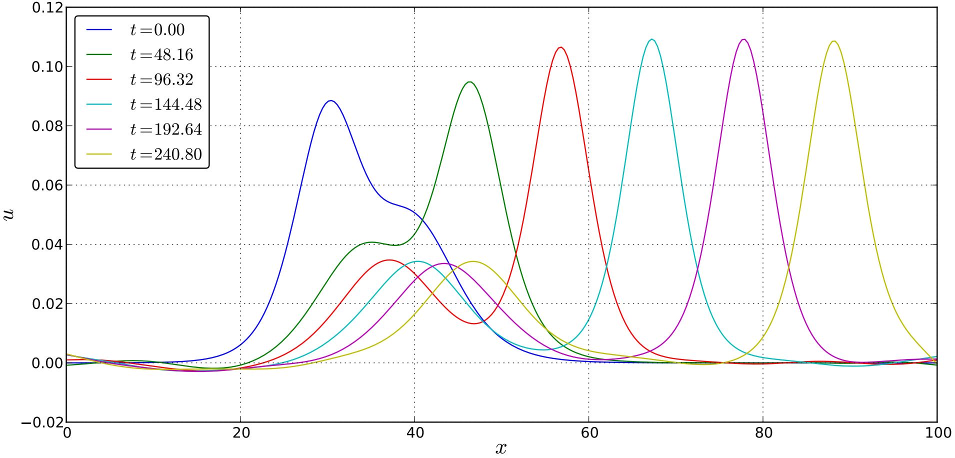
The l.h.s. is a difference Gröbner basis (no nontrivial S-polynomials), and by the construction it is consistent with KdV in the limit $h \to 0, \tau \to 0$.

Therefore FDA is s-consistent [3][Gerdt'2012].



IVP (soliton solution) $\quad f(t,x,k,\omega) = \dfrac{\omega}{6k} + \dfrac{4}{3}k^2 - 2k^2\tanh^2(kx - \omega t)$

$$u(0,x) = f(0, x+30, 0.2, 4\cdot 0.2^3)$$



Double soliton solution with $f(t,x,k,\omega) = \dfrac{\omega}{6k} + \dfrac{4}{3}k^2 - 2k^2\tanh^2(kx - \omega t)$

$$u(0,x) = f(0, x+30, 0.2, 4\cdot 0.2^3) + f(0, x+40, 0.15, 4\cdot 0.15^3)$$

KdV-Like Equations

Now we consider the following 3-parametric family of PDEs

$$\frac{\partial}{\partial t}\,\mathrm{u} + 6\,\mathrm{u}\,\frac{\partial}{\partial x}\,\mathrm{u} + \frac{\partial^3}{\partial x^3}\,\mathrm{u} - s_1\,\mathrm{u}^2\,\frac{\partial}{\partial x}\,\mathrm{u} - s_2\,\frac{\partial^2}{\partial x^2}\,\mathrm{u} - s\,\mathrm{u} = 0\,,$$

$$s,\ s_1,\ s_2 \text{ are constants}\,.$$

This family describes propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid [?][Mogilevich, Blinkova, Ivanov'2011].

$s = s_1 = s_2 = 0 \implies$ KdV equation.

The sign of s characterizes the shell material:

- $s > 0$ – nonorganic,
- $s < 0$ – living organisms,
- $s = 0$ – rubber.

Contents

One-Parametric Family

$s = s_1 = 0,\ s_2 \neq 0$

equation: $$\frac{\partial}{\partial t} u + 6\, u \frac{\partial}{\partial x} u + \frac{\partial^3}{\partial x^3} u - s_2 \frac{\partial^2}{\partial x^2} u = 0\,,$$

exact solutions:

$$\theta = \frac{1}{10} s_2 x - \omega t\,,$$
$$u = \frac{5}{3}\frac{\omega}{s_2} + \frac{1}{50} s_2^2 - \frac{1}{25} s_2^2 \tanh(\theta) - \frac{1}{50} s_2^2 \tanh^2(\theta)\ ,$$

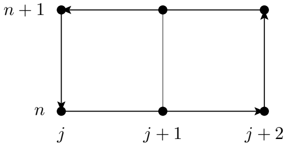
and

$$\theta = -\frac{1}{10} s_2 x - \omega t\,,$$
$$u = -\frac{5}{3}\frac{\omega}{s_2} + \frac{1}{50} s_2^2 + \frac{1}{25} s_2^2 \tanh(\theta) - \frac{1}{50} s_2^2 \tanh^2(\theta)\ .$$

Discretization

$$u_t + 6uu_x + u_{xxx} - s_2 u_{xx} = 0\,,$$

$$\oint\limits_{\partial\Omega} -(3u^2 + u_{xx} - s_2 u_x)\,dt + u\,dx = 0\,,$$



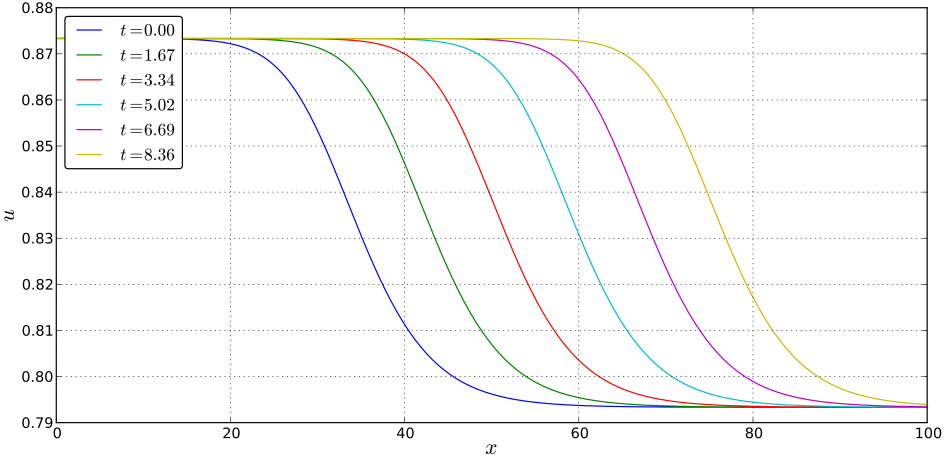
$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j)\,, \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j)\,.$$

$$\begin{cases} -(1+\theta_t - \theta_x^2 - \theta_t\theta_x^2)\circ(3u^2 + u_{xx} - s_2 u_x)\cdot\dfrac{\tau}{2} + \\ \qquad + (\theta_x\theta_t - \theta_x)\circ u\cdot 2h = 0\,, \\ (\theta_x + 1)\circ u_x\cdot\dfrac{h}{2} = (\theta_x - 1)\circ u \\ \theta_x\circ u_{xx}\cdot 2h = (\theta_x^2 - 1)\circ u_x\,. \end{cases}$$

Finite Difference Approximation

S-consistent FDA:

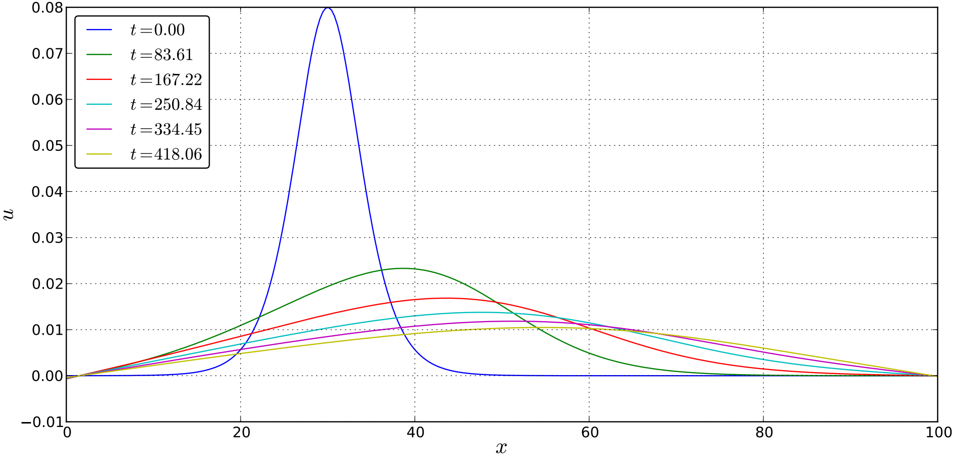
$$\frac{u_j^{n+1} - u_j^n}{\tau} +$$
$$+\,3\,\frac{({u^2}_{j+1}^{n+1} - {u^2}_{j-1}^{n+1}) + ({u^2}_{j+1}^{n} - {u^2}_{j-1}^{n})}{4h} +$$
$$+\,\frac{(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) + (u_{j+2}^{n} - 2u_{j+1}^{n} + 2u_{j-1}^{n} - u_{j-2}^{n})}{4h^3} -$$
$$-\,s_2\,\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n})}{2h^2} = 0\,.$$



Initial data $f(t, x, s_2, \omega) = \dfrac{5}{3}\dfrac{\omega}{s_2} + \dfrac{1}{50}s_2^2 -$

$$- \frac{1}{25}s_2^2 \tanh\left(\frac{1}{10}s_2 x - \omega t\right) - \frac{1}{50}s_2^2 \tanh^2\left(\frac{1}{10}s_2 x - \omega t\right)$$

$$u(0, x) = f(0, x + 30, 1.0, 0.5)$$



Soliton dynamics (blooming) for IVP with $s_2 = 1.0$ and

$$f(t,x,k,\omega) = \frac{\omega}{6k} + \frac{4}{3}k^2 - 2k^2 \tanh^2(kx - \omega t)$$

$$u(0,x) = f(0, x+30, 0.2, 4\cdot 0.2^3)$$

Contents

Two-Parametric Family

equation: $$\frac{\partial}{\partial t}\,\mathrm{u} + 6\,\mathrm{u}\,\frac{\partial}{\partial x}\,\mathrm{u} + \frac{\partial^3}{\partial x^3}\,\mathrm{u} - s_1\,\mathrm{u}^2\,\frac{\partial}{\partial x}\,\mathrm{u} - s_2\frac{\partial^2}{\partial x^2}\,\mathrm{u} = 0\,,$$

exact solution:

$$\theta = kx + t\left(-9\frac{k}{s_1} + \frac{1}{6}ks_2^2 + 2k^3\right)\,,$$

$$\mathrm{u} = \frac{3}{s_1} \pm \frac{s_2\sqrt{6}}{6\sqrt{s_1}} \pm \frac{k\sqrt{6}}{\sqrt{s_1}}\tanh(\theta)\,.$$

one-parametric subfamily:

$$\frac{\partial}{\partial t}\,\mathrm{u} + 6\,\mathrm{u}\,\frac{\partial}{\partial x}\,\mathrm{u} + \frac{\partial^3}{\partial x^3}\,\mathrm{u} - \frac{54}{s_2^2}\,\mathrm{u}^2\,\frac{\partial}{\partial x}\,\mathrm{u} - s_2\frac{\partial^2}{\partial x^2}\,\mathrm{u} = 0\,,$$

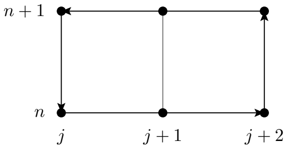
$$\theta = kx + 2tk^3\,,$$

$$\mathrm{u} = -\frac{1}{3}ks_2\tanh(\theta)\,.$$

Discretization

$$u_t + 6uu_x + u_{xxx} - s_1 u^2 u_{xx} - s_2 u_{xx} = 0\,,$$

$$\oint_{\partial\Omega} -(3u^2 + u_{xx} - \frac{s_1}{3}u^3 - s_2 u_x)\,dt + u\,dx = 0\,,$$



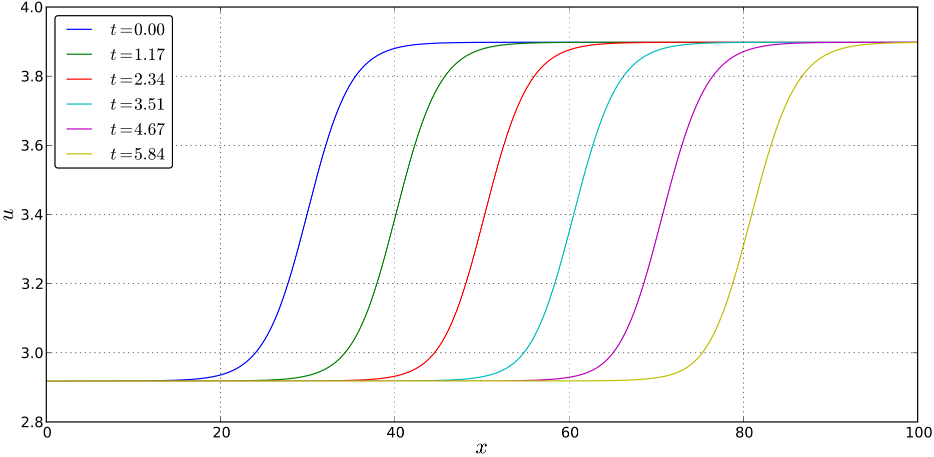
$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j)\,, \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j)\,.$$

$$\begin{cases} -(1+\theta_t - \theta_x^2 - \theta_t\theta_x^2)\circ(3u^2 + u_{xx} - \frac{s_1}{3}u^3 - s_2 u_x)\cdot\frac{\tau}{2} + \\ \qquad + (\theta_x\theta_t - \theta_x)\circ u\cdot 2h = 0\,, \\ (\theta_x + 1)\circ u_x\cdot\frac{h}{2} = (\theta_x - 1)\circ u\,, \\ \theta_x\circ u_{xx}\cdot 2h = (\theta_x^2 - 1)\circ u_x\,. \end{cases}$$

Finite Difference Approximation

S-consistent FDA:

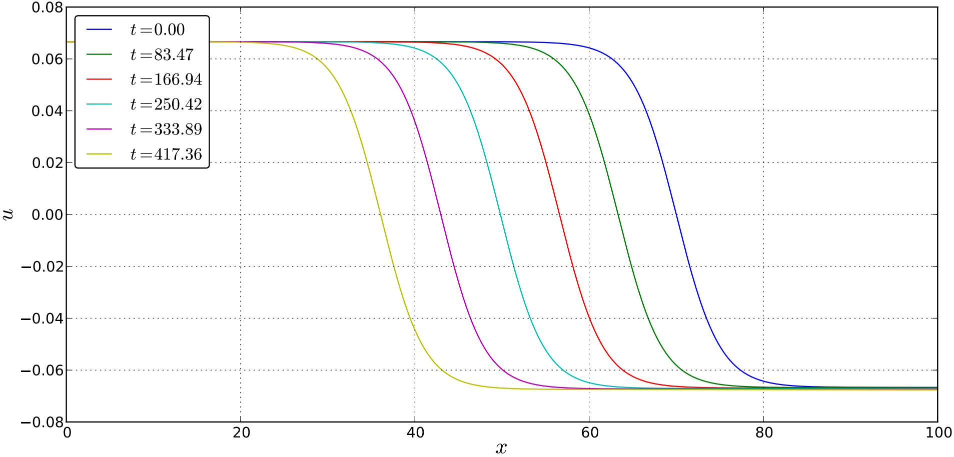
$$
\begin{gathered}
\frac{u_j^{n+1}-u_j^n}{\tau}+ \\
+3\,\frac{({u^2}_{j+1}^{n+1}-{u^2}_{j-1}^{n+1})+({u^2}_{j+1}^{n}-{u^2}_{j-1}^{n})}{4h}+ \\
+\frac{(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+(u_{j+2}^{n}-2u_{j+1}^{n}+2u_{j-1}^{n}-u_{j-2}^{n})}{4h^3}- \\
-\frac{s_1}{3}\,\frac{({u^3}_{j+1}^{n+1}-{u^3}_{j-1}^{n+1})+({u^3}_{j+1}^{n}-{u^3}_{j-1}^{n})}{4h}- \\
-s_2\,\frac{(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n})}{2h^2}=0\,.
\end{gathered}
$$



IVP for the exact solution with

$$f(t, x, k, s_1, s_2) = \frac{3}{s_1} + \frac{s_2\sqrt{6}}{6\sqrt{s_1}} + \frac{k\sqrt{6}}{\sqrt{s_1}} \tanh\left(kx + t\left(-9\frac{k}{s_1} + \frac{1}{6}ks_2^2 + 2k^3\right)\right)$$

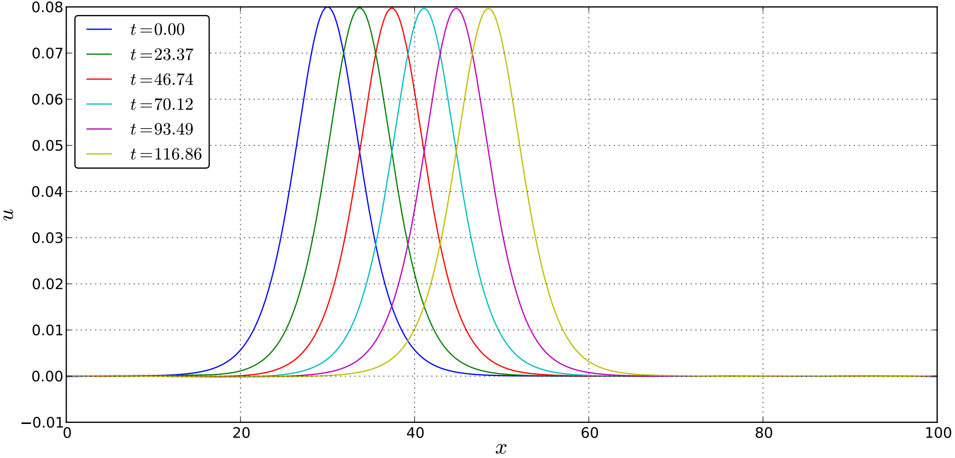
$$u(0, x) = f(0, x + 30, 0.2, 1.0, 1.0)$$



IVP for the exact solution with $s_1 = 54/s_2^2$ and

$$f(t, x, k, s_2) = -\frac{1}{3} k s_2 \tanh\left(kx + 2tk^3\right)$$

$$u(0, x) = f(0, x + 70, 0.2, 1.0)$$



Soliton dynamics for $s_1 = 1.0$, $s_2 = 0.0$ and

$$f(t, x, k, \omega) = \frac{\omega}{6k} + \frac{4}{3}k^2 - 2k^2 \tanh^2(kx - \omega t)$$

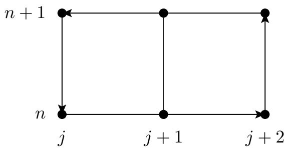
$$u(0, x) = f(0, x + 30, 0.2, 4 \cdot 0.2^3)$$

Contents

Discretization

$$u_t + 6uu_x + u_{xxx} - s_1 u^2 u_{xx} - s_2 u_{xx} - su = 0\,,$$

$$\oint_{\partial\Omega} -(3u^2 + u_{xx} - \frac{s_1}{3}u^3 - s_2 u_x)\,dt + u\,dx - - s\iint_{\Omega} u\,dt\,dx = 0\,,$$



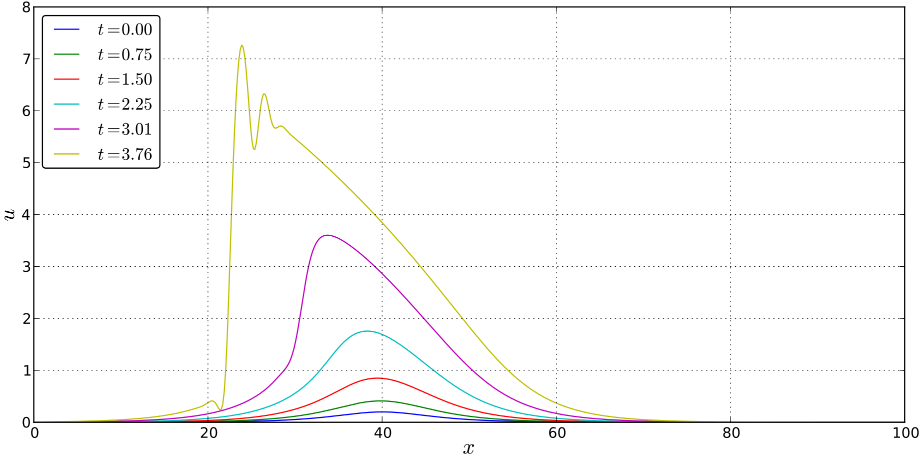
$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j)\,, \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j)\,.$$

$$\begin{cases} -(1+\theta_t-\theta_x^2-\theta_t\theta_x^2)\circ(3u^2+u_{xx}-\frac{s_1}{3}u^3-s_2u_x)\cdot\frac{\tau}{2} + \\ \qquad + (\theta_x\theta_t-\theta_x)\circ u\cdot 2h - s(\theta_x\theta_t+\theta_x)\circ u\cdot h\tau = 0\,, \\ (\theta_x+1)\circ u_x\cdot\frac{h}{2} = (\theta_x-1)\circ u\,, \\ \theta_x\circ u_{xx}\cdot 2h = (\theta_x^2-1)\circ u_x\,. \end{cases}$$

Finite-Difference Approximation

S-consistent FDA:

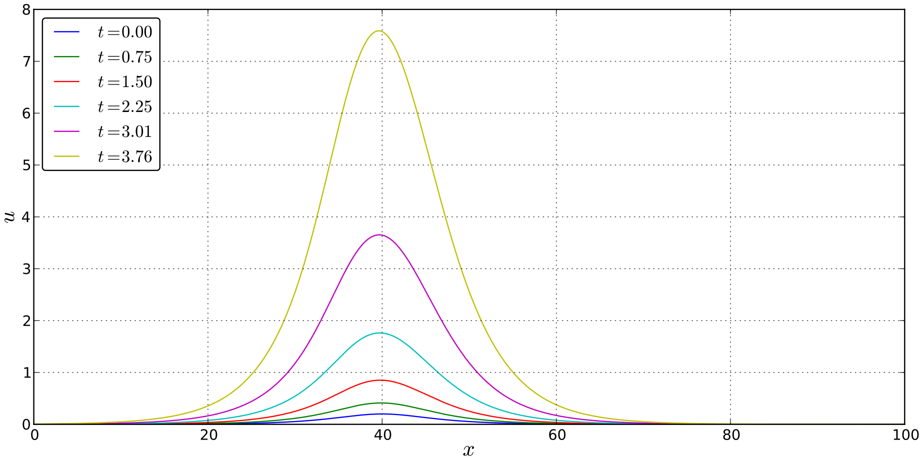
$$
\begin{gathered}
\frac{u_j^{n+1}-u_j^n}{\tau} + \\
+\,3\,\frac{({u^2}_{j+1}^{n+1}-{u^2}_{j-1}^{n+1})+({u^2}_{j+1}^{n}-{u^2}_{j-1}^{n})}{4h} + \\
+\,\frac{(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+(u_{j+2}^{n}-2u_{j+1}^{n}+2u_{j-1}^{n}-u_{j-2}^{n})}{4h^3} - \\
-\,\frac{s_1}{3}\,\frac{({u^3}_{j+1}^{n+1}-{u^3}_{j-1}^{n+1})+({u^3}_{j+1}^{n}-{u^3}_{j-1}^{n})}{4h} - \\
-\,s_2\,\frac{(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n})}{2h^2} - \\
-\,s(u_j^{n+1}+u_j^n)\frac{1}{2}=0\,.
\end{gathered}
$$



Soliton dynamics for $s_1 = 1.0,\ s_2 = 1.0\, s = 1.0$

$$f(t,x,k,\omega) = \frac{\omega}{6k} + \frac{4}{3}k^2 - 2k^2 \tanh^2(kx - \omega t)$$

$$u(0,x) = f(0, x+40, 0.2, 4 \cdot 0.2^3)$$



Soliton dynamics for $s_1 = 0.0$, $s_2 = 1.0\, s = 1.0$

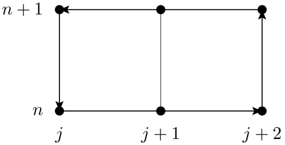
$$f(t, x, k, \omega) = \frac{\omega}{6k} + \frac{4}{3}k^2 - 2k^2 \tanh^2 (kx - \omega t)$$

$$u(0, x) = f(0, x + 40, 0.2, 4 \cdot 0.2^3)$$

Contents

System of KdV-Like Equations

$$\begin{cases} {u_1}_t + 6u_1 {u_1}_x + {u_1}_{xxx} - s_1 {u_1}^2 {u_1}_{xx} - s_2 {u_1}_{xx} + u_1 - u_2 = 0\,, \\ {u_2}_t + 6u_2 {u_2}_x + {u_2}_{xxx} - s_1 {u_2}^2 {u_2}_{xx} - s_2 {u_2}_{xx} + u_2 - u_1 = 0\,. \end{cases}$$



$$\begin{cases} \oint_{\partial\Omega} -(3{u_1}^2 + {u_1}_{xx} - \frac{s_1}{3}{u_1}^3 - s_2 {u_1}_x)\,dt + u_1\,dx + \\ \quad + \iint_{\Omega} (u_1 - u_2)\,dt\,dx = 0\,, \\ \oint_{\partial\Omega} -(3{u_2}^2 + {u_2}_{xx} - \frac{s_1}{3}{u_2}^3 - s_2 {u_2}_x)\,dt + u_2\,dx + \\ \quad + \iint_{\Omega} (u_2 - u_1)\,dt\,dx = 0\,. \end{cases}$$

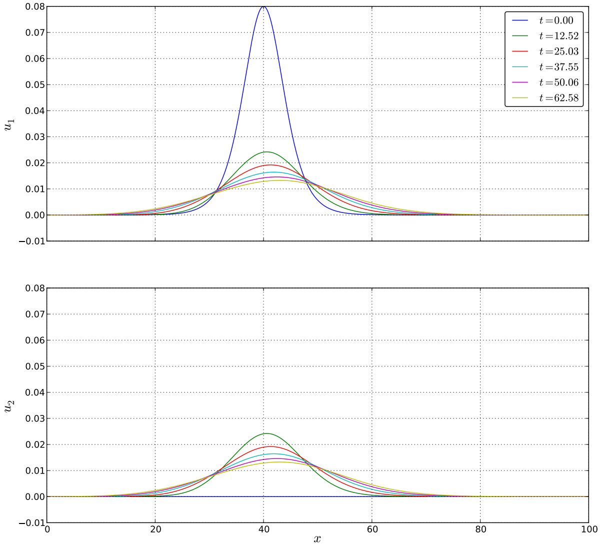
$$\int_{x_j}^{x_{j+1}} {u_1}_x\,dx = u_1(t, x_{j+1}) - u_1(t, x_j)\,, \quad \int_{x_j}^{x_{j+1}} {u_1}_{xx}\,dx = {u_1}_x(t, x_{j+1}) - {u_1}_x(t, x_j)\,,$$

$$\int_{x_j}^{x_{j+1}} {u_2}_x\,dx = u_2(t, x_{j+1}) - u_2(t, x_j)\,, \quad \int_{x_j}^{x_{j+1}} {u_2}_{xx}\,dx = {u_2}_x(t, x_{j+1}) - {u_2}_x(t, x_j)\,.$$

Discretization

$$
\begin{cases}
-(1+\theta_t-\theta_x^2-\theta_t\theta_x^2)\circ(3{u_1}^2+u_{1\,xx}-\frac{s_1}{3}{u_1}^3-s_2u_{1\,x})\cdot\frac{\tau}{2}+ \\
\qquad +(\theta_x\theta_t-\theta_x)\circ u_1\cdot 2h+(\theta_x\theta_t-\theta_x)\circ(u_1-u_2)\cdot h\tau=0\,, \\
-(1+\theta_t-\theta_x^2-\theta_t\theta_x^2)\circ(3{u_2}^2+u_{2\,xx}-\frac{s_1}{3}{u_2}^3-s_2u_{2\,x})\cdot\frac{\tau}{2}+ \\
\qquad +(\theta_x\theta_t-\theta_x)\circ u_2\cdot 2h+(\theta_x\theta_t-\theta_x)\circ(u_2-u_1)\cdot h\tau=0\,, \\
(\theta_x+1)\circ u_{1\,x}\cdot\frac{h}{2}=(\theta_x-1)\circ u_1\,, \\
\theta_x\circ u_{1\,xx}\cdot 2h=(\theta_x^2-1)\circ u_{1\,x}\,, \\
(\theta_x+1)\circ u_{2\,x}\cdot\frac{h}{2}=(\theta_x-1)\circ u_2\,, \\
\theta_x\circ u_{2\,xx}\cdot 2h=(\theta_x^2-1)\circ u_{2\,x}\,.
\end{cases}
$$

$$
\mathrm{FDA} = \begin{cases}
\dfrac{{u_1}_j^{n+1} - {u_1}_j^n}{\tau} + 3\,\dfrac{({u_1^2}_{j+1}^{n+1} - {u_1^2}_{j-1}^{n+1}) + ({u_1^2}_{j+1}^{n} - {u_1^2}_{j-1}^{n})}{4h} + \\
+ \dfrac{({u_1}_{j+2}^{n+1} - 2{u_1}_{j+1}^{n+1} + 2{u_1}_{j-1}^{n+1} - {u_1}_{j-2}^{n+1}) + ({u_1}_{j+2}^{n} - 2{u_1}_{j+1}^{n} + 2{u_1}_{j-1}^{n} - {u_1}_{j-2}^{n})}{4h^3} - \\
- \dfrac{s_1}{3}\,\dfrac{({u_1^3}_{j+1}^{n+1} - {u_1^3}_{j-1}^{n+1}) + ({u_1^3}_{j+1}^{n} - {u_1^3}_{j-1}^{n})}{4h} - \\
- s_2\,\dfrac{({u_1}_{j+1}^{n+1} - 2{u_1}_{j}^{n+1} + {u_1}_{j-1}^{n+1}) + ({u_1}_{j+1}^{n} - 2{u_1}_{j}^{n} + {u_1}_{j-1}^{n})}{2h^2} + \\
+ ({u_1}_j^{n+1} + {u_1}_j^n)\dfrac{1}{2} - ({u_2}_j^{n+1} + {u_2}_j^n)\dfrac{1}{2} = 0\,, \\
\dfrac{{u_2}_j^{n+1} - {u_2}_j^n}{\tau} + 3\,\dfrac{({u_2^2}_{j+1}^{n+1} - {u_2^2}_{j-1}^{n+1}) + ({u_2^2}_{j+1}^{n} - {u_2^2}_{j-1}^{n})}{4h} + \\
+ \dfrac{({u_2}_{j+2}^{n+1} - 2{u_2}_{j+1}^{n+1} + 2{u_2}_{j-1}^{n+1} - {u_2}_{j-2}^{n+1}) + ({u_2}_{j+2}^{n} - 2{u_2}_{j+1}^{n} + 2{u_2}_{j-1}^{n} - {u_2}_{j-2}^{n})}{4h^3} - \\
- \dfrac{s_1}{3}\,\dfrac{({u_2^3}_{j+1}^{n+1} - {u_2^3}_{j-1}^{n+1}) + ({u_2^3}_{j+1}^{n} - {u_2^3}_{j-1}^{n})}{4h} - \\
- s_2\,\dfrac{({u_2}_{j+1}^{n+1} - 2{u_2}_{j}^{n+1} + {u_2}_{j-1}^{n+1}) + ({u_2}_{j+1}^{n} - 2{u_2}_{j}^{n} + {u_2}_{j-1}^{n})}{2h^2} + \\
+ ({u_2}_j^{n+1} + {u_2}_j^n)\dfrac{1}{2} - ({u_1}_j^{n+1} + {u_1}_j^n)\dfrac{1}{2} = 0\,.
\end{cases}
$$



Soliton dynamics for $s_1 = 0.0$, $s_2 = 1.0$

$$f(t, x, k) = 2k^2(1 - \tanh(kx - 4k^3 t)^2)$$

$$u_1(0, x) = f(0, x + 40, 0.2), \quad u_2(0, x) = 0$$

Conclusions

- We applied our computer algebra assisted approach [1][Gerdt, Blinkov, Mozzhilkin'2006] based on finite volume method and difference elimination to obtain FDA to the KdV equation itself and to some KdV-like PDEs that model dynamics of elastic cylinder shells containing viscous incompressible liquid.
- The structure of a FDA depends on the numerical methods used to approximate integrals.
- All obtained FDAs are s-consistent.
- By numerical experiments we shown that the obtained FDAs nicely describe the dynamics of solitons and other exact solutions.
- Gröbner bases techniques can be used for difference elimination and for verification of s-consistency [3][Gerdt'2012].
- Difference Gröbner bases algorithms are described in [2][Gerdt, Robertz'2012], [3][Gerdt'2012] and [4][Scala'2011].

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