AN EFFICIENT ALGORITHM FOR COMPUTING RATIONAL FIRST INTEGRALS OF POLYNOMIAL VECTOR FIELDS

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During this talk we wil consider a planar polynomial vector field

$$(S)\ :\quad \left\{ \begin{array}{lcl} \dot{x} & = & A(x,y), \\ \dot{y} & = & B(x,y), \end{array} \right. \qquad A,\, B \in \mathbb{K}[x,y],$$

where $\mathbb{K}$ is a field of characteristic zero, and discuss the problem of computing rational first integrals of (S), *i.e.*, rational functions $F \in \mathbb{K}(x,y)$ that are constant on the solutions $(x(t), y(t))$ of (S). More precisely, the present paper is concerned with the following algorithmic problem:

$(\mathcal{P}_N)$: given a degree bound $N \in \mathbb{N}$, either compute a non-trivial rational first integral $F \in \mathbb{K}(x,y)$ of (S) of total degree at most N if it exists, or prove that no such F exists.

This old problem was already studied by Darboux in 1878 ([2]). It has been the subject of numerous works leading either to quadratic equations in the coefficients of F, or methods using what we called nowadays Darboux polynomials in the spirit of the celebrated Prelle-Singer's method. Recently, Chèze has shown in [1] that we can solve problem $(\mathcal{P}_N)$ in polynomial time. Unfortunately, this result is theoretical since the exponent of the polynomial in the complexity estimate is bigger than 10.
Our starting point was the article [3] of Ferragut and Giacomini. Their observation is that, (S) has a rational first integral if and only if all power series solutions of the first order non-linear differential equation

$$(E)\ :\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)},$$

are algebraic. Furthermore, minimal polynomials of these algebraic functions leads to non-trivial rational first integrals. However, they still need to solve quadratic equations.

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The main contributions of our work are the following. We push the observation of Ferragut and Giacomini further so as to give fast algorithms solving Problem $(\mathcal{P}_N)$. In particular, we prove that this can be done by considering only *linear* systems instead of quadratic systems. We develop a probabilistic algorithm using at most $\tilde{\mathcal{O}}(N^{2\omega})$ arithmetic operations, where d is the maximum of the degree of A and B, $N \geq d$ and $2 \leq \omega \leq 3$. This probabilistic algorithm is then turned into a deterministic one solving Problem $(\mathcal{P}_N)$ in at most $\tilde{\mathcal{O}}(N^{2\omega+3})$ arithmetic operations. This is to compare with the previous polynomial time algorithm that was given in [1] which uses at least $\tilde{\mathcal{O}}(d^{\omega+1}N^{4\omega+4})$ arithmetic operations.

References

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