Homomorphism between two difference operators

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Notations

- τ is a shift operator, $x \mapsto x+1$
- $D := \mathbb{C}(x)[\tau]$ is a ring of difference operators.

$$\tau \cdot f(x) = f(x+1)\tau, \quad \tau \cdot \tau^i = \tau^{i+1}$$

- $L := \tau^d + \sum_{i=0}^{d-1} a_i(x)\tau^i \in D$ corresponds to a recurrence equation

$$u(x+d) + \cdots + a_0(x)u(x) = 0$$

  we assume $a_0(x) \neq 0$ and say $\mathrm{ord}(L) = d$
- Let V be the universal extension of D and $V(L) = \ker(L, V)$.
- $L^* := \sum_{i=0}^{d} a_{d-i}(x+i-1)\tau^i$ is called the adjoint of L.

$\mathrm{Hom}(L_1, L_2)$

We want to compute

$$\mathrm{Hom}(L_1, L_2) := \{G \in \mathbb{C}(x)[\tau] \mid \mathrm{ord}(G) < \mathrm{ord}(L_1), G(V(L_1)) \subseteq V(L_2)\}.$$

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Idea:

$$\begin{aligned}\mathrm{Hom}(L_1, L_2) \subseteq \mathrm{Hom}_{\mathbb{C}}(V(L_1), V(L_2)) \\ &\cong V(L_1)^* \otimes_{\mathbb{C}} V(L_2) \\ &\cong V(L_1^*) \otimes_{\mathbb{C}} V(L_2)\end{aligned}$$

Definition

The symmetric product, $N \circledS M$, of N and $M \in D$ is an order-minimal and monic operator such that $\nu\mu \in V(N \circledS M)$ for all $\nu \in V(N)$ and $\mu \in V(M)$.

Suppose $\mathrm{ord}(N) = d_1, \mathrm{ord}(M) = d_2$,

$V(N) := \mathrm{span}_{\mathbb{C}}\{\nu_1, \ldots, \nu_{d_1}\}$ and

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$\Psi : V(N) \otimes V(M) \to V(N \circledS M)$

$\Psi(\sum a_{i,j}\nu_i \otimes \mu_j) = \sum a_{i,j}\nu_i\mu_j$ is an onto map.

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$$V(N) \otimes V(M) \ncong V(N \circledS M)$$

For each ν_i and μ_j we define,

$$\mathcal{M}(\nu_i,\mu_j) = \begin{pmatrix} \nu_i\mu_j & \nu_i\tau(\mu_j) & \cdots & \nu_i\tau^{d_2-1}(\mu_j) \\ \tau(\nu_i)\mu_j & \tau(\nu_i)\tau(\mu_j) & \cdots & \tau(\nu_i)\tau^{d_2-1}(\mu_j) \\ \vdots & \vdots & \ddots & \vdots \\ \tau^{d_1-1}(\nu_i)\mu_j & \tau^{d_1-1}(\nu_i)\tau(\mu_j) & \cdots & \tau^{d_1-1}(\nu_i)\tau^{d_2-1}(\mu_j) \end{pmatrix},$$

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Then

$$V(N) \otimes V(M) \cong Mat(N, M)$$

What is rational in $V(N) \otimes V(M)$?

For $W = \sum a_{i,j} \nu_i \otimes \mu_j \in V(N) \otimes V(M)$ define
$\Phi(W) := \sum a_{i,j} \mathcal{M}(\nu_i, \mu_j) \in Mat(N, M)$.

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Theorem

Rational elements of $V(L_1^*) \otimes_{\mathbb{C}} V(L_2)$ correspond bijectively to elements of $\mathrm{Hom}(L_1, L_2)$.

How to compute rational elements of $Mat(N, M)$

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\begin{array}{cccccc}
 & \vdots & \vdots & & \vdots & \\
\cdots & {\color{green}*} & * & \cdots & * & \cdots \\
\cdots & * & {\color{blue}*} & \cdots & * & \cdots \\
 & \vdots & \vdots & \ddots & \vdots & \\
\cdots & * & * & \cdots & * & \cdots \\
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\end{array}
$$

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$$\tau(\Phi(W)_{k,l}) = \Phi(W)_{k+1,l+1}$$

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How to compute rational elements of $Mat(N, M)$

We need $\min\{\mathrm{ord}(N), \mathrm{ord}(M)\}$ consecutive elements of $\Phi(W)$ to get all elements in $\Phi(W)$.
Suppose $\mathrm{ord}(N) = 3$ and $\mathrm{ord}(M) = 5$ then

$$* \quad * \quad *$$

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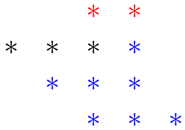

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How to compute rational elements of $Mat(N, M)$

- rational solution
  - $d(x) \in \mathbb{C}(x)$ is a denominator bound of $f(x) \in \mathbb{C}(x)$ if there exists $n(x) \in \mathbb{C}[x]$ such that $f(x) = d(x)n(x)$
  - degree of $n(x)$ is called the numerator bound of $f(x)$.

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  - degree of $n(x)$ is called the numerator bound of $f(x)$.
  - $d(x)$ can be computed from valuation growth
  - degree of $n(x)$ can be computed from generalized exponents.

How to compute rational elements of $Mat(N, M)$

Let $m \in Mat(N, M)$ be a matrix with rational entries.

$$m_{1,1} \quad m_{1,2} \quad m_{1,3} \quad \dots \quad m_{1,d_2-1}$$

- characteristic data
  - a Compute valuation growth of N and M, then we get denominator bound $d_i(n)$ for each $m_{1,i}, i \in \{1, \dots, d_2 - 1\}$
  - b Compute generalized exponent of M, then we get numerator bound j for each $m_{1,i}, i \in \{1, \dots, d_2 - 1\}$.
  - c for each $m_{1,i}$ we get $d_i(x) \sum_j c_{i,j} x^j$.
  - d generate m_{1,d_2} with shift and N and apply M to find the coefficients.

Applications of $\mathrm{Hom}(L_1, L_2)$

1. Gauge Transformation
   $L_1, L_2 \in D, \mathrm{ord}(L_1) = \mathrm{ord}(L_2)$, are said to be gauge equivalent, $L_1 \sim_g L_2$, if there exist $G \in D$ that bijectively maps $V(L_1) \to V(L_2)$.

2. Factoring
   For $L \in D$, suppose we can compute $\tilde{M}$ which is gauge equivalent to a right hand factor M of L. Then by applying $G \in \mathrm{Hom}(\tilde{M}, L)$ to $\tilde{M}$, we get M.

- $LbIK = z\tau^2 + (2 + 2v + 2x)\tau - z$
  Solutions: Modified Bessel functions of the first and second kind, $I_{v+x}(z)$ and $K_{v+x}(-z)$
- $LbJY = z\tau^2 - (2 + 2v + 2x)\tau + z$
  Solutions: Bessel functions of the first and second kind, $J_{v+x}(z)$ and $Y_{v+x}(z)$
- $LWW = \tau^2 + (z - 2v - 2x - 2)\tau - v - x - \frac{1}{4} - v^2 - 2vx - x^2 + n^2$
  Solution: Whittaker function $W_{x,n}(z)$
- $LWM = \tau^2(2n + 2v + 3 + 2x) + (2z - 4v - 4x - 4)\tau - 2n + 1 + 2v + 2x$
  Solution: Whittaker function $M_{x,n}(z)$
- $L2F1 = (z - 1)(a + x + 1)\tau^2 + (-z + 2 - za - zx + 2a + 2x + zb - c)\tau - a + c - 1 - x$
  Solution: Hypergeometric function ${}_2F_1(a + x, b; c; z)$
- $Ljc = \tau^2 - \frac{1}{2}\frac{(2x+3+a+b)(a^2-b^2+(2x+a+b+2)(2x+4+a+b)z)}{(x+2)(x+2+a+b)(2x+a+b+2)}\tau + \frac{(x+1+a)(x+1+b)(2x+4+a+b)}{(x+2)(x+2+a+b)(2x+a+b+2)}$
  Solution: Jacobian polynomial $P_x^{a,b}(z)$
- $Lgd = \tau^2 - \frac{(2x+3)z}{x+2}\tau + \frac{x+1}{x+2}$
  Solution: Legendre functions $P_x(z)$ and $Q_x(z)$
- $Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2}\tau + \frac{x+1+\alpha}{x+2}$
  Solution: Laguerre polynomial $L_x^{(\alpha)}(z)$
- $Lgb = \tau^2 - \frac{2z(m+x+1)}{x+2}\tau - \frac{2m+x}{x+2}$
  Solution: Gegenbauer polynomial $C_x^m(z)$
- $Lgr1 = (x + 2)\tau^2 + (x + z - b + 1)\tau + z$
  Solution: Laguerre polynomial $L_x^{(b-x)}(z)$
- $Lkm = (a + x + 1)\tau^2 + (-2a - 2x - 2 + b - c)\tau + a + x + 1 - b$
  Solution: Kummer's function $M(a + x, b, c)$
- $L2F0 = \tau^2 + (-zb + zx + z + za - 1)\tau + z(b - x - 1)$
  Solution: Hypergeometric function ${}_2F_0(a, b - x; ; z)$
- $Lge = (x + 2)\tau^2 + (-ab - d + (a + 1)(1 + x))\tau + ax - a(b + d)$
  Solution: Sequences whose ordinary generating function is $(1 + ax)^b(1 + bx)^d$

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Thank you for your attention