A DifferentialAlgebra Package in Sage

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In this talk, we present a new Sage [7] package[1], called *DifferentialAlgebra*, dedicated to the differential algebra theory, initiated by Ritt and Kolchin [6, 5]. The Sage package is written in Cython. It is mostly an interface for the open source BLAD and BMI libraries [1], which underlie the MAPLE *DifferentialAlgebra* package.

The main functionality of the package is a simplification tool for systems of polynomial differential equations [3, Algorithm *Rosenfeld-Gröbner*]. With more technical words, this simplifier takes as input a system of polynomial differential equations, a ranking [5, chapter I, section 8], and computes a representation of the radical of the differential ideal generated by the input system as an intersection of differential ideals presented by regular differential chains [2, definition 3.1], with respect to the given ranking. The decomposition can be performed over differential base fields presented by generators and relations. The simplifier is also able to take advantage of theorems which only apply in some specific cases: lower bounds on the dimension of the irreducible components in the non-differential case, Low Power Theorem [6, chapter III] in the case of a system made of a single differential equation.

During the talk, some applications of the package will be sketched among the following ones: the analysis of a differential index 2 system of differential algebraic equations ; the quasi-steady state approximation of a chemical reaction system ; the computation of an input-output equation of a dynamical system. As a starter, here is an academic, simple but famous example of Ritt [6, chapter II], which is a good starting point for beginners in differential algebra:

Consider the differential equation $y'^2 - 4\,y = 0$. Its solution set can be decomposed into two parts: the general solution $y(x) = (x+c)^2$ where c denotes some constant ; and a singular solution $y(x) = 0$. Observe that the singular solution is not a particular case of the general one, since $(x + c)^2 \neq 0$ for any value of c. As a polynomial in the two indeterminates y and y', the equation is

[1]At the moment this abstract is written, the package is still being polished. It may not be included in Sage at conference time.

irreducible. However, its first derivative factors: $(y'^2 - 4y)' = 2\,y'\,(y'' - 2)$. The differential ideal generated by the equation is the intersection of two differential ideals presented by regular differential chains, called essential components in [4]:

```
sage: from sage.libs.blad.DifferentialAlgebra import DifferentialRing
sage: x = var('x')
sage: y = function ('y')
sage: R = DifferentialRing (derivations = [x], blocks = [y])
sage: eqn = diff(y(x),x)**2-4*y(x) == 0
sage: L = R.RosenfeldGroebner ([eqn], singsol='essential')
sage: [ C.equations (solved=true) for C in L ]
[[D[0](y)(x)^2 == 4*y(x)], [y(x) == 0]]
```

Consider now the differential equation $y'^2 - 4\,y^3 = 0$. Its general solution is $y(x) = 1/(x+c)^2$. The function $y(x) = 0$ is still a solution (a particular solution) but not a singular solution for it is the limit of the general solution when c tends towards infinity. The equation and its first derivative are irreducible. It turns out that the differential ideal generated by the equation is prime:

```
sage: eqn = diff(y(x),x)**2-4*y(x)**3 == 0
sage: L = R.RosenfeldGroebner ([eqn], singsol='essential')
sage: [ C.equations (solved=true) for C in L ]
[[D[0](y)(x)^2 == 4*y(x)^3]]
```

References

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