Operational Methods in the Environment of a Computer Algebra System

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Survey of our joint research with Prof. Ivan Dimovski

Main aspects

- Operational calculus (OC) approach
- Classical methods and algorithms, based on the OC of Heaviside–Mikusinski
- Operational calculi of Mikusinski type, developed by I. Dimovski, for solving boundary value problems (BVPs)
- Program implementation of the considered algorithms using the computer algebra system *Mathematica*

Operational calculus

- Main idea of the operational calculus
  Transformation of calculus problems to algebraic problems, treating the differentiation operator as algebraic object.
- Historical notes
  (works of Leibnitz, Euler, Cauchy, …)
  - Oliver Heaviside (1850 - 1925)
  - Jan Mikusinski (1913 - 1987)
  - L. Berg, I.H. Dimovski, V.A. Ditkin, A.P. Prudnikov, K. Yosida, etc.

Heaviside-Mikusinski Operational Calculus and its applications

- The approach of Mikusinski
- Heaviside algorithm for solving Cauchy problems for linear ordinary differential equations with constant coefficients
- Program implementation of the algorithm using *Mathematica* system

The approach of Mikusinski

Mikusinski started from the classical Duhamel convolution

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

considering the space $C[0,\infty)$ of the continuous functions on $[0,\infty)$ as a ring on $\mathbb{R}$ or $\mathbb{C}$

In the same way, as the ring $\mathbb{Z}$ of the integers is extended to the field $\mathbb{Q}$ of the rational numbers, Mikusiński extended the ring $(C[0,\infty),*)$ to the smallest field $\mathcal{M}$ containing the initial ring. We denote it by $\mathcal{M}$ and name it Mikusiński's field. The elements of $\mathcal{M}$ are convolution fractions

$$\frac{f}{g} = \frac{\{f(t)\}}{\{g(t)\}},$$

called "operators".

The basic operator in the Mikusiński approach is the integration operator

$$lf(t) = \int\limits_0^t f(\tau)d\tau.$$

In fact, l is the convolution operator $l = \{1\} *$.

The algebraic analogue of the differentiation operator $D = \dfrac{d}{dt}$ is the convolution fraction

$$s = \frac{1}{l},$$

which is not an operator in the proper sense of the word, but an algebraic object.

Specific notation: $f = \{f(x)\}$

The relation between the derivative $f'(t)$ and the product $s\{f(t)\}$ is given by the basic formula of Mikusiński's operational calculus

$$\{f'(t)\} = s\{f(t)\} - f(0)$$

If the function $f = \{f(t)\}$ has continuous derivatives up to n-th order for $0 \le t < \infty$, a more general formula can be derived:

$$f^{(n)} = s^n f - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0), \; n = 1,2,3,\ldots$$

Application for solving initial value problems for Linear Ordinary Differential Equations (LODE) with constant coefficients!

The so-called Heaviside algorithm

- It is intended for solving initial value problems for LODE with constant coefficients using the operational calculus approach; we use it in the frames of the Mikusinski's operational calculus.
- Consider the following Cauchy problem (such problems are known as initial values problems as well):

$$P\left(\frac{d}{dt}\right) y = f(t),\ y(0) = \gamma_0,\ y'(0) = \gamma_1,\ \cdots,\ y^{(n-1)}(0) = \gamma_{n-1}$$

- Main steps of the algorithm: algebraization of the problem using the basic formula of the Mikusinski's operational calculus and thus reducing it to one algebraic equation; obtaining solution of this equation; expansion of the rational expressions in this solution into elementary fractions and interpretation of these fractions as functions, etc. (more details on the next slide)

Main steps of the Heaviside algorithm

- Algebraization of the problem

$$P(s)y = f + Q(s)$$

- Solution of the algebraic equation

$$y = \frac{1}{P(s)} f + \frac{Q(s)}{P(s)}$$

- Factorization of P and decomposition of 1/P and Q/P into elementary fractions

$$\frac{1}{(s-\alpha)^n} = \left\{ \frac{t^{n-1}}{(n-1)!} e^{\alpha t} \right\}, \; n = 1, 2, \ldots$$

- Interpretation of the elementary fractions as functions using table of formulae

$$\frac{s-\alpha}{(s-\alpha)^2 + \beta^2} = \left\{ e^{\alpha t} \cos \beta t \right\}, \; etc.$$

- Denotation of the obtained functions

$$1/P(s) = G(t), \;\; Q(s)/P(s) = R(t)$$

- General form of the solution; computation of the Duhamel convolution in it

$$y = G(t) * f(t) + R(t)$$

- Returning the result or a message that the problem can not be solved.
- About the program implementation of the above steps.

Features of our program implementation of the Heaviside algorithm

- A program package for *Mathematica* system is developed; it provides use of the Heaviside algorithm for solving Cauchy problems for one LODE with constant coefficients or a system of such equations (examples are given on the next slide).
- Another complete implementation of the algorithm has not been published yet.
- Some advantages of our package in comparison with other implemented algorithms for solving Cauchy problems were achieved.

Example 1. Initial value problem for one LODE with constant coefficients

```
DSolveOC[{y'''''[t] + 2 y'''[t] + y'[t] == α t + β Sin[t] + γ Cos[t],
  y[0] == a0, y'[0] == a1, y''[0] == a2, y'''[0] == a3, y''''[0] == a4}, y[t], t]
```

$$y[t] \to \frac{1}{8}\left(4\left(-4+t^2\right)\alpha + 8\beta + \left(16\alpha + \left(-8+t^2\right)\beta - 3t\gamma\right)\mathrm{Cos}[t] + \left(3\gamma - t\left(-4\alpha + 5\beta + t\gamma\right)\right)\mathrm{Sin}[t]\right) + a_0 +$$
$$\mathrm{Sin}[t]\, a_1 + 2a_2 - \frac{1}{2}\left(t\,\mathrm{Cos}[t] - \mathrm{Sin}[t]\right)\left(a_1 + a_3\right) + a_4 - \frac{1}{2}t\,\mathrm{Sin}[t]\left(a_2 + a_4\right) - \mathrm{Cos}[t]\left(2a_2 + a_4\right)$$

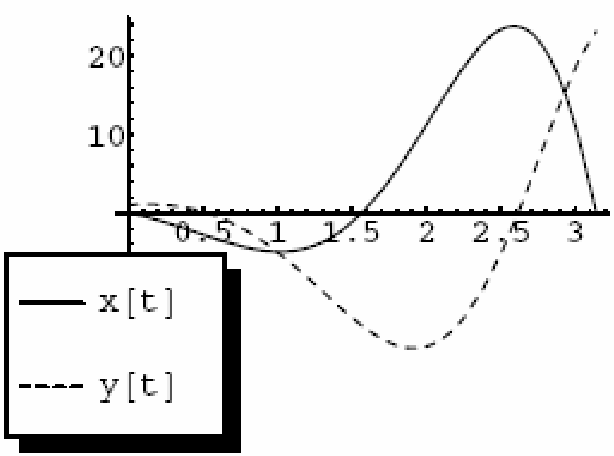
Example 2. Initial value problem for a system of LODE with constant coefficients; an option for visualization of the solution is used

```
mysyst = {-x[t] + 2 y[t] + x'[t] == -2 e^t, -2 x[t] - y[t] + y'[t] == 0, x[0] == 0, y[0] == 1};

DSolveOC[mysyst, {x[t], y[t]}, t, GraphInterval → {0, π}]
```

$$\{x[t] \to -4e^t\,\mathrm{Cos}[t]\,\mathrm{Sin}[t],\ y[t] \to e^t\left(-1 + 2\,\mathrm{Cos}[2t]\right)\}$$

Visualization of the solution:



Extension of the Heaviside algorithm for solving nonlocal BVPs

- An extension of the Heaviside algorithm is developed by I. Dimovski and S. Grozdev, for finding periodic solutions of linear ordinary differential equations with constant coefficients both in the non-resonance and in the resonance cases. The problem is reduced to a nonlocal BVP.
- Program implementation of the algorithm is developed.

The problem

Consider the LODE with constant coefficients:

$$P\left(\frac{d}{dt}\right) y = f(t), \quad -\infty < t < \infty$$

We are looking for a periodic solution $y(t)$ with period T of this equation, i.e. a solution satisfying the identity:

$$y(t+T) = y(t), \quad -\infty < t < \infty$$

An obvious necessary condition:

$$f(t+T) = f(t)$$

"Boundary" conditions:

$$y(T) - y(0) = 0,\ y'(T) - y'(0) = 0,\ \ldots\ y^{(n-1)}(T) - y^{(n-1)}(0) = 0$$

Equivalent problem

- The problem of finding periodic solutions is equivalent to the following BVP:

$$P\left(\frac{d}{dt}\right)y = f(t)$$
$$y^{(k)}(T) - y^{(k)}(0) = 0,\ k = 0, 1, \ldots n-1$$

- An intermediate (auxiliary) problem, considered by Dimovski and Grozdev:

$$P\left(\frac{d}{dt}\right)y = f(t),\quad -\infty < t < \infty$$
$$\int_0^T y(\tau)\,d\tau = \alpha_0,\ \ y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}$$
$$k = 0, 1, \ldots n-2,\ \ n = \deg P$$

Operational calculus of Dimovski for solving nonlocal BVP

- Convolution of Dimovski and Grozdev:

$$(f * g)(t) = \frac{f(t)}{T} \int_0^T g(\tau)\, d\tau + \frac{g(t)}{T} \int_0^T f(\tau)\, d\tau$$

$$-\frac{1}{T}\int_0^t f(t-\tau)\, g(\tau)\, d\tau - \frac{1}{T}\int_t^T f(t+T-\tau)\, g(\tau)\, d\tau$$

- Algebraic analogon of the operator d/dt:

$$S = \frac{1}{L} \qquad Lf = \left\{ t - \frac{T}{2} \right\} * f$$

- Basic formula of the operational calculus of Dimovski:

$$\{f'(t)\} = S\{f(t)\} - \frac{1}{T}\int_0^T f(\tau)d\tau$$

Solving the auxiliary problem

- Extention of the classical Heaviside algorithm, considered above, with:
  - verification of conditions for the so-called resonance case;
  - use of new formulae for interpretation of the fractions in the algebraic solution.
- The general solution is a sum of the non-resonance and the resonance parts of the solution (it is possible, of course, each of these parts to be equal to zero).

Additional considerations

- Reducing the initial problem to the auxiliary problem
- Derivation of new interpretation formulae
  (a part of these formulae is given below)
- Development of a *Mathematica* program package
- Advantages in comparison with other algorithms for finding periodic solutions

  (for ex., Mathematica is unable to resolve some of the arbitrary constants in the general solution of the resonance problems)

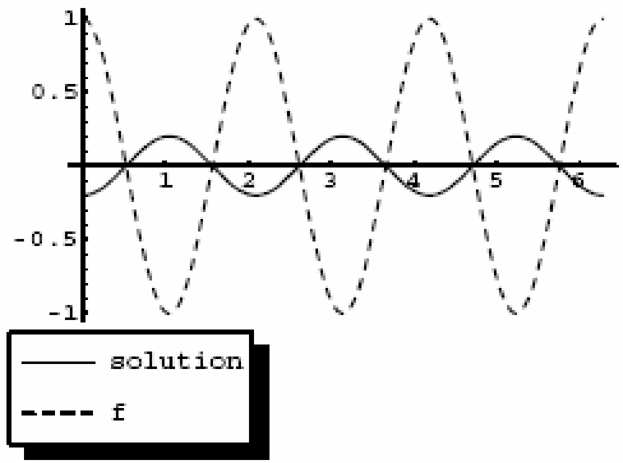
Two examples – respectively of resonance and “mixed” cases. The periodic solutions are found by means of our package

Example1 :

```
DSolveOCP[{y'(t) + 4 y(t) == Cos[3 t], α[1] == 0},
 y[t], t, 2 π, Graph → True]
```

$y[t] \to -\frac{1}{5}\,\mathrm{Cos}[3\,t]$

Visualization of the solution :



Example2 :

```
de = y'''[t] + y''[t] + 4 y'[t] + 4 y[t] == Cos[5 t];

DSolveOCP[{de, α[1] == 0, α[2] == 0}, y[t], t, 2 π]
```

$y[t] \to \frac{1}{546}\,(-\mathrm{Cos}[5\,t] - 5\,\mathrm{Sin}[5\,t])$

Main part of the interpretation formulae used by our program implementation:

For the non-resonance case:

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{T\, \partial_{\{a,-1+m\}} \frac{e^{t a}}{-1+e^{T a}}}{(-1+m)\,!}$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} - \frac{T\, \mathrm{Cos}[a\, t - \frac{a T}{2}]\, \mathrm{Csc}[\frac{a T}{2}]}{2\, a}$$

$$\frac{S}{S^2+a^2} = \frac{1}{2}\, T\, \mathrm{Csc}\left[\frac{a T}{2}\right] \mathrm{Sin}\left[a\, t - \frac{a T}{2}\right]$$

$$\frac{c\, S + d}{S^2+a^2} = c\left(\frac{1}{a^2} - \frac{T\, \mathrm{Cos}[a\, t - \frac{a T}{2}]\, \mathrm{Csc}[\frac{a T}{2}]}{2\, a}\right) + d\left(\frac{1}{2}\, T\, \mathrm{Csc}\left[\frac{a T}{2}\right] \mathrm{Sin}\left[a\, t - \frac{a T}{2}\right]\right)$$

$$\frac{1}{S^2+p\, S+q} = \frac{1}{q} + \frac{\left(\frac{e^{\frac{1}{2}\left(-p+\sqrt{p^2-4 q}\right) t}}{-1+e^{\frac{1}{2}\left(-p+\sqrt{p^2-4 q}\right) T}} - \frac{e^{-\frac{1}{2}\left(p+\sqrt{p^2-4 q}\right) t}}{-1+e^{-\frac{1}{2}\left(p+\sqrt{p^2-4 q}\right) T}}\right) T}{\sqrt{p^2-4 q}}, \quad p^2 - 4 q \neq 0$$

For $\sqrt{p^2-4 q} = \delta,\ p+\delta = \alpha,\ -p+\delta = \beta$:

$$\frac{1}{S^2+p\, S+q} = \frac{1}{q} + \frac{T}{\delta}\, \frac{\left(-\frac{e^{-\frac{t \alpha}{2}}}{-1+e^{-\frac{T \alpha}{2}}} + \frac{e^{\frac{t \beta}{2}}}{-1+e^{\frac{T \beta}{2}}}\right)}{\sqrt{p^2-4 q}}$$

$$\frac{S}{S^2+p\, S+q} = \frac{T}{\delta}\left(\frac{e^{-\frac{t \alpha}{2}}\, \alpha}{2\left(-1+e^{-\frac{T \alpha}{2}}\right)} + \frac{e^{\frac{t \beta}{2}}\, \beta}{2\left(-1+e^{\frac{T \beta}{2}}\right)}\right)$$

For the resonance case:

$$\frac{1}{S^m} = \frac{T^m}{m!} B\left[m, \frac{t}{T}\right]$$

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{e^{a t}}{m!} T^m B\left[m, \frac{t}{T}\right]$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} + \frac{t \,\mathrm{Sin}[a\, t]}{a} - \frac{T \,\mathrm{Sin}[a\, t]}{2\, a}$$

$$\frac{S}{S^2+a^2} = t \,\mathrm{Cos}[a\, t] - \frac{1}{2} T \,\mathrm{Cos}[a\, t] + \frac{\mathrm{Sin}[a\, t]}{a}$$

$$\frac{c\, S + d}{S^2+a^2} = c\left(\frac{1}{a^2} + \frac{t \,\mathrm{Sin}[a\, t]}{a} - \frac{T \,\mathrm{Sin}[a\, t]}{2\, a}\right) + d\left(t \,\mathrm{Cos}[a\, t] - \frac{1}{2} T \,\mathrm{Cos}[a\, t] + \frac{\mathrm{Sin}[a\, t]}{a}\right)$$

$$\frac{1}{S^2 + p\, S + q} = \frac{e^{-\frac{1}{2} t\, (p+\delta)}}{2\, q\, \delta}\left(e^{t\, \delta} q\, (2\, t - T) + q\, (-2\, t + T) + 2\, e^{\frac{1}{2} t\, (p+\delta)}\, \delta\right), \quad p^2 - 4\, q \neq 0$$

where $B[m, \frac{t}{T}]$ is the polynomial of Bernoulli of degree m.

Operational calculus approach for solving BVPs for some partial differential equations

- Application of the operational calculus approach for solving local and nonlocal BVPs for three classical equations of mathematical physics in finite domains:
  - the heat equation
  - the wave equation
  - the equation of a free supported beam.
- Program packages for *Mathematica* system are developed for solving the problems.

About the approach

- Extension of the Duhamel principle for the space variable.
- Use of an approach, suggested by I. Dimovski, for extension of the OC of Heaviside-Mikusinski for functions of two variables.
- Application of this approach to local and non-local boundary value problems.

Duhamel principle

Duhamel formulated his principle in 1830. Due to this principle the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \; u(1,t) = \varphi(t), \; u(x,0) = 0$$

can be obtained for arbitrary $\varphi(t)$, if a solution $U(x,t)$ of the same problem but for a special choice of $\varphi(t)$, namely for $\varphi(t) \equiv 1$, is available. Then the general solution has the form:

$$u(x,t) = \frac{\partial}{\partial t} \int_0^t U(x, t-\tau)\varphi(\tau)d\tau$$

for $0 \leq x \leq 1, \; 0 \leq t$.

The special solution can be obtained using the Fourier method

Convolutions
for BVP for the considered equations

- Convolution for the differentiation operator

  Basic BVP: $y' = f(t),\ \chi(y) = 0$

  (χ is a linear functional on $C[0,\infty)$)

  The solution $y = l\,f(t)$:

$$l\,f(t) = \int_0^t f(\tau)\,d\tau - \chi_\tau\{\int_0^t f(\tau)\,d\tau\}$$

  Basic convolution:

$$(f * g)(t) = \chi_\tau\{\int_\tau^t f(t-\sigma+\tau)\,g(\sigma)\,d\sigma\}$$

- Convolution for the square of the differentiation operator

  Simple non-local BVP: $y'' = f(x),\ \ y(0) = 0,\ \ \Phi\{y\} = 0$

  (Φ is a linear functional on $C^1[0,a]$)

  The solution $y = L f(x)$:

$$L f(x) = \int_0^x (x-\xi)\ f(\xi)\,d\xi - x\,\Phi_\xi\{\int_0^\xi (\xi-\eta)\ f(\eta)\,d\eta\}$$

  Basic convolution: $(f*g)(x) = -\frac{1}{2}\Phi_\xi\left\{\int_0^\xi h(x,\eta)\,d\eta\right\}$

where

$$h(x,\eta) = \int_x^\eta f(\eta+x-\zeta)\,g(\zeta)\,d\zeta - \int_{-x}^\eta f(|\eta-x-\zeta|)\,g(|\zeta|)\,\mathrm{sgn}(\eta-x-\zeta)\,\zeta\,d\zeta$$

One more convolution

- Two-variate convolution

$$\{u(x,t)\} * \{v(x,t)\} = -\frac{1}{2}\tilde{\Phi}_\xi \chi_\tau \{h(x,t;\xi,\tau)\}$$

where

$$h(x,t;\xi,\tau) = \int_\xi^x \int_\tau^t u(x+\xi-\eta, t+\tau-\sigma)\, v(\eta,\sigma)\, d\sigma d\eta -$$

$$- \int_{-\xi}^x \int_\tau^t u(|x-\xi-\eta|, t+\tau-\sigma)\, v(|\eta|,\sigma)\, \mathrm{sgn}\,[(x-\xi-\eta)\eta]\; d\sigma\, d\eta$$

Operational calculus for the operators *L* and *l*

- Basic formulae of the operational calculus for *l* and *L*

$$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x,\tau)\}]_t \qquad \frac{\partial^2 u}{\partial x^2} = Su - [\Phi_\xi\{u(\xi,t)\}]_x$$

where the indices t and x mean that the corresponding functions of t and x are considered "partial" numerical operators.

These formulae express the relation between the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ and the products su and Su, with $s = \frac{1}{l}$, $S = \frac{1}{L}$.

In order to illustrate the application of the OC, briefly described above, let's consider the following class of BVP:

$$
\begin{aligned}
&u_t = u_{xx} + F(x,t),\ \ 0 < x < a,\ t > 0\\
&u(0,t) = 0,\ \ \Phi_\xi\{u(\xi,t)\} = 0\\
&\chi_\tau\{u(x,\tau)\} = f(x),
\end{aligned}
$$

where Φ and χ are linear functionals respectively in $C^1[0,a]$ and $C[0,\infty]$.

Using the main formulae, we reduce the problem to the single equation:

$$(s-S)u \ = \ [f(x)]_t + \{F(x,t)\}$$

Assuming that $s-S$ is not a divisor of 0 (this assumption is equivalent to the requirement for uniqueness of the solution), we can write the following form of the solution:

$$u \ = \ \frac{1}{s-S}\,[f(x)]_t + \frac{1}{s-S}\{F(x,t)\}$$

Consider the partial solution $\Omega(x,t)$ of the equation for $F(x,t) \equiv 0$ and $f(x) \equiv x$. This solution is an algebraic object and it has the form:

$$\Omega = \frac{1}{S(s-S)},$$

since $[f(x)]_t = [x]_t = \dfrac{1}{S}$.

Theorem. *If $\Omega(x,t)$ is a function in $C(\Delta)$, the problem*

$$u_t = u_{xx}, \;\; u(0,t) = 0, \;\; \Phi_\xi\{u(\xi,t)\} = 0, \;\; \chi_\tau\{u(x,\tau)\} = f(x)$$

with $f(0) = 0$, $\Phi\{f\} = 0$ and $f \in C^2[0,a]$ has a classical solution $u(x,t)$ of the form

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{(x)}{*} f(x)\right\}.$$

Having in mind this theorem, we can obtain a representation of the solution of given BVP for the heat equation.

Local BVP for the heat equation

- BVP:

$$u_t = u_{xx}, \;\; 0 \le x \le 1, \; t \ge 0, \;\; u(0,t) = 0, \; u(1,t) = 0, \;\; u(x,0) = f(x)$$

- Form of the solution:

$$u(x,t) = \int_0^1 \left[\Omega(1 - x - \xi, t) - \Omega(1 + x - \xi, t)\right] f(\xi) d\xi,$$

where

$$\Omega(x,t) = \sum_{n=1}^{\infty} (-1)^n \exp(-n^2 \pi^2 t) \cos n\pi x$$

Non-local BVP for the heat equation

- BVP (Samarskii – Ionkin problem)

$$u_t = u_{xx},\ u(0,t) = 0,\ \int_0^1 u(x,\tau)d\tau = 0,\ u(x,0) = f(x)$$

$$u(x,t) = -2\int_0^x \Omega(x-\xi,t)f(\xi)d\xi - \int_x^1 \Omega(1+x-\xi,t)f(\xi)d\xi \\ + \int_{-x}^1 \Omega(1-x-\xi,t)f(|\xi|)\mathrm{sgn}\xi d\xi,$$

$$\Omega(x,t) = \sum_{n=1}^{\infty}\{-2x\cos 2n\pi x + 8\pi nt\sin 2n\pi x\}\, e^{-4n^2\pi^2 t}$$

Other BVPs

- In a similar way we can obtain formulae for the solutions of BVPs for the wave equation and for the equation of a free supported beam.
- Examples of derived representations of the solutions of BVP with local and non-local boundary value conditions are given on the next slides.
- Program packages for the computer algebra system *Mathematica* were developed (for everyone of the considered equations). They provide numerical computation and visualization of the solutions of given (local or non-local) BVPs, using our representations.

Example of a BVP for the wave equation

- Non-local BVP:

$$u_{tt} = u_{xx}, \quad 0 < x < 1,\ 0 < t < \infty$$
$$u(0,t) = 0,\ \int_0^1 u(\xi,t)d\xi = 0$$
$$u(x,0) = f(x),\ \ u_t(x,0) = g(x)$$

- Form of the solution

Case 1. $f(x) \equiv 0,\ g(x) \neq 0.$

We use the solution $\Omega(x,t)$ for $g(x) \equiv x^3/6 - x/12$ and $f(x) \equiv 0$; we have:

$$\Omega(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{x\cos(2\,n\,\pi\,x)\,\sin(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} + \left(\frac{t\cos(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} - \frac{3\,\sin(2\,n\,\pi\,t)}{8\,n^4\,\pi^4} \right) \sin(2\,n\,\pi\,x) \right\}$$

$$u(x,t) = -2\int_0^x \Omega_x(x-\xi,t)g'(\xi)d\xi - \int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)d\xi\text{-} \ + \int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)d\xi$$

Visualization of the solution

- A table of numerical values of the solution is obtained using the package for the wave equation, for the following boundary function:

$$g(x) = 2\pi x \cos 2\pi x + \frac{3}{2} \sin 2\pi x.$$

- visualization of the solution:

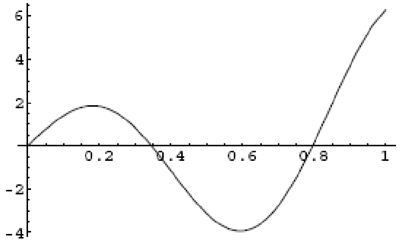


Figure 7: $g(x)$

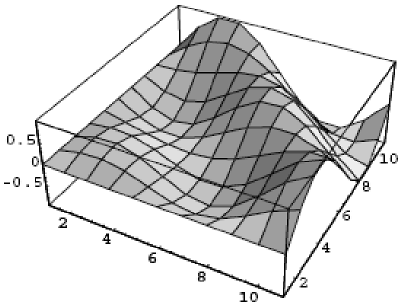


Figure 8: Relief of the solution

Local BVP for the Beam equation

- BVP

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4},\ 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(1,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

$$u(x,t) = -\frac{1}{2}\int_x^1 \Omega_x(1+x-\xi,t)\, g(\xi)\, d\xi + \frac{1}{2}\int_{-x}^1 \Omega_x(1-x-\xi,t) g(|\xi|)\, \mathrm{sgn}\, \xi\, d\xi,$$

where

$$\Omega_x(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin(n\pi)^2 t \cos n\pi x$$

Non-local BVP for the Beam equation

- BVP

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4},\ 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0$$
$$\int_0^1 u(\xi,t)d\xi = 0,\ u_x(1,t) - u_x(0,t) = 0$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

- Form of the solution for $f(x) \neq 0,\ g(x) \equiv 0$

$$u(x,t) = -2\int_0^x \tilde{\Omega}_{xx}(x-\xi,t)f^{iv}(\xi)d\xi + \int_x^1 \tilde{\Omega}_{xx}(1+x-\xi,t)f^{iv}(\xi)d\xi -$$
$$\int_{-x}^1 \tilde{\Omega}_{xx}(1-x-\xi,t)f^{iv}(|\xi|)\mathrm{sign}(\xi)d\xi + 2f^{iv}(\xi)(\tilde{\Omega}_x(0,t) - \tilde{\Omega}_x(1,t)) + f(x),$$

More about the above representation

$$\tilde{\Omega}_{xx}(x,t) = \Big\{ -\frac{x\cos(2n\pi x)\sin(2n^2\pi^2 t)^2}{4n^4\pi^4} - \frac{\left(4n^2\pi^2 t\cos(2n^2\pi^2 t) - 3\sin(2n^2\pi^2 t)\right)\sin(2n^2\pi^2 t)\sin(2\pi x)}{4n^7\pi^5} - \frac{\sin(2n^2\pi^2 t)^2\sin(2n\pi x)}{4n^5\pi^5} \Big\}$$

$$\tilde{\Omega}(x,t) = \int_0^t \Omega_x(x,\tau)\,d\tau$$

- An example for this case (on the next slide) illustrates the use of our package for the beam equation; a comparison with the exact solution is made (required by an optional argument).

```
f1[x_] := -2 Sin[4 π x]; uexact = -2 Cos[16 π^2 t] Sin[4 π x];
```

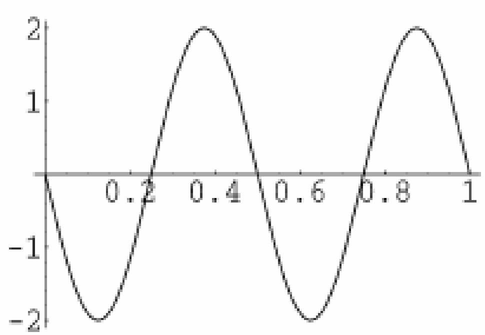
```
<< DSolveOCBeam`
```

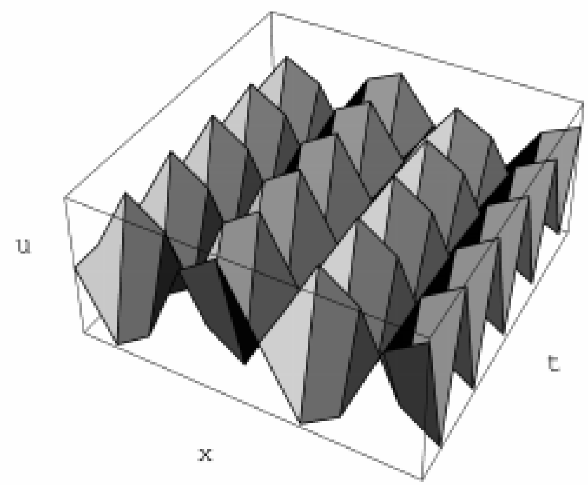
```
DSolveOCBeamN[{f1, 0}, u, x, t, {0.0000001, 1, 0.1}, {0, 1, 0.1}, 3, uexact]
```

Numerical values of the solution:

-2.51327×10^{-6}	-1.90211	-1.17557	1.17557	1.90211	-2.51327×10^{-6}
2.50454×10^{-6}	1.8955	1.17148	-1.17149	-1.8955	2.50454×10^{-6}
-2.47839×10^{-6}	-1.87571	-1.15925	1.15926	1.87571	-2.47839×10^{-6}
2.43501×10^{-6}	1.84288	1.13896	-1.13897	-1.84288	2.43501×10^{-6}
-2.3747×10^{-6}	-1.79724	-1.11075	1.11076	1.79724	-2.3747×10^{-6}
2.29789×10^{-6}	1.7391	1.07482	-1.07483	-1.7391	2.29789×10^{-6}
-2.2051×10^{-6}	-1.66888	-1.03142	1.03142	1.66887	-2.2051×10^{-6}
2.09697×10^{-6}	1.58705	0.980847	-0.98085	-1.58705	2.09697×10^{-6}
-1.97427×10^{-6}	-1.49418	-0.923454	0.923458	1.49418	-1.97427×10^{-6}
1.83785×10^{-6}	1.39093	0.859642	-0.859645	-1.39093	1.83785×10^{-6}
-1.68865×10^{-6}	-1.27801	-0.789854	0.789856	1.27801	-1.68865×10^{-6}

Visualization of function f and the solution:





{MinError, MaxError}:

{-1.98508×10^{-13}, 1.98394×10^{-9}}

Concluding remarks

- More details - in a paper, dedicated to the 75-th anniversary of Prof. I. Dimovski:

  M. Spiridonova. Operational methods in the environment of a computer algebra system. Serdica Journal of Computing 3 (2009), Bulgarian Acad. of Sci., Inst. of Mathematics and Informatics, 381 – 424.

- Further research and some applications are under way.