Partial Integro-Differential Operators

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(Integro-)Differential Operators

Ordinary differential operators $\mathcal{F}[\partial]$	$\subseteq$	Partial differential operators $\mathcal{F}[\partial_x, \partial_y]$
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Classical two-point boundary functionals ($a_i, b_i \in \mathbb{C}$):

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Regular boundary problem: $\forall f\ \exists! u$

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Simple example (more details later):

$$\boxed{\begin{aligned} &u_{tt} - u_{xx} = f \\ &u(x,0) = u_t(x,0) = u(0,t) = u(1,t) = 0 \end{aligned}}$$

Multiplying and Factoring BndProb

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Theorem

Let $(T, \mathcal{B})$ be a regular boundary problem. A given factorization $T = T_1 T_2$ can be **lifted** to a factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ of regular boundary problems with $\mathcal{B}_2 \leq \mathcal{B}$.

Factorization Example of Partial BndProb

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Unbounded wave equation:

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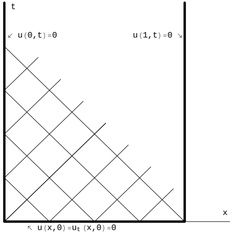

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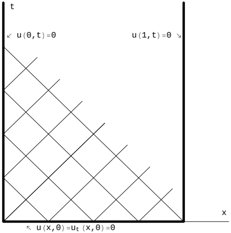

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$$(D_{tt} - D_{xx}, [L_t, L_t D_t, L_x, R_x]) = (D_t - D_x, [L_t, S]) \cdot (D_t + D_x, [L_t, L_x])$$

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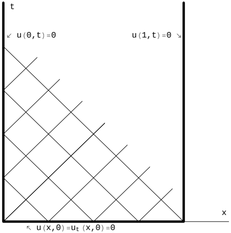

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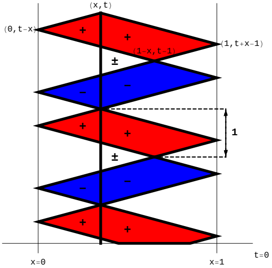

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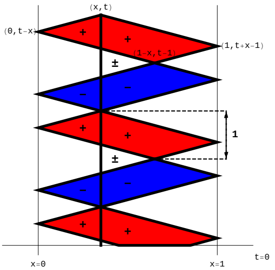

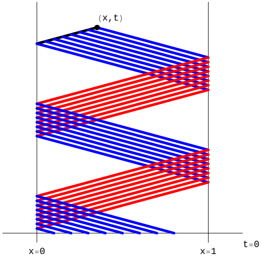

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Integral operators	$\int^x, \int^y$	$\int_0^x f(\xi, y)\, d\xi, \int_0^y f(x, \eta)\, d\eta$
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Integral operators	$\int^x$, $\int^y$	$\int_0^x f(\xi, y)\, d\xi$, $\int_0^y f(x, \eta)\, d\eta$
Evaluation operators	L_x, L_y	$f(0, y)$, $f(x, 0)$
Substitution operators	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \in \mathrm{GL}_2(\mathbb{R})$	$f(ax + by, cx + dy)$

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Back to Unbounded Wave Equation

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That’s all folks. . .

THANK YOU