Linear Differential Elimination for Analytic Functions

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Outline

- Problem: Decide whether given analytic function u can be written as $f_1(\alpha_1(z))g_1(z) + \ldots + f_k(\alpha_k(z))g_k(z)$

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- Examples / Applications

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$$u(x,y) = f_1(x) \cdot (\cos(x+y))^2 + f_2(y) \cdot \cos(2x+y)$$

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Q: Is $(\sin(x))^2$ of this form?

A: Yes: $\quad f_1(x)\equiv 1$, $\quad f_2(y) = -\cos(y)$ works!

$$\begin{array}{rcl} 1 & \cdot & (\cos(x)\cos(y)-\sin(x)\sin(y))^2 \\ -\cos(y) & \cdot & (2(\cos(x))^2\cos(y)-\cos(y)-2\cos(x)\sin(x)\sin(y)) \end{array}$$

How can we give an answer to such a question in general?

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Determine $f(y)$!

$f(y) = \exp(x+y)/\exp(x) \quad \forall x, \qquad x = 0 \quad \Rightarrow \quad f(y) = \exp(y).$

q.e.d.

Notation

$\Omega \subseteq \mathbb{C}^n$ open and connected

$K :=$ field of meromorphic functions on Ω

$z_1, \ldots, z_n$ coordinates for $\mathbb{C}^n$

$R := K\langle \partial_1, \ldots, \partial_n \rangle$ ring of differential operators

∂_i partial derivative w.r.t. z_i

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$$u(z) = f_1(\alpha_1(z))g_1(z) + \ldots + f_k(\alpha_k(z))g_k(z)$$

$g_1, \ldots, g_k : \Omega \to \mathbb{C}$ non-zero, analytic, $\nu(i) < n, \quad i = 1, \ldots, k$

$\alpha_i : \Omega \to \mathbb{C}^{\nu(i)}$ analytic with Jacobian of rank $\nu(i)$ throughout Ω

(α, g)-representability

Def. $u : \Omega \to \mathbb{C}$ analytic is called (α, g)-*representable* if

$$\exists\ f_i : \alpha_i(\Omega) \to \mathbb{C}, \quad i = 1, \ldots, k, \qquad (\alpha_i(\Omega) \subseteq \mathbb{C}^{\nu(i)})$$

such that $f_i \circ \alpha_i$ is analytic, $i = 1, \ldots, k$, and

$$u(z) = f_1(\alpha_1(z))g_1(z) + \ldots + f_k(\alpha_k(z))g_k(z) \qquad \text{for all } z \in \Omega.$$

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Special cases:

all $\nu(i) = n$: linear algebra over K

all $\nu(i) = 0$: linear algebra over $\mathbb{C}$

$0 < \nu(i) < n$: case treated here

essential (α, g)-representability

Let $u : \Omega \to \mathbb{C}$ be analytic.

Def.

- u is (α, g)-*representable around* $p_0 \in \Omega$ if

  $\exists$ open neighborhood $\Omega' \subseteq \Omega$ of p_0 such that

  $u|_{\Omega'}$ is $(\alpha|_{\Omega'}, g|_{\Omega'})$-representable.

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Example. $\Omega = \mathbb{C}$. Is x of the form $f(x^2)$?

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fix $1 \leq i \leq k$

recall: $\alpha_i : \Omega \to \mathbb{C}^{\nu(i)}$ has Jacobian of rank $\nu(i)$ throughout Ω

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$\implies$ around any point in Ω, $\exists\ n - \nu(i)$ independent variables

$\beta_{i,\nu(i)+1}, \ldots, \beta_{i,n}$ from $z_1, \ldots, z_n$ such that

$\beta = (\alpha_i, \beta_{i,\nu(i)+1}, \ldots, \beta_{i,n})$ is an analytic diffeomorphism

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$\rightsquigarrow$ $n - \nu(i)$ first order linear PDEs with coeff. in K

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Jacobian $\left(\frac{\partial \alpha_{i,j}}{\partial z_r}\right)_{1 \le j \le \nu(i),\, 1 \le r \le n}$ of α_i has rank $\nu(i)$

$B \in K^{n \times (n-\nu(i))}$ whose columns are a basis for the kernel

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Lemma. u is ess. (α_i, g_i)-repres. $\iff$ u is annihilated by

$$I(\alpha_i, g_i) := \left\langle \sum_{r=1}^{n} B_{r,s}\, \partial_r \circ \frac{1}{g_i(z)} \mid 1 \le s \le n - \nu(i) \right\rangle \le R.$$

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Examples. 1) $u(x,y) = f(x+y)\cdot 1,\quad \left(\frac{\partial \alpha_{1,j}}{\partial z_r}\right) = (1 \quad 1),$
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2) $u(x,y) = f(x^3 - xy) \cdot \sin(x)$, $\left(\frac{\partial \alpha_{1,j}}{\partial z_r}\right) = (3x^2 - y \quad -x)$,
$I(\alpha_1, g_1) = \langle \left(x \frac{\partial}{\partial x} + (3x^2 - y) \frac{\partial}{\partial y}\right) \circ \frac{1}{\sin(x)} \rangle$

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Let $u : \Omega \to \mathbb{C}$ be analytic.

Theorem.

$$\begin{aligned} & u \text{ is essentially } (\alpha, g)\text{-representable} \\ \iff \quad & u \text{ is annihilated by } I(\alpha, g) := \bigcap_{i=1}^{k} I(\alpha_i, g_i). \end{aligned}$$

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More efficient:
$$I_{1,2} := I(\alpha_1, g_1) \cap I(\alpha_2, g_2), \ \dots$$
$$I_{k-1,k} := I(\alpha_{k-1}, g_{k-1}) \cap I(\alpha_k, g_k)$$
$$I_{1,2,3} := I_{1,2} \cap I_{2,3}, \ \dots, \ I_{k-2,k-1,k} := I_{k-2,k-1} \cap I_{k-1,k}$$
$$\dots$$
$$I(\alpha, g) = I_{1,\dots,k-1} \cap I_{2,\dots,k}$$

Janet Bases

Janet basis $\{ (p_1, \mu_1), \ldots, (p_r, \mu_r) \}$ for $I \leq K\langle \partial_1, \ldots, \partial_n \rangle$

generating set for I s. t. every $p \in I$ can be written *uniquely* as

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- can read off a free resolution of R/I
- every Janet basis is a Gröbner basis

Linear algebra and jets

Use *jet variables* to write the partial derivatives of

$$u(z) = f_1(\alpha_1(z))g_1(z) + \ldots + f_k(\alpha_k(z))g_k(z).$$

$$\begin{array}{lcll} u_\mu & \longleftrightarrow & \partial_\mu u, & \mu \in (\mathbb{Z}_{\geq 0})^n \\ f_{i,\mu} & \longleftrightarrow & (\partial_\mu f_i) \circ \alpha_i, & \mu \in (\mathbb{Z}_{\geq 0})^{\nu(i)}, \quad i = 1, \ldots, k \end{array}$$

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$\rightsquigarrow \quad j_d(u) = \Delta_d(\alpha, g) \cdot j_d(\alpha, f)$ (up to order d)

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Example. $u(x,y) = x\, f_1(y + x^2) + y\, f_2(x + y^2)$

$$\Delta_2(\alpha, g) = \left(\begin{array}{cc|cc|cc} x & y & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & x & 2\,y^2 & 0 & 0 \\ 1 & 0 & 2\,x^2 & y & 0 & 0 \\ \hline 0 & 0 & 0 & 6\,y & x & 4\,y^3 \\ 0 & 0 & 1 & 1 & 2\,x^2 & 2\,y^2 \\ 0 & 0 & 6\,x & 0 & 4\,x^3 & y \end{array}\right) \qquad \begin{array}{c} u \\ u_{(0,1)} \\ u_{(1,0)} \\ u_{(0,2)} \\ u_{(1,1)} \\ u_{(2,0)} \end{array}$$

Linear algebra and jets

$\Delta_d(\alpha, g)$ has a block structure:

$$\Delta_d(\alpha,g) = \left(\begin{array}{c|c|c|c|c} \Delta_0(\alpha,g) & 0 & 0 & \dots & 0 \\ \hline \Delta_d^{(1,0)}(\alpha,g) & \Delta_d^{(1,1)}(\alpha,g) & 0 & \dots & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \hline \Delta_d^{(i,0)}(\alpha,g) & \Delta_d^{(i,1)}(\alpha,g) & \Delta_d^{(i,2)}(\alpha,g) & \dots & \Delta_d^{(i,i)}(\alpha,g) \end{array}\right)$$

Linear algebra and jets

Let $\Delta_d(\alpha, g) \in K^{\zeta(d)\times\sigma(d)}$.

row rank: $\operatorname{rank} \Delta_d(\alpha, g) = \zeta(d) - \dim I(\alpha, g)_{\le d}$

column rank: $\operatorname{rank} \Delta_d(\alpha, g) \le \sigma(d) - \tau(d)$

where $\tau(d)$ = number of parametric derivatives up to order d

for $\bigoplus_{i=1}^{k} R\, u_i / \langle I(\alpha_1, g_1) \oplus \ldots \oplus I(\alpha_k, g_k),\, u_1 + \ldots + u_k \rangle$

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(Note: sum of ideals is much easier to compute than intersection!)

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Strategy: for $d = 1,\ 2,\ 3,\ \ldots$
add new row relations to Janet basis for $I(\alpha, g)_{\leq d}$
compare number of parametric derivatives
to estimate for column rank until we have equality

Example

$$
\begin{aligned}
u(x,y) &= f_1(x+y^2) + f_2(x+y) \\
0 &= f_1(x+y^2) + f_2(x+y) \quad \Rightarrow \quad f_1 = f_2 = \text{const.}
\end{aligned}
$$

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#rows	$\frac{1}{(1-t)^2} \cdot \frac{1}{(1-t)}$	$= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \dots$
#columns	$\frac{2}{(1-t)} \cdot \frac{1}{(1-t)}$	$= 2 + 4t + 6t^2 + 8t^3 + 10t^4 + \dots$
column rank $\leq$	$\left(\frac{2}{(1-t)} - 1\right) \cdot \frac{1}{(1-t)}$	$= 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \dots$

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$$\Delta_2(\alpha, g) = \left(\begin{array}{cc|cc|cc} 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2y & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 4y^2 & 1 \\ 0 & 0 & 0 & 0 & 2y & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{array}{c} u \\ u_{(0,1)} \\ u_{(1,0)} \\ u_{(0,2)} \\ u_{(1,1)} \\ u_{(2,0)} \end{array}$$

Example

linear dependence among rows:

$$\{\ (y - \tfrac{1}{2})u_{(0,2)} + (-2y^2 + \tfrac{1}{2})u_{(1,1)} + (2y^2 - y)u_{(2,0)} - u_{(0,1)} + u_{(1,0)}\ \}$$

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Hilbert series (counting parametric derivatives): $1 + \frac{2t}{1-t}$

#rows	$\frac{1}{(1-t)^2} \cdot \frac{1}{(1-t)}$	$=$	$1 + 3t + 6t^2 + 10t^3 + 15t^4 + \ldots$
#columns	$\frac{2}{(1-t)} \cdot \frac{1}{(1-t)}$	$=$	$2 + 4t + 6t^2 + 8t^3 + 10t^4 + \ldots$
column rank $\leq$	$\left(\frac{2}{(1-t)} - 1\right) \cdot \frac{1}{(1-t)}$	$=$	$1 + 3t + 5t^2 + 7t^3 + 9t^4 + \ldots$
param. deriv.	$\left(1 + \frac{2t}{1-t}\right) \cdot \frac{1}{(1-t)}$	$=$	$1 + 3t + 5t^2 + 7t^3 + 9t^4 + \ldots$

Computing a representation

$$u(z) = \Delta_0(\alpha, g) \cdot j_0(\alpha, f) = (g_1 \ \dots \ g_k) \begin{pmatrix} f_1(\alpha_1(z)) \\ \vdots \\ f_k(\alpha_k(z)) \end{pmatrix}$$

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$u(z)$ representable $\iff$ $\exists\ a_1(z), \dots, a_{k-1}(z) \in K$ such that

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substitute i-th entry of r.h.s. into linear PDE in $I(g_i, \alpha_i)$

$\rightsquigarrow$ determining linear PDEs for $a_1(z), \ldots, a_{k-1}(z) \in K$

Application 1

Is given $u(x, y, z)$ a sum of analytic functions of two variables?

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Example. Spherical coordinates r, φ, θ

$\rightsquigarrow$ three commuting differential operators

$$\partial_r \;:=\; \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x\,\partial_x + y\,\partial_y + z\,\partial_z), \qquad \partial_\varphi \;:=\; -y\,\partial_x + x\,\partial_y,$$

$$\partial_\theta \;:=\; \frac{1}{\sqrt{x^2+y^2}}\,(-xz\partial_x - yz\partial_y) + \sqrt{x^2+y^2}\,\partial_z.$$

$\partial_r\,\partial_\varphi\,\partial_\theta$ characterizes sums of analytic functions depending only on two of r, φ, θ.

Application 2

Apply PDE solver to:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

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characterizing PDE: $\frac{\partial^2 v}{\partial x\, \partial y} = 0$

$u(x, y) = (x + y + 1)e^{-y}$ is a solution of (1),

but is *not* of the form $F_1(x) + F_2(y)$!

Application 3

Is there a representation $\rho : \mathbb{R} \to \mathrm{GL}(3, \mathbb{C})$ of the Lie group $\mathbb{R}$ with prescribed first row $\gamma(x) := (\gamma_1(x), \gamma_2(x), \gamma_3(x))$?

We should have $\gamma(0) = (1, 0, 0)$. Moreover,

$$\rho_{1,1}(x + y) = (\rho(x)\,\rho(y))_{1,1},$$

i.e.

$$\gamma_1(x + y) = \gamma_1(x)\,\gamma_1(y) + \gamma_2(x)\,\rho_{2,1}(y) + \gamma_3(x)\,\rho_{3,1}(y).$$

i.e. check (α, g)-representability of $\gamma_1(x + y) - \gamma_1(x)\,\gamma_1(y)$ with $g := (\gamma_2(x), \gamma_3(x))$ and $\alpha = (\alpha_1, \alpha_2) := (y, y)$!

Paper

For more details, see

W. Plesken, D. Robertz,
Linear Differential Elimination for Analytic Functions,
to appear in

Mathematics in Computer Science,
Special Issue AADIOS,
G. Regensburger, M. Rosenkranz, W. Sit (eds.)

References

M. Janet,
Leçons sur les systémes des équationes aux dérivées partielles,
Gauthiers-Villars, 1929

C. Méray,
Démonstration générale de l'existence des intégrales des équations aux dérivées partielles,
J. de mathématiques pures et appliquées, 3e série, tome VI, 1880

C. Riquier,
Les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, 1910

References

J. F. Ritt,
Differential Algebra,
Dover, 1966

J. M. Thomas,
Differential Systems,
AMS Colloquium Publications, vol. XXI, 1937

E. R. Kolchin,
Differential algebra and algebraic groups,
Academic Press, 1973

References

W. Plesken, D. Robertz,
Linear Differential Elimination for Analytic Functions,
to appear in Special Issue AADIOS, Mathematics in Computer Science

W. Plesken, D. Robertz,
Janet's approach to presentations and resolutions for polynomials and linear pdes,
Archiv der Mathematik, 84 (1), 2005

Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz,
The Maple Package "Janet": I. Polynomial Systems and
II. Linear Partial Differential Equations,
CASC 2003

D. Robertz,
Formal Computational Methods for Control Theory,
PhD thesis, RWTH Aachen University, 2006

References

V. P. Gerdt, Y. A. Blinkov,
Involutive bases of polynomial ideals. Minimal involutive bases,
Mathematics and Computers in Simulation, 45, 1998

Y. A. Blinkov, V. P. Gerdt, D. A. Yanovich,
Construction of Janet Bases, I. Monomial Bases, II. Polynomial Bases,
CASC 2001

V. P. Gerdt,
Involutive Algorithms for Computing Gröbner Bases,
Proc. "Computational commutative and non-commutative algebraic geometry" (Chishinau, June 6-11, 2004), IOS Press, 2005

V. P. Gerdt, Y. A. Blinkov, V. V. Mozzhilkin,
Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations,
Symmetry, Integrability and Geometry: Methods and Applications, 2006

References

T. Wolf, A. Brand, M. Mohammadzadeh,
Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions,
J. Symbolic Comput. **27**:2 (1999), 221–238

T. Wolf,
Applications of CRACK in the classification of integrable systems.
In *Superintegrability in classical and quantum systems*,
CRM Proc. Lecture Notes, Vol. 37, Amer. Math. Soc., Providence, RI, 2004, 283–300