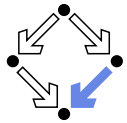
Converting matrices of Ore polynomials from Popov form to Hermite form using the FGLM algorithm

Johannes Middeke



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Linz, Austria

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Motivation/Overview

Ore polynomials are a **generalisation of** ordinary polynomials, linear **differential operators** and linear **difference operators**.

Matrices are used to represent linear systems of operators.

Normal forms capture important properties of matrices and the systems they represent.

We will present in this talk:

- Normal forms are connected to Gröbner bases (generalising [Kojima et al., 2007]).
- The FGLM algorithm can be used to convert normal forms into each other.

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Ore polynomials

Let K be a skew field with automorphism $\sigma\colon K \to K$ and (σ-) derivative $\vartheta\colon K \to K$.

Ore polynomials are expressions

$$a_n\partial^n + \ldots + a_1\partial + a_0 \quad \text{with } a_0, \ldots, a_n \in K.$$

We denote the set of all Ore polynomials by $R = K[\partial; \sigma, \vartheta]$.

Addition is as usual and multiplication is given by the commutation rule

$$\partial a = \sigma(a)\partial + \vartheta(a) \quad \text{for all } a \in K.$$

Examples are

- ordinary (commutative) polynomials ($\sigma = \mathrm{id}$ and $\vartheta = 0$),
- differential operators ($\sigma = \mathrm{id}$), and
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Matrices

We denote the ring of $m \times n$ **matrices** over R by ${}^mR^n$.

An important notion will be the degree of a matrix $M \in {}^mR^n$

$$\deg M = \max\{\deg M_{i,j} \mid 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}.$$

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Hermite normal form

A matrix in **Hermite normal form** is essentially in **upper row echelon form** with some restrictions on the degrees.

For example is the matrix

$$H = \begin{pmatrix} 1 & \partial & x\partial - 1 \\ 0 & \partial^2 - 1 & x + 1 \end{pmatrix} \in {}^2\mathbb{Q}(x)[\partial; \mathrm{id}, d/dx]^3$$

in Hermite normal form.

The left-most non-zero entries of each row are called pivots.

Every matrix can be brought to a unique Hermite normal form by elementary row operations.

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Gröbner bases of modules

There is a theory of (left) **Gröbner bases for Ore polynomials** that looks **"almost identical" to** that of **ordinary polynomials**.

Also the extension to modules is as in the commutative case.

In particular, are Monomials of the form

$$\partial^i \mathfrak{e}_j, \qquad \text{where } 0 \leqslant i \text{ and } 1 \leqslant j \leqslant n$$

and where $\mathfrak{e}_j$ is the j^{th} canonical basis vector.

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Monomial orderings

There are two important **admissible monomial orderings**:

Definition (position over term ordering)

$$\partial^i \mathfrak{e}_j <_{\mathrm{POT}} \partial^s \mathfrak{e}_t \iff j > t \vee (j = t \wedge i < s)$$

For $n = 2$ this yields

$$(0,1) < (0,\partial) < (0,\partial^2) < \ldots < (1,0) < (\partial,0) < (\partial^2,0) < \ldots$$

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Example

Let $v = (\partial^2 + 1, x - \partial) \in \mathbb{Q}(x)[\partial; \mathrm{id}, d/dx]^2$.

The term over position leading term is $(\partial^2, 0)$.

Let $w = (x\partial, 1)$ with leading term $(x\partial, 0)$.

Then v is reducible by $\{w\}$ and

$$v \longrightarrow_{\{w\}} v - \frac{1}{x}\partial \cdot w = (1 - \tfrac{1}{x}\partial, x - (1 + \tfrac{1}{x})\partial).$$

But v is not reducible by $\{(1, x - \partial^2)\}$ since the leading term is $-(0, \partial^2)$.

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Normal forms & Gröbner bases

We may generalise [Kojima et al., Prop. 2 and 4] to Ore polynomials

Theorem

*A matrix $H \in {}^mR^n$ is in **Hermite normal form** if and only if its rows form a **reduced position over term Gröbner basis** for $R^m H$.*

Theorem

A matrix $P \in {}^mR^n$ is in Popov normal form if and only if its rows form a reduced term over position Gröbner basis for $R^m P$.

In both cases do the pivots correspond to the leading terms.

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In both cases do the **pivots** correspond to the **leading terms**.

The FGLM algorithm

The FGLM algorithm may be used to **convert Gröbner bases from one monomial ordering to another one**.

The original algorithm was created for ideals of commutative polynomials having a finite dimensional quotient.

It translates the algebraic problem to a linear algebra problem.

In our case, the FLGM algorithm needs to work in the K-vector space

$$\frac{R^n}{R^m M} \quad \text{where } M \in {}^m R^n.$$

Unfortunately, this space is in general not finite dimensional.

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Degree bounds

We can generalise [Giesbrecht et al., 2009] to non-square matrices of Ore polynomials

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$$\deg H \leqslant m \deg M.$$

Similarly, we obtain

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The FGLM revisited

Using the **degree bounds**, for FGLM we only need to consider a **finite dimensional subspace** of $R^n/R^m M$.

Given a matrix $P \in {}^mR^n$ in Popov normal form, we want to compute the corresponding Hermite normal form H.

With FGLM, we test a monomials in ascending position over term order for whether they are

- leading monomial of a position over term Gröbner basis element, or
- element of a basis of $R^n/R^m P$.

The checks are done using the Gröbner basis P.

We can ignore all those monomials of degree larger than $m \deg P$.

The complexity (in field operations) amounts to $\mathcal{O}((mn \deg P)^4)$ —which drops to $\mathcal{O}((m \deg P)^4)$ if P is square.

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The FGLM revisited

Using the degree bounds, for FGLM we only need to consider a finite dimensional subspace of $R^n/R^m M$.

Given a matrix $P \in {}^mR^n$ in Popov normal form, we want to compute the corresponding Hermite normal form H.

With FGLM, we test a monomials in ascending position over term order for whether they are

- leading monomial of a position over term Gröbner basis element, or
- element of a basis of $R^n/R^m P$.

The checks are done using the Gröbner basis P.

We can ignore all those monomials of degree larger than $m \deg P$.

The **complexity** (in field operations) amounts to $\mathcal{O}((mn \deg P)^4)$ —which drops to $\mathcal{O}((m \deg P)^4)$ if P is **square**.

Example

Consider the matrix

$$M = \begin{pmatrix} 1 & \partial & x\partial - 1 \\ \frac{1}{x}\partial & \frac{1}{x} & \partial^2 - \frac{x+1}{x} \end{pmatrix} \in {}^2\mathbb{Q}(x)[\partial; \mathrm{id}, d/dx]^3.$$

The degree bound is 4 and we work in the space generated by the irreducible monomials of degree less or equal to 4

$$\overline{\mathfrak{e}_1}, \overline{\partial\mathfrak{e}_1}, \overline{\partial^2\mathfrak{e}_1}, \overline{\partial^3\mathfrak{e}_1}, \overline{\partial^4\mathfrak{e}_1}, \ \overline{\mathfrak{e}_2}, \ \overline{\mathfrak{e}_3}, \overline{\partial\mathfrak{e}_3}.$$

The smallest position over term monomials are

$$\mathfrak{e}_3, \partial\mathfrak{e}_3, \partial^2\mathfrak{e}_3, \partial^3\mathfrak{e}_3, \partial^4\mathfrak{e}_3,$$

and their residue classes are linearly independent.

Since we reached the degree bound, the next elements are $\mathfrak{e}_2, \partial\mathfrak{e}_2$ whose residue classes are still linearly independent.

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Since we **reached the degree bound**, the next elements are $\mathfrak{e}_2, \partial\mathfrak{e}_2$ whose residue classes are still **linearly independent**.

Example (cont.)

The residue class of the next monomial $\partial^2 \mathfrak{e}_2$ **depends linearly** on the previous ones.

We have

$$\partial^2 \mathfrak{e}_2 \equiv \mathfrak{e}_2 - (x+1)\mathfrak{e}_3 \pmod{P}.$$

Thus the first position over term Gröbner basis element is

$$(0, \partial^2 - 1, x+1).$$

Continuing in this manner we obtain the Hermite normal form of

$$\begin{pmatrix} 1 & \partial & x\partial - 1 \\ \frac{1}{x}\partial & \frac{1}{x} & \partial^2 - \frac{x+1}{x} \end{pmatrix}$$

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In this talk we have considered **matrices of Ore polynomials**.

We connected the Hermite normal form and the Popov normal form to Gröbner bases.

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We connected the Hermite normal form and the Popov normal form to Gröbner bases.

We adapted the FGLM algorithm to convert between Popov normal form and Hermite normal form.

Thank you very much!

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Hermite normal form

Definition (Hermite form)

A matrix $M \in {}^mR^n$ is in **Hermite form** if there exist column indices $j_1 > j_2 > \ldots > j_m$ which we call **pivot indices** such that

1. $M_{i,k} = 0$ if $k < j_i$,
2. the entries M_{i,j_i} are monic, and
3. $\deg M_{i,j_i} > M_{i,k}$ for all $k \neq j_i$.

Popov normal form

Definition (Popov form)

A matrix $M \in {}^{m}R^{n}$ is said to be in **Popov form**, if

1. M is row-reduced and $\operatorname{rdeg}_i M \leqslant \operatorname{rdeg}_{i+1} M$ for all i.
2. for the i^{th} row there exists a column index j_i (the **pivot index**) such that
   - 2.1 M_{i,j_i} is monic and $\deg M_{i,j_i} = \operatorname{rdeg}_i M$;
   - 2.2 $\deg M_{i,k} < \operatorname{rdeg}_i M$ if $k < j_i$;
   - 2.3 $\deg M_{k,j_i} < \operatorname{rdeg}_i M$ if $k \neq i$; and
   - 2.4 if $\operatorname{rdeg}_i M = \operatorname{rdeg}_k M$ and $i < k$ then $j_i < j_k$ (i. e., pivot indices are ordered increasingly).

Row-reducedness

Definition (Leading row coefficient matrix)

Let $M \in {}^mR^n$. The **leading row coefficient matrix** of M is defined by

$$\mathrm{LC}_{\mathrm{row}}(M) = \left(\sigma^{\deg M - \deg M_{i,\bullet}}(\mathrm{coeff}(\deg M_{i,\bullet}, M_{i,j}))\right)_{i,j} \in {}^mK^n.$$

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A matrix $M \in {}^mR^n$ is row-reduced if $\mathrm{LC}_{\mathrm{row}}(M)$ has left row-rank m.

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