Integro-differential Operators via Normal Forms in MAPLE

Anja Korporal

Research Institute for Symbolic Computation
Johannes Kepler University
Linz, Austria

Applications of Computer Algebra
Vlora, June 24, 2010

Outline

Available features

We show how our MAPLE package `IntDiffOp` can

- perform arithmetic operations in the algebra of integro-differential operators,
- compute solution operators (Green's operators) for boundary problems for LODEs from a given fundamental system,
- multiply boundary problems in a way corresponding to multiplication of their solution operators,
- lift given factorizations of differential operators to boundary problems.

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Integro-differential Algebras

Definition

An integro-differential algebra $(\mathcal{F}, \partial, \int)$ consists of a commutative differential algebra $(\mathcal{F}, \partial)$ over a field K and

- a K-linear right inverse $\int$ of ∂
- $(\int f')(\int g') = (\int f')g + f(\int g') - \int(fg)$ (differential Baxter axiom)
- We always have the evaluation $\mathrm{E} = 1 - \int\partial$.

Example

$\mathcal{C}^\infty(\mathbb{R})$ together with usual derivation and integration $\int = \int_0^x$.
In this case, the formulae above and the evaluation are just

- fundamental theorem of calculus
- a version of integration by parts
- $\mathrm{E}\colon f \mapsto f(0)$.

Integro-differential Operators

Definition

The integro-differential operators $\mathcal{F}[\partial, \int]$ is the free K-algebra in

- the symbols ∂ and $\int$,
- the functions $f \in \mathcal{F}$,
- a collection of multiplicative functionals $\Phi \subseteq \mathcal{F}^*$,

modulo the Noetherian and confluent rewrite system given by

$$
\begin{array}{lll|lll}
fg & \rightarrow & f \bullet g & \partial f & \rightarrow & \partial \bullet f + f\partial \\
\varphi\psi & \rightarrow & \psi & \partial\varphi & \rightarrow & 0 \\
\varphi f & \rightarrow & (\varphi \bullet f)\,\varphi & \partial\int & \rightarrow & 1 \\
\hline
\int f \int & \rightarrow & \multicolumn{4}{l}{(\int \bullet f) \int - \int (\int \bullet f)} \\
\int f \partial & \rightarrow & \multicolumn{4}{l}{f - \int(\partial \bullet f) - (\mathrm{E} \bullet f)\, \mathrm{E}} \\
\int f \varphi & \rightarrow & \multicolumn{4}{l}{(\int \bullet f)\, \varphi}
\end{array}
$$

Implementation in General

General setting

- As $\mathcal{F}$, we use "$\mathcal{C}^\infty(\mathbb{R})$" functions, that are representable in MAPLE.
- $\partial, \int$ are the usual derivation and integration.
- $\Phi \subset \mathcal{F}^*$ consists of arbitrary point evaluations.

Notation

- The function parameter is denoted by x.
- The differential operator ∂ is represented by D.
- The integral operator $\int = \int_0^x$ is represented by A.
- Point evaluations are denoted by E[c], where $c \in \mathbb{R}$ is the evaluation point.

Normal forms of Integro-differential Operators

Basic strategy

We implemented arithmetic operations directly in normal forms.

Normal forms

- Direct sum of a differential, an integral, and a so-called Stieltjes boundary operator.
- Differential operators are sums of terms of the form $f\partial^i$ with $f \in \mathcal{F}$.
- Integral Operators are sums of terms of the form $f\int g$ with $f, g \in \mathcal{F}$.

Data Structures

- $\texttt{INTDIFFOP}(\texttt{DIFFOP}(\ldots), \texttt{INTOP}(\ldots), \texttt{BOUNDOP}(\ldots))$
- $\texttt{DIFFOP}(f_0, f_1, f_2, \ldots)$, where f_i are the coefficients of D^i
- $\texttt{INTOP}(\texttt{INTTERM}(f_1, g_1), \texttt{INTTERM}(f_2, g_2), \ldots)$, where $\texttt{INTTERM}(f, g)$ represents the term $f \,\mathrm{A}\, g$

Normal forms of Integro-differential Operators

Normal Forms of Boundary Operators

Boundary operators are sums of terms having the form $f\varphi\partial^i$ or $f\varphi\int g$ with $\varphi \in \Phi$ and $f, g \in \mathcal{F}$.

Data Structures for Boundary Operators

- $\mathtt{BOUNDOP}(\mathtt{EVOP}, \mathtt{EVOP}, ...)$ as a tuple of evaluations at different points
- $\mathtt{EVOP}(c, \mathtt{EVDIFFOP}(\ldots), \mathtt{EVINTOP}(\ldots))$ where c is the evaluation point
- $\mathtt{EVDIFFOP}$ and $\mathtt{EVINTOP}$ similar to $\mathtt{DIFFOP}$ and $\mathtt{INTOP}$
  - $\mathtt{EVDIFFOP}(f_0, f_1, f_2, ...)$, with f_i the coefficients of $\mathrm{E}\,\mathrm{D}^i$
  - $\mathtt{EVINTOP}(\mathtt{EVINTTERM}(f_1, g_1), ...)$, for $f\,\mathrm{E}\,\mathrm{A}\,g + \ldots$.

Example

Arithmetic of Integro-differential Operators

The following MAPLE session shows the multiplication rule for integral operators,

$$\int \cdot f \int = (\int \bullet f) \int - \int (\int \bullet f),$$

with $\bullet$ meaning the application of the integral operator.

```
> with(IntDiffOp):
> op1 := INTOP(INTTERM(1,1)):
> op2 := INTOP(INTTERM(f(x), 1)):
> multiply(op1, op2);
```

$$\left(\left(\int_0^x f(x)\,\mathrm{d}x\right) . \mathsf{A}\right) - \left(\mathsf{A} . \left(\int_0^x f(x)\,\mathrm{d}x\right)\right)$$

Boundary Problems

Example

$$
\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}
$$

General notation

Given

- a forcing function $f \in \mathcal{F}$,
- a monic differential operator $T \in \mathcal{F}[\partial]$ of order n,
- n linear functionals $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$.

Find $u \in \mathcal{F}$ such that

$$
\begin{array}{l} Tu = f \\ \beta_1 u = \cdots = \beta_n u = 0 \end{array}
$$

Representation of Boundary Problems

In General

A boundary problem $(T, [\beta_1, \ldots, \beta_n])$ is represented by

- $\texttt{BP}(\texttt{DIFFOP}(T), \texttt{BC}(\ldots))$
  - $\texttt{BC}(\texttt{BOUNDOP}(\beta_1), \ldots \texttt{BOUNDOP}(\beta_n))$

Example

Back to the previous example.

```
> T := DIFFOP(0,0,1):
> b1 := BOUNDOP(EVOP(0, EVDIFFOP(1), EVINTOP())):
> b2 := BOUNDOP(EVOP(1, EVDIFFOP(1), EVINTOP())):
> Bp := BP(T, BC(b1, b2));
```

$$Bp := \texttt{BP}(\mathrm{D}^2, \texttt{BC}(\mathrm{E}[0], \mathrm{E}[1]))$$

Green's Operators

Definition

A boundary problem is called regular, if it has a unique solution for each forcing function. The operator $G\colon \mathcal{F} \to \mathcal{F}$, $f \mapsto u$ is called Green's operator.

Computation

```
> Bp := BP(T, BC(b1, b2));
```

$$Bp := \mathrm{BP}(\mathrm{D}^2, \mathrm{BC}(\mathrm{E}[0], \mathrm{E}[1]))$$

```
> greens_op(Bp);
```

$$(x.\,\mathrm{A}) - (\mathrm{A}\,.x) - ((x\,\mathrm{E}[1]).\,\mathrm{A}) + ((x\,\mathrm{E}[1]).\,\mathrm{A}\,.x)$$

Advantages of our package

Example

Consider the boundary problem

$$\boxed{\begin{array}{l} u''' - (e^x + 2)u'' - u' + (e^x + 2)u = f \\ u(0) = u(1) = u'(1) + u''(0) = 0 \end{array}}$$

- MAPLE can solve the differential equation and
- solve it together with some easy boundary conditions, but
- MAPLE cannot solve this boundary problem systemetically.
- The Green's operator computed by our package is given in our example worksheet.

Composition and Factorization of Boundary Problems

Definition

The composition of two boundary problems $(T_1, [\beta_1, \ldots, \beta_n])$ and $(T_2, [\gamma_1, \ldots, \gamma_m])$ is given by

$$(T_1 T_2, [\beta_1 T_2, \ldots, \beta_n T_2, \gamma_1, \ldots, \gamma_m]).$$

Theorem

For a regular boundary problem $(T, [\beta_1, \ldots, \beta_n])$ every factorization of the differential operator $T = T_1 T_2$ can be lifted to a factorization into regular lower-order boundary problems.

Note

For factoring the differential operator, we use the function `DFactor` by Mark van Hoeij in the MAPLE package `DEtools`.

Example

Easy Example

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}} = \boxed{\begin{array}{l} u' = f \\ \int_0^1 u(\xi) d\xi = 0 \end{array}} \circ \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}}$$

Sample Computation

```
> Bp := BP(T, BC(b1, b2));
```

$$Bp := \mathrm{BP}(\mathrm{D}^2, \mathrm{BC}(\mathrm{E}[0], \mathrm{E}[1]))$$

```
> f1, f2 := factor_bp(Bp);
```

$$f_1, f_2 := \mathrm{BP}(\mathrm{D}, \mathrm{BC}(\mathrm{E}[1].\mathrm{A})), \mathrm{BP}(\mathrm{D}, \mathrm{BC}(\mathrm{E}[0]))$$

Outlook

Future Work

- Computation of Green's functions
- Extension for singular boundary problems
- Extension for multivariate functions

Thank you for your attention!

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