Discrete Burgers' equation, Binomial coefficients and Mandala

Hiroshi Umemura
Graduate School of Mathematics
Nagoya University
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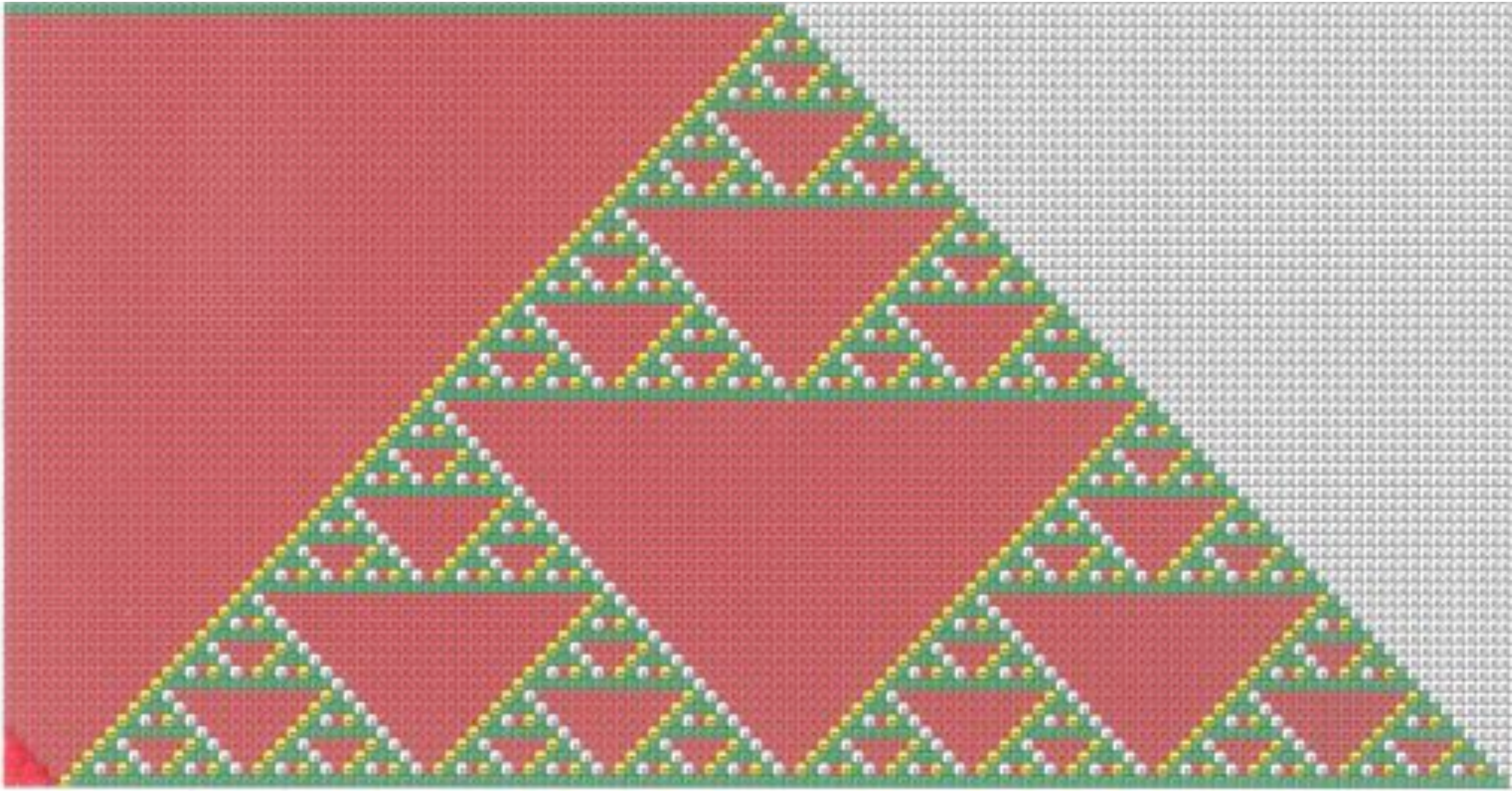
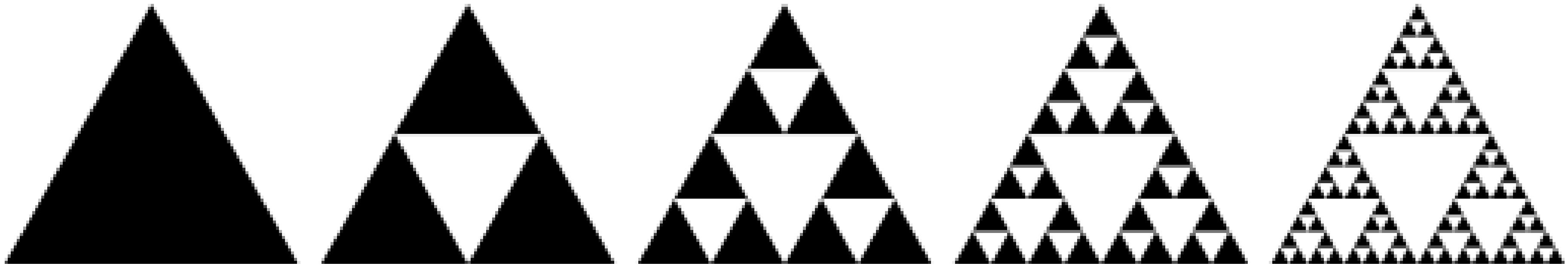
Burgers' equation is a fundamental partial differential equation in fluid mechanics. Gas dynamics, trafic flow.

$$u_t = 2uu_x + u_{xx}.$$





Joint work with Shuji Morikawa, Taihei Takeuchi.

- §1 Cellular Automaton CA

An infinite one dimensional array of cells

$\cdots$					$\cdots$

The j-th cell with state a_j

$\cdots$	a_{j-1}	a_j	a_{j+1}	a_{j+2}	$\cdots$

or simply

$$\cdots \quad a_{j-1} \quad a_j \quad a_{j+1} \quad a_{j+2} \quad \cdots$$

States a_j's change according to discrete time $t = 0, 1, 2, \ldots$. So we write

$$\cdots \quad a_{j-1}(n) \quad a_j(n) \quad a_{j+1}(n) \quad a_{j+2}(n) \quad \cdots$$

to show the states at time n.

Definition of Cellular Automaton. Assume that states $a_j(l)$ for $j \in \mathbb{Z}$ and $l \le n-1$ determine the states $a_j(n)$ for all $j \in \mathbb{Z}$. In other words,

$$a_j(n) = a_j(n)(a_i(l))_{l \le n-1,\, i \in \mathbb{Z}}.$$

Namely states $a_j(n)$ at time n are determined by the previous states $a_i(l)$, $l \le n-1$ of previous stages.

N.B. We also denote $a_j(n)$ by a_j^n.

Example. Rule 90 (Wolfram).

$$\text{For } \forall n \in \mathbb{Z},\, m \in \mathbb{N}, \qquad a_n(m) = 0,\, 1$$

satisfying

$$a_j(n+1) \equiv a_{j-1}(n) + a_{j+1}(n) \qquad \bmod 2,$$

for every $j \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$\begin{matrix} 1 & 1 & 1, \\ & 0 & \end{matrix} \qquad \begin{matrix} 1 & 1 & 0, \\ & 1 & \end{matrix} \qquad \begin{matrix} 1 & 0 & 1, \ldots \\ & 0 & \end{matrix}$$

Your state of tomorrow totally depends on your two neighbors!

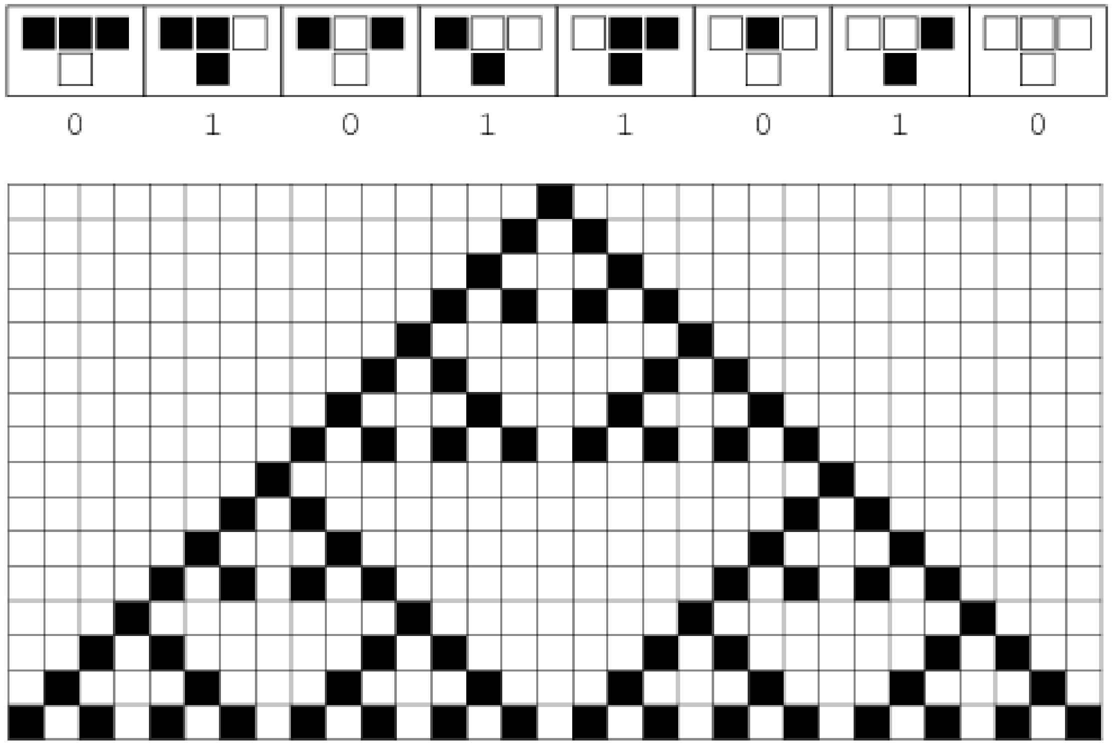
Evolution of the sequence

$$\ldots 0\,0\,1\,0\,0 \ldots$$

according to Rule 90

$$\begin{matrix}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \ldots
\end{matrix}$$

rule 90



Pascal's triangle = binomial coefficients mod 2.

Conclusion. Evolution of the sequence

$$\ldots 0\,0\,1\,0\,0 \ldots$$

acording to Rule 90 is nothig but binomial coefficients mod 2 giving Sierpinski triangle, pre-Mandala.

§2 Burgers' equation

Burgers' equation is a fundamental partial differential equation in fluid mechanics. Gas dynamics, trafic flow.

$$u_t = 2uu_x + u_{xx}.$$

Where does it come from?

Consider a linear partial differential equation

$$f_t = f_{xx}. \qquad \text{Heat equation}$$

that describes distribution of heat on the line $\mathbb{R}$ with coordinate x over time t.

Cole-Hopf transformation

$$u := \frac{f_x}{f}.$$

Then u satisfies Burgers' equation. f is the τ-function of Burgers' equation.

Discretize Burgers' equation.

$$\text{Burgers' equation} \qquad u_t = 2uu_x + u_{xx}. \qquad (1)$$

$$\text{Discrete Burgers' equation} \qquad v_j^{n+1} = v_{j-1}^n \frac{v_j^n v_{j+1}^n + 1}{v_{j-1}^n v_j^n + 1} \qquad (2)$$

Here $j \in \mathbb{Z}$, $n \in \mathbb{N}$, $v_j^n := v_j(n)$.

Discretization Procedure

Discretization through linear equation.

	discretization	
Linear differential equation	$\longrightarrow$	Linear difference equation
$\downarrow$ Cole-Hopf transformation		$\downarrow$ Difference Cole-Hopf transform
Burgers' equation	$\cdots\longrightarrow$	*Difference Burgers' equation*

Want to know the discrete nature of Burgers' equation. In discrete Burgers' equation, parameters $(j, n) \in \mathbb{Z} \times \mathbb{N}$ are discrete but the values or the states $v_n^j \in \mathbb{R}$ are not discrete.
To choose either
(1) Ultra-discretize difference Burgers' equation or
(2) Study difference Burgers' equation over a finite field.

(1) was studied by Hirota and Takahashi. Rule 184 (Wolfram) describing trafic flows.
Let us adopt (2).
The simplest case is over the finite field $\mathbb{F}_2$.

Numerical experiments with initial conditions $v_j^0 = 0, 1$ suggest first case to study is the initial conditions

$$\ldots 000\overbrace{111\ldots1}^{k}000\ldots.$$

Moreover the case $k = 2^l$ is fundamental.
Finally

$$\ldots 11111110000000\ldots. \tag{3}$$

to get Mandala.

Be careful

$$\frac{0}{0}$$

happens.
We should define

$$\frac{0}{0} = \text{indeterminate}$$

denoted by i.

Notation

$\frac{0}{1} = 0$,	$\frac{1}{1} = 1$,	$\frac{1}{0} = \infty$,	$\frac{0}{0} =$ indeterminate,
White	Green	Yellow	Red.

How to calculate the evolution.

We had better first work over $\mathbb{Z}$. Starting from the initial condition (3), numerical experiments suggest

Lemma 1.

$$v_j^n = \frac{v_{j-n}^0 \left(1 + \sum_{k=1}^n \begin{pmatrix} n \\ k \end{pmatrix} \prod_{l=1}^{2k} v_{j-n+l}^0\right)}{1 + \sum_{k=1}^n \begin{pmatrix} n \\ k \end{pmatrix} \prod_{l=1}^{2k} v_{j-n-1+l}^0}$$

Proof Induction on n.

It is convenient to set

$$g_j^n = 1 + \sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \prod_{l=1}^{2k} v_{j-n-1+l}^0 \text{ for } j \in \mathbb{Z},\ n = 1,\ 2,\ 3,\ \ldots$$

and

$$g_j^0 = 1 \text{ for } j \in \mathbb{Z}.$$

So we have

$$v_n^j = \frac{v_{j-n}^0 g_{j+1}^n}{g_j^n} \text{ for } n \in N,\ j \in \mathbb{Z}.$$

Lemma 2. $g_j^n \in \mathbb{Z}[v_j^0]_{j\in\mathbb{Z}}$ (resp. $\mathbb{F}_2[v_j^0]_{j\in\mathbb{Z}}$) is an irreducible polynomial.

Lemma 3. g_j^n satisfies the recurrence relation

$$g_j^{n+1} = g_{j-1}^n + v_{j-n-1}^0 v_{j-n}^0 g_{j+1}^n \text{ for } n \in \mathbb{N},\ j \in \mathbb{Z}.$$

Proposition 1. Let us work over $\mathbb{F}_2$. Initial condition

$$v_j^0 = \begin{cases} 1 \in \mathbb{F}_2, & \text{for } J \leq 0, \\ 0 \in \mathbb{F}_2, & \text{for } J \geq 1. \end{cases}$$

$$\ldots 11111110000000 \ldots.$$

We get for every integer $m \geq 0$

$$v_j^n = \begin{cases} \text{indeterminate}, & \text{for } j \leq -n, \\ \infty, & \text{for } j = -n+1, \\ 1, & \text{for } n+2 \leq j \leq n, \\ 0, & \text{for } j \geq n+1, \end{cases}$$

where $n = 2^m$. Namely

$$\ldots iiii\infty \overbrace{111\ldots1}^{2n-1} 0000 \ldots$$

§3 Burgers' equation and binomial coefficients

Discretization of the Heat equation

$$f_t = f_{xx}.$$

Discrete Heat equation

$$f_j^{n+1} = c(f_{j-1} + f_{j+1}), \tag{4}$$

c being a constant. Discrete Cole-Hopf transformation

$$v_j^n := f_{j+1}^n / f_j^n.$$

Then v_j^n satisfies the discrete Burgers' equation.
So we may assume $c = 1$

Binomial coefficients satisfies the discrete Heat equation with $c = 1$.

$$\begin{pmatrix} n \\ j \end{pmatrix} = \begin{pmatrix} n-1 \\ j-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ j+1 \end{pmatrix}.$$

Conclusion Assume $c = 1$.
(1) The binomial coefficients satisfies the discrete Heat equation.
(2) Every solution of the discrete Heat equation is obtained as a super-position of the binomial coefficients.

Proposition 2 We starting from the initial condition

$$\ldots 000011000 \ldots,$$

Rule CA 90 gives f_j^n. Then the solution

$$v_j^n := f_{j+1}/f_j^n$$

to discrete Burgers' equation describes Mandala.

We may say that the Mandala or discretized Burgers' equation mod 2 is a friend of Rule CA 90.
On the other hand, the ultra-discretization of Burgers' equation

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On the other hand, the ultra-discretization of Burgers' equation

gives CA of trafic flow or Rule CA 184.

How do we explain it?
Observation from general Galois theory?

4 Differential analogue of binomial coefficients

We work over $\mathbb{Z}$. Consider the binomial coefficients

$$\frac{1}{2^n}\binom{n}{j}$$

for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. In fact, we choose $c = 1/2$ in the recurrence relation

$$f_j^{n+1} = \frac{1}{2}\left(f_{j-1}^{n-1} + f_{j+1}^{n-1}\right).$$

When the time n is fixed, the sequence

$$\frac{1}{2^n}\binom{n}{j}, \qquad j = 0, 1, 2, \ldots.$$

the normal distribution or Gaussian distribution.

The differential analogue of the solution to the discrete Heat equation is $f(x,t)$ defined on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ such that

$$f_t = f_{xx} \text{ with } f(x, 0) = \delta(x),$$

where $\delta(x)$ is the delta function. Physically heat is given on the origin $x = 0$ at time 0. $f(x, t)$ is the temperature on the point x at time t.

The answer is Gaussian density

$$f(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

with deviation $\sqrt{t}$. It scatters as time goes on. So the corresponding solution to the Burgers' equation is

$$u(x, t) := f_x/f = -\frac{x}{2t}.$$

When t is fixed, $y = -\frac{x}{2t}$ is a line.

Can you see Madala here?