A normal form for 2-dimensional linear functional systems

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Introduction

- In this talk, we study the integration of 2-dimensional linear functional systems (e.g., PDEs in 2 variables).

- Normal forms for 1-dimensional linear functional systems with coefficients in a skew field K (over $K[\partial; \alpha, \beta]$, α injective) exist:

Smith and Jacobson normal forms.

- In this talk, we develop

"a normal normal form" for 2-dimensional linear systems.

- It is more intricate than Smith and Jacobson normal forms.

- We need to use "sophisticated algebraic tools" such as algebraic analysis, pure modules, filtration by purity, $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$.

Introduction

- The main idea is to integrate the system in "cascade":

$$R\,\eta = 0 \quad \Leftrightarrow \quad \begin{cases} S\,\zeta = B\,\tau, \\ S''\,\tau = A\,\upsilon, \\ S'\,\upsilon = 0, \end{cases}$$

$S'\,\upsilon = 0$ 0-dimensional, $S''\,\tau = 0$ 1-dimensional,

$S\,\zeta = 0$ 2-dimensional.

- In some particular cases, we can take $A = 0$ and $B = 0$

$\Rightarrow$ decouple the i^{th}-dimensional parts.

Introduction

- How can we constructively find the 0-dimensional, 1-dimensional and 2-dimensional parts of a given linear functional system?

$$\begin{cases} \partial_2^2\,\theta = 0, \\ \partial_1\,\partial_2\,\theta = 0. \end{cases}$$

- $z_1 = \partial_1\,\theta$ satisfies $\partial_2\,z_1 = 0$, i.e., $z_1 = f(x_1)$: 1-dimensional.
- $z_2 = \partial_2\,\theta$ satisfies:

$$\begin{cases} \partial_2\,z_2 = 0, \\ \partial_1\,z_2 = 0, \end{cases}$$

i.e., $z_2 = c$ is an arbitrary constant: 0-dimensional.

$$\Rightarrow\ \partial_2\,\theta = c\ \Rightarrow\ \theta = c\,x_2 + f(x_1).$$

- 2-dimensional (underdetermined systems):

$$\partial_1\,\sigma_1 + \partial_2\,\sigma_2 = 0\ \Rightarrow\ \sigma_1 = -\int_0^{x_1} \partial_2\,\sigma_2(x_1, x_2)\,dx_1 + \phi(x_2).$$

Notations

- Let D be a noetherian noncommutative polynomial ring.
- Let $R \in D^{q \times p}$ be a $q \times p$-matrix with coefficients in D.
- $D^{1 \times q} = \{\lambda = (\lambda_1 \ \ldots \ \lambda_q) \mid \lambda_i \in D, \ i = 1, \ldots, q\}$.
- $D^q = \{\eta = (\eta_1 \ \ldots \ \eta_q)^T \mid \eta_i \in D, \ i = 1, \ldots, q\}$.
- Let us consider the left (resp., right) D-homomorphism:

$$\begin{array}{ccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} \\ \lambda & \longmapsto & \lambda\, R, \end{array} \qquad \begin{array}{ccc} D^{p} & \xrightarrow{R.} & D^{q} \\ \eta & \longmapsto & R\, \eta. \end{array}$$

- Let us consider the left D-module $M = D^{1 \times p} / (D^{1 \times q}\, R)$.

$$0 \longrightarrow \ker_D(.R) \longrightarrow D^{1 \times q} \xrightarrow{.R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0.$$

$$\forall \ \lambda \in \ker_D(.R) : \ R\, \eta = \zeta \ \Rightarrow \ \lambda\, \zeta = (\lambda\, R)\, \eta = 0.$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\mathrm{gld}(D) = 2 \quad \Rightarrow \quad \ker_D(.R) = D^{1\times r}\, R_2, \quad \ker_D(.R_2) = 0,$$

i.e., the inhomogeneous linear system

$$R\,\eta = \zeta,$$

admits a set of left D-linearly independent compatibility conditions.

$D = k[\partial_1, \partial_2]$, $A_2(k) = k[x_1, x_2][\partial_1, \partial_2]$, $B_2(k) = k(x_1, x_2)[\partial_1, \partial_2]$.

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R.} D^p$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$$

$$\ker_D(R.) = Q\,D^m$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$$

$$D^r \xleftarrow{R_2.} D^q$$

$$\mathrm{ext}^2_D(M, D) = \mathrm{coker}_D(R_2.) = D^r/(R_2\, D^q)$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m$$

$$D^r \xleftarrow{R_2.} D^q \xleftarrow{R'.} D^{p'}$$

$$\ker_D(R_2.) = R'\, D^{p'}$$

$$R_2\, R = 0 \quad \Rightarrow \quad \mathrm{im}_D(R.) = R\, D^p \subseteq \ker_D(R_2.) = R'\, D^{p'}$$

$$\Rightarrow \quad \exists\, R'' \in D^{p'\times p} : \; R = R'\, R''.$$

$$\mathrm{ext}^1_D(M, D) = \ker_D(R_2.)/\mathrm{im}_D(R.) = (R'\, D^{p'})/(R\, D^p)$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccccc} & & D^q & \xleftarrow{R.} & D^p & \xleftarrow{Q.} & D^m \\ & & \| & & \downarrow R''. & & \\ D^r & \xleftarrow{R_2.} & D^q & \xleftarrow{R'.} & D^{p'} & & \end{array}$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccccc} & & D^q & \xleftarrow{R.} & D^p & \xleftarrow{Q.} & D^m \\ & & \| & & \downarrow R''. & & \\ D^r & \xleftarrow{R_2.} & D^q & \xleftarrow{R'.} & D^{p'} & \xleftarrow{Q'.} & D^{m'} \end{array}$$

$$\ker_D(R'.) = Q'\, D^{m'}$$

Duality & $\mathrm{ext}^i_D(M, D)$

$$0 \longrightarrow D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0$$

$$\begin{array}{ccccccccc} & & D^q & \xleftarrow{R.} & D^p & \xleftarrow{Q.} & D^m \\ & & \| & & \downarrow R''. & & \downarrow Q''. \\ D^r & \xleftarrow{R_2.} & D^q & \xleftarrow{R'.} & D^{p'} & \xleftarrow{Q'.} & D^{m'} \end{array}$$

$$R'\, R''\, Q = R\, Q = 0$$

$$\Rightarrow \ \mathrm{im}_D((R''\, Q).) = (R''\, Q)\, D^p \subseteq \ker_D(R'.) = Q'\, D^{m'}$$

$$\Rightarrow \quad \exists\ Q'' \in D^{m\times m'} : \ R''\, Q = Q'\, Q''.$$

Biduality & $\mathrm{ext}^j_D(\mathrm{ext}^i_D(M, D), D)$

$$\begin{array}{ccc} D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} \\ \uparrow .R'' & & \uparrow .Q'' \\ D^{1\times p'} & \xrightarrow{.Q'} & D^{1\times m'} \end{array}$$

$$R''\,Q = Q'\,Q''$$

Biduality & $\mathrm{ext}_D^j(\mathrm{ext}_D^i(M, D), D)$

$$\begin{array}{ccccc} D^{1\times t} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} \\ \uparrow .S'' & & \uparrow .R'' & & \uparrow .Q'' \\ D^{1\times t'} & \xrightarrow{.S'} & D^{1\times p'} & \xrightarrow{.Q'} & D^{1\times m'} \end{array}$$

$$\ker_D(.Q) = D^{1\times t}\, S, \quad \ker_D(.Q') = D^{1\times t'}\, S', \quad S''\, S = S'\, R''$$

Biduality & $\mathrm{ext}_D^j(\mathrm{ext}_D^i(M, D), D)$

$$\begin{array}{ccccccc}
D^{1\times u} & \xrightarrow{.T} & D^{1\times t} & \xrightarrow{.S} & D^{1\times p} & \xrightarrow{.Q} & D^{1\times m} \\
\uparrow .T'' & & \uparrow .S'' & & \uparrow .R'' & & \uparrow .Q'' \\
D^{1\times u'} & \xrightarrow{.T'} & D^{1\times t'} & \xrightarrow{.S'} & D^{1\times p'} & \xrightarrow{.Q'} & D^{1\times m'}
\end{array}$$

$$\ker_D(.S) = D^{1\times u}\, T, \quad \ker_D(.S') = D^{1\times u'}\, T', \quad T''\, T = T'\, S''.$$

$$R\, Q = 0 \;\Rightarrow\; \mathrm{im}_D(.R) = D^{1\times q}\, R \subseteq \ker_D(.Q) = D^{1\times t}\, S$$
$$\Rightarrow\; \exists\, F \in D^{q\times t} : \; R = F\, S.$$

$$R'\, Q' = 0 \;\Rightarrow\; \mathrm{im}_D(.R') = D^{1\times q}\, R' \subseteq \ker_D(.Q') = D^{1\times t'}\, S'$$
$$\Rightarrow\; \exists\, F' \in D^{q\times t'} : \; R' = F'\, S'.$$

Biduality & $\mathrm{ext}^i_D(\mathrm{ext}^i_D(M, D), D)$

- $t(M) = \{m \in D \mid \exists\, 0 \neq d \in D : \; d\, m = 0\}$ submodule of M.

$$\begin{array}{rcl}
\mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) & \cong & D^{1\times t} \Big/ \left(D^{1\times(t'+u)} \begin{pmatrix} S'' \\ T \end{pmatrix} \right), \\
\mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) & \cong & D^{1\times t'} \Big/ \left(D^{1\times(q+u')} \begin{pmatrix} F' \\ T' \end{pmatrix} \right), \\
t(M) & \cong & D^{1\times t} \Big/ \left(D^{1\times(q+u)} \begin{pmatrix} F \\ T \end{pmatrix} \right), \\
M/t(M) & \cong & D^{1\times p}/(D^{1\times t}\, S).
\end{array}$$

- We have the following short exact sequences ($\mathrm{gl}(D) = 2$):

$$0 \longrightarrow t(M) \xrightarrow{\iota} M \xrightarrow{p} M/t(M) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) \xrightarrow{\alpha} t(M) \xrightarrow{\beta} \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \longrightarrow 0.$$

Dimensions

- For $D = k[x_1, x_2]$, $k[\partial_1, \partial_2]$, $B_2(k) = k(x_1, x_2)[\partial_1, \partial_2]$ (k field):

$$\begin{cases} \dim_D(\mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D)) = 0, \\ \dim_D(\mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D)) = 1, \\ \dim_D(M/t(M)) = 2. \end{cases}$$

- For $D = A_2(k) = k[x_1, x_2][\partial_1, \partial_2]$, $A[\partial_1, \partial_2]$, where $A = k[[x_1, x_2]]$ or $\mathbb{C}\{x_1, x_2\}$ and k a field of characteristic 0:

$$\begin{cases} \dim_D(\mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D)) = 2, \\ \dim_D(\mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D)) = 3, \\ \dim_D(M/t(M)) = 4. \end{cases}$$

First main theorem

- Theorem: We have $t(M) \cong D^{1\times(t+t')}/(D^{1\times(t'+u+q+u')}\,U)$,

$$U = \begin{pmatrix} S'' & -A^{\natural} \\ T & -A^{\flat} \\ 0 & F' \\ 0 & T' \end{pmatrix}, \quad A^{\natural} \in D^{t'\times t'}, \quad A^{\flat} \in D^{u\times t'},$$

with $\begin{pmatrix} A^{\natural} \\ A^{\flat} \end{pmatrix} = \begin{pmatrix} I_{t'} \\ 0 \end{pmatrix} + \begin{pmatrix} S'' \\ T \end{pmatrix} B + C \begin{pmatrix} F' \\ T' \end{pmatrix} \in D^{(t'+u)\times t'}$,

is any representative of the residue class of $(I_{t'}^T \quad 0^T)^T$ in:

$$\Pi = D^{(t'+u)\times t'} / \left(\begin{pmatrix} S'' \\ T \end{pmatrix} D^{t\times t'} + D^{(t'+u)\times(q+u')} \begin{pmatrix} F' \\ T' \end{pmatrix} \right).$$

In particular, we can take $A^{\natural} = I_{t'}$ and $A^{\flat} = 0$.

First main theorem

- If there exist $B \in D^{t \times t'}$ and $C \in D^{(t'+u)\times(q+u')}$ such that

$$\begin{pmatrix} I_{t'} \\ 0 \end{pmatrix} + \begin{pmatrix} S'' \\ T \end{pmatrix} B + C \begin{pmatrix} F' \\ T' \end{pmatrix} = 0, \quad (\star)$$

then we can take $A^{\natural} = 0$ and $A^{\flat} = 0$ and get:

$$t(M) \cong \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \oplus \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D).$$

- If $D = k[x_1, x_2]$ or $D = k[\partial_1, \partial_2]$, then $(\star)$ is equivalent to

$$\mathrm{row}\left(\begin{pmatrix} I_{t'} \\ 0 \end{pmatrix}\right) = -(\mathrm{row}(B) \quad \mathrm{row}(C)) \begin{pmatrix} \begin{pmatrix} S'' \\ T \end{pmatrix}^T \otimes I_{t'} \\ I_{t'+u} \otimes \begin{pmatrix} F' \\ T' \end{pmatrix} \end{pmatrix},$$

where $\otimes$ is denotes the Kronecker product (tensor product).

Systemic interpretation

- Corollary: If we denote by $V = (F^T \quad T^T)^T \in D^{(q+u)\times t}$ and $\mathcal{F}$ is a left D-module, then $t(M) \cong D^{1\times t}/(D^{1\times (q+u)}\, V)$ and

$$\left\{ \begin{array}{l} F\,\theta = 0, \\ T\,\theta = 0, \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} S''\,\tau - A^{\natural}\,\upsilon = 0, \\ T\,\tau - A^{\flat}\,\upsilon = 0, \\ F'\,\upsilon = 0, \\ T'\,\upsilon = 0, \end{array} \right.$$

$$\begin{array}{rcl} \delta : \ker_{\mathcal{F}}(U.) & \longrightarrow & \ker_{\mathcal{F}}(V.) \\ \left(\begin{array}{c} \tau \\ \upsilon \end{array} \right) & \longmapsto & \theta = \tau - B\,\upsilon, \end{array}$$

$$\begin{array}{rcl} \delta^{-1} : \ker_{\mathcal{F}}(V.) & \longrightarrow & \ker_{\mathcal{F}}(U.) \\ \theta & \longmapsto & \left(\begin{array}{c} \tau \\ \upsilon \end{array} \right) = \left(\begin{array}{c} \theta + B\,S''\,\theta \\ S''\,\theta \end{array} \right). \end{array}$$

In particular, we can take $A^{\natural} = I_{t'}$, $A^{\flat} = 0$ and $B = 0$.

Example I

- We consider $D = \mathbb{Q}[\partial_1, \partial_2]$ and $R = (\partial_2^2 \quad \partial_1\,\partial_2)^T$.

Let us integrate the following linear system of PDEs (torsion):

$$\begin{cases} \partial_2^2\,\theta = 0, \\ \partial_1\,\partial_2\,\theta = 0. \end{cases}$$

- Denoting $R_2 = (\partial_1 \quad -\partial_2)$, we have the long exact sequence:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0.$$

- We can easily compute the different matrices

$$R' = (\partial_2 \quad \partial_1)^T,\ R'' = \partial_2,\ Q = 0,\ Q' = 0,$$
$$S = 1,\ S' = 1,\ T = 0,\ T' = 0,\ F' = R',\ S'' = R'',$$

$$\Rightarrow \quad \begin{cases} \mathrm{ext}_D^2(\mathrm{ext}_D^2(M, D), D) \cong D/(\partial_1, \partial_2), \\ \mathrm{ext}_D^1(\mathrm{ext}_D^1(M, D), D) \cong D/(D\,\partial_2). \end{cases}$$

Example I

- Hence, we obtain the matrix

$$U = \begin{pmatrix} \partial_2 & -A^\natural \\ 0 & \partial_2 \\ 0 & \partial_1 \end{pmatrix},$$

where $A^\natural = 1 + (B + C_{11})\,\partial_2 + C_{12}\,\partial_1$, where $B,\ C_{11},\ C_{12} \in D$,

$$\begin{cases} \partial_2^2\,\theta = 0, \\ \partial_1\,\partial_2\,\theta = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \partial_2\,\tau - \upsilon = 0, \\ \partial_2\,\upsilon = 0, \\ \partial_1\,\upsilon = 0, \end{cases} \qquad (\star)$$

where $\theta = \tau$ and $\upsilon = \partial_2\,\tau$.

- We first integrate the last two PDEs to get $\upsilon = c$, where c is an arbitrary constant, and then the inhomogeneous PDE $\partial_2\,\tau = c$,

$$\theta = \tau = c\,x_2 + f(x_1),$$

where f is an arbitrary smooth function.

Example II

- Let us consider $D = \mathbb{Q}[\partial_1, \partial_2]$ and the matrix:

$$R = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 \\ \partial_1 & -\partial_1 & -2\,\partial_1 \end{pmatrix}.$$

- Denoting $R_2 = (\partial_1 \quad -\partial_1 \quad \partial_2)$, we have the exact sequence:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M \longrightarrow 0.$$

Example II

- We can easily compute the different matrices:

$$Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad T = 0,$$

$$R' = \begin{pmatrix} 1 & 0 \\ 1 & -\partial_2 \\ 0 & -\partial_1 \end{pmatrix}, \quad Q' = 0, \quad S' = I_2, \quad T' = 0,$$

$$R'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 \\ -1 & 1 & 2 \end{pmatrix}, \quad S'' = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ -1 & 1 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ \partial_2 & -\partial_1 \\ \partial_1 & -\partial_1 \end{pmatrix}, \quad F' = R'.$$

Example II

- Hence, we get the following modules:

$$\begin{cases} \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \cong D^{1\times 2}/(D^{1\times 2}\, S''), \\ \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) \cong D^{1\times 2}/(D^{1\times 3}\, R'), \\ t(M) \cong D^{1\times 2}/(D^{1\times 3}\, F), \\ M/t(M) \cong D^{1\times 3}/(D^{1\times 2}\, S). \end{cases}$$

- In particular, we have:

$$\begin{cases} \dim_D(\mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D)) = 0, \\ \dim_D(\mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D)) = 1, \\ \dim_D(M/t(M)) = 2. \end{cases}$$

Example II

- Hence, we get the following matrix

$$U = \begin{pmatrix} 0 & \partial_2 - \partial_1 & -A^\natural_{11} & -A^\natural_{12} \\ -1 & 1 & -A^\natural_{21} & -A^\natural_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\partial_2 \\ 0 & 0 & 0 & -\partial_1 \end{pmatrix},$$

where $A^\natural_{11}$, $A^\natural_{12}$, $A^\natural_{21}$ and $A^\natural_{22}$ are defined by

$$\begin{pmatrix} A^\natural_{11} & A^\natural_{12} \\ A^\natural_{21} & A^\natural_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial_2 - \partial_1 \\ -1 & 1 \end{pmatrix} B + C \begin{pmatrix} 1 & 0 \\ 1 & -\partial_2 \\ 0 & -\partial_1 \end{pmatrix},$$

for all $B \in D^{2\times 2}$ and all $C \in D^{3\times 3}$.

$$B = -I_2, \quad C = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad A^\natural = 0.$$

$$\Rightarrow \; t(M) \cong \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \oplus \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D).$$

Example II

$$\begin{cases} \partial_2\,\theta_2 - \partial_1\,\theta_2 = 0, \\ \partial_2\,\theta_1 - \partial_1\,\theta_2 = 0, \\ \partial_1\,\theta_1 - \partial_1\,\theta_2 = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ \upsilon_1 = 0, \\ \upsilon_1 - \partial_2\,\upsilon_2 = 0, \\ -\partial_1\,\upsilon_2 = 0, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \tau_1 = \tau_2, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ \upsilon_1 = 0, \\ \partial_1\,\upsilon_2 = 0, \\ \partial_2\,\upsilon_2 = 0. \end{cases}$$

Hence, $\tau_1 = \tau_2 = f(x_1 + x_2)$, $\upsilon_1 = 0$ and $\upsilon_2 = c$, where f is an arbitrary smooth function and c an arbitrary constant.

Example II

- Finally, the integration of the sub-system of $R\,\eta = 0$

$$\begin{cases} \partial_2\,\theta_2 - \partial_1\,\theta_2 = 0, \\ \partial_2\,\theta_1 - \partial_1\,\theta_2 = 0, \\ \partial_1\,\theta_1 - \partial_1\,\theta_2 = 0, \end{cases}$$

corresponding to $t(M)$, gives the following general solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} = \begin{pmatrix} f\,(x_1 + x_2) \\ f\,(x_1 + x_2) + c \end{pmatrix},$$

where f (resp., c) is an arbitrary smooth function (resp., constant).

Second main theorem

- Theorem: We have $M \cong D^{1\times(p+t+t')}/(D^{1\times(t+t'+u+q+u')}\, P)$,

$$P = \begin{pmatrix} S & -G^{\natural} & -G^{\flat} \\ 0 & S'' & -A^{\natural} \\ 0 & T & -A^{\flat} \\ 0 & 0 & F' \\ 0 & 0 & T' \end{pmatrix},$$

with $(G^{\natural} \quad G^{\flat}) = (I_t \quad -B) + S\, J + K\, U \in D^{t\times(t+t')}$,

is any representative of the residue class of $(I_t \quad -B)$ in

$$\Omega = D^{t\times(t+t')}/\left(S\, D^{p\times(t+t')} + D^{t\times(t'+u+q+u')}\, U\right).$$

In particular, we can take:

$$A^{\natural} = I_{t'}, \quad A^{\flat} = 0, \quad B = 0, \quad C = 0, \quad G^{\natural} = I_t, \quad G^{\flat} = 0.$$

Second main theorem

- If there exist $J \in D^{p\times(t+t')}$ and $K \in D^{t\times(t'+u+q+u')}$ such that

$$(I_t \quad -B) + S\,J + K\,U = 0, \quad (\star)$$

then we can take $G^\natural = 0$ and $G^\flat = 0$ and get:

$$M \cong t(M) \oplus M/t(M).$$

Then, we have:

$$R\,\eta = 0 \;\Leftrightarrow\; \eta = (Q \quad -J)\begin{pmatrix} \xi \\ \tau \\ \upsilon \end{pmatrix},\; U\begin{pmatrix} \tau \\ \upsilon \end{pmatrix} = 0,\; \forall\, \xi \in \mathcal{F}^m.$$

- if D is a commutative ring, then $(\star)$ is equivalent to:

$$\mathrm{row}((I_t \quad -B)) = -(\mathrm{row}(J) \quad \mathrm{row}(K))\begin{pmatrix} S^T \otimes I_{t+t'} \\ I_t \otimes U \end{pmatrix}.$$

- Example: $D = k[x_1, x_2]$ or $k[\partial_1, \partial_2]$, where k is a field.

Systemic interpretation

- Corollary: We have the following equivalence:

$$R\,\eta = 0 \quad \Leftrightarrow \quad \begin{cases} S\,\zeta - G^{\natural}\,\tau - G^{\flat}\,\upsilon = 0, \\ S''\,\tau - A^{\natural}\,\upsilon = 0, \\ T\,\tau - A^{\flat}\,\upsilon = 0, \\ F'\,\upsilon = 0, \\ T'\,\upsilon = 0. \end{cases}$$

In particular, we can take:

$$A^{\natural} = I_{t'}, \quad A^{\flat} = 0, \quad B = 0, \quad C = 0, \quad G^{\natural} = I_t, \quad G^{\flat} = 0.$$

Systemic interpretation

- If $\mathcal{F}$ is a left D-module, then we have:

$$\begin{array}{ccc} \ker_{\mathcal{F}}(P.) & \xrightarrow{\varpi} & \ker_{\mathcal{F}}(R.) \\ \left(\begin{array}{c} \zeta \\ \tau \\ \upsilon \end{array}\right) & \longmapsto & \eta = \zeta - J \left(\begin{array}{c} \tau \\ \upsilon \end{array}\right), \end{array}$$

$$\begin{array}{ccc} \ker_{\mathcal{F}}(R.) & \xrightarrow{\varpi^{-1}} & \ker_{\mathcal{F}}(P.) \\ \eta & \longmapsto & \left(\begin{array}{c} \zeta \\ \tau \\ \upsilon \end{array}\right) = \left(\begin{array}{c} I_p + J \left(\begin{array}{c} (I_t + B\, S'')\, S \\ S''\, S \end{array}\right) \\ (I_t + B\, S'')\, S \\ S''\, S \end{array}\right) \eta. \end{array}$$

Example II (continued)

$$\begin{cases} \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \zeta_1 - \zeta_3 - G^{\natural}_{11}\,\tau_1 - G^{\natural}_{12}\,\tau_2 - G^{\flat}_{11}\,\upsilon_1 - G^{\flat}_{12}\,\upsilon_2 = 0, \\ \zeta_2 + \zeta_3 - G^{\natural}_{21}\,\tau_1 - G^{\natural}_{22}\,\tau_2 - G^{\flat}_{21}\,\upsilon_1 - G^{\flat}_{22}\,\upsilon_2 = 0, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ \upsilon_1 = 0, \\ \upsilon_1 - \partial_2\,\upsilon_2 = 0, \\ -\partial_1\,\upsilon_2 = 0. \end{cases}$$

- We can take:

$$G^{\natural}_{11} = 1, \; G^{\natural}_{12} = G^{\natural}_{21} = G^{\natural}_{22} = G^{\flat}_{11} = G^{\flat}_{12} = G^{\flat}_{21} = G^{\flat}_{22} = 0.$$

Example II (continued)

- If we consider the matrices

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $(G^{\natural} \quad G^{\flat}) = (I_t \quad -B) + S\,J + K\,U = 0$ and:

$$M \cong \mathrm{ext}^1_D(\mathrm{ext}^1_D(M, D), D) \oplus \mathrm{ext}^2_D(\mathrm{ext}^2_D(M, D), D) \oplus M/t(M).$$

$$\left\{\begin{array}{l} \zeta_1 - \zeta_3 = 0, \\ \zeta_2 + \zeta_3 = 0, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ -\tau_1 + \tau_2 = 0, \\ \upsilon_1 = 0, \\ \upsilon_1 - \partial_2\,\upsilon_2 = 0, \\ -\partial_1\,\upsilon_2 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \zeta_1 = \zeta_3, \\ \zeta_2 = -\zeta_3, \\ \tau_1 = \tau_2, \\ \partial_2\,\tau_2 - \partial_1\,\tau_2 = 0, \\ \upsilon_1 = 0, \\ \partial_2\,\upsilon_2 = 0, \\ \partial_1\,\upsilon_2 = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \zeta_1 = \zeta_3, \\ \zeta_2 = -\zeta_3, \\ \tau_1 = f(x_1 + x_2), \\ \tau_2 = f(x_1 + x_2), \\ \upsilon_1 = 0, \\ \upsilon_2 = c. \end{array}\right.$$

Example II (continued)

- A general solution of the following linear PD system

$$\begin{cases} \partial_2\,\eta_2 - \partial_1\,\eta_2 + \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_2\,\eta_1 - \partial_1\,\eta_2 - \partial_2\,\eta_3 - \partial_1\,\eta_3 = 0, \\ \partial_1\,\eta_1 - \partial_1\,\eta_2 - 2\,\partial_1\,\eta_3 = 0, \end{cases}$$

is

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \zeta_3 \\ -\zeta_3 \\ \zeta_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(x_1 + x_2) \\ f(x_1 + x_2) \\ 0 \\ c \end{pmatrix}$$

$$= \begin{pmatrix} \zeta_3 \\ -\zeta_3 + 2\,f(x_1 + x_2) \\ \zeta_3 - f(x_1 + x_2) + c \end{pmatrix},$$

where ζ_3 (resp., f) is an arbitrary smooth function of $\mathbb{R}^2$ (resp., $\mathbb{R}$) and c an arbitrary real constant.

A few more results

- The left D-module $M/t(M)$ is torsion-free. Since $\mathrm{gl}(D) = 2$:

1. $M/t(M)$ is either purely torsion-free,
2. or purely projective, i.e., stably free,
3. or free.

- If D is a commutative polynomial ring with coefficients in a field k, then projective modules are free (Quillen-Suslin theorem).

- If $D = A_2(k)$ or $B_2(k)$, k a field of characteristic 0, then projective modules of rank at least 2 are free (Stafford theorem).

- Corollary: If $M/t(M)$ is a projective left D-module, namely, there exist $X \in D^{p \times t}$ and $Y \in D^{t \times u}$ such that $S\,X + Y\,T = I_t$, then:

$$R\,\eta = 0 \quad \Leftrightarrow \quad \eta = (X \quad Q) \begin{pmatrix} \tau - B\,\upsilon \\ \xi \end{pmatrix}, \quad \forall\, \xi \in \mathcal{F}^m.$$

Example II (continued)

- We have $M = D^{1\times 3}/(D^{1\times 2}\, S) \cong D^{1\times 3}\, Q$, where:

$$S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

- S admits a right-inverse X, i.e., $S\, X = I_2$, defined by:

$$X = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

- $B = -I_2$, $\tau = (f(x_1, x_2) \quad f(x_1, x_2))^T$, $\tau = (0 \quad c)^T$, we get:

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(x_1, x_2) \\ f(x_1, x_2) + c \\ \xi \end{pmatrix} = \begin{pmatrix} \xi \\ 2\, f(x_1, x_2) + c - \xi \\ -f(x_1, x_2) + \xi \end{pmatrix}.$$