A difference operators attack on hard combinatorial problems

Christoph Koutschan

(joint work with Manuel Kauers and Doron Zeilberger)

Research Institute for Symbolic Computation
Johannes Kepler Universität Linz, Austria

25 June 2009
ACA 2009

Topics

This talk contains (but is not restricted to)

- discrete analogue of differential operators
- nontrivial application of operator algebra (ACA!)
- plane partitions
- a new proof of Stembridge's TSPP theorem
- big operators, heavy computations

Continuous vs. Discrete

Continuous	Discrete
"functions"	sequences
$x \in \mathbb{R}$, $x \in \mathbb{C}$	$n \in \mathbb{N}$, $n \in \mathbb{Z}$
differential equations	recurrences
differential operator D_x	shift operator S_n
$D_x \bullet f(x) = f'(x)$	$S_n \bullet f(n) = f(n+1)$
$D_x x = x D_x + 1$	$S_n n = n S_n + S_n$

Note: We will consider recurrences as members of "rational Ore algebras", i.e., as polynomials in the respective operators, say $S_{n_1}, \ldots, S_{n_d}$, with coefficients being rational functions in the variables $n_1, \ldots, n_d$.

Plane Partitions

Definition: A *plane partition* π of some integer n is a two-dimensional array

$$\pi = (\pi_{ij}), \quad \pi_{ij} \in \mathbb{N} \text{ for integers } i, j \geq 1$$

with finite sum $n = \sum_{i,j\geq 1} \pi_{ij}$ which is weakly decreasing in rows and columns, or more precisely

$$\pi_{i+1,j} \leq \pi_{ij} \quad \text{and} \quad \pi_{i,j+1} \leq \pi_{ij} \quad \text{for all } i, j \geq 1.$$

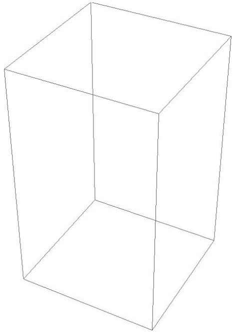
Example:

5	4	1
3	2	1
1		

To each plane partition we can draw its 3D Ferrers diagram by stacking π_{ij} unit cubes on top of the location (i, j).

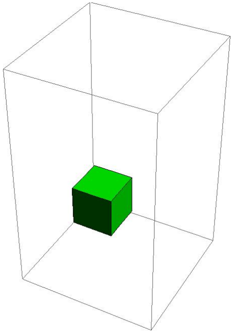
3D Ferrers diagram

5	4	1
3	2	1
1		



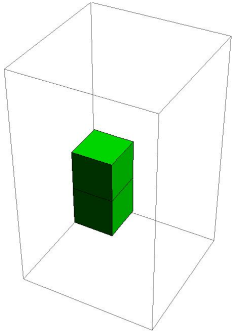
3D Ferrers diagram

5	4	1
3	2	1
1		



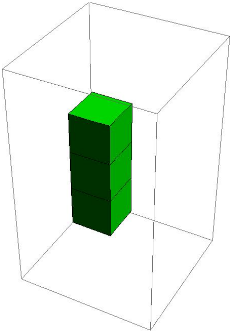
3D Ferrers diagram

5	4	1
3	2	1
1		



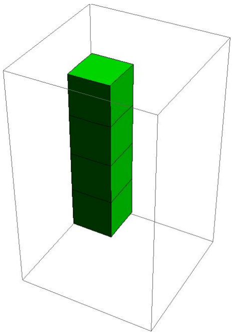
3D Ferrers diagram

5	4	1
3	2	1
1		



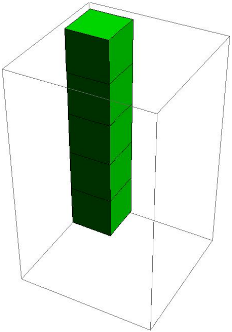
3D Ferrers diagram

5	4	1
3	2	1
1		



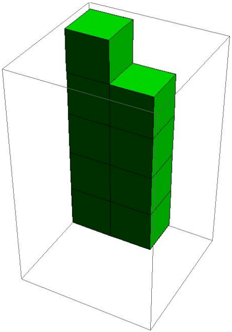
3D Ferrers diagram

5	4	1
3	2	1
1		



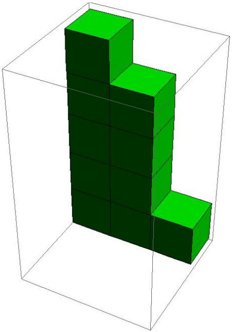
3D Ferrers diagram

5	4	1
3	2	1
1		



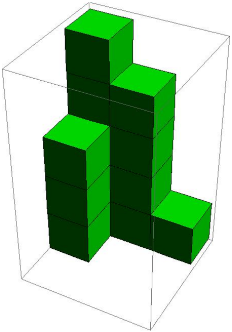
3D Ferrers diagram

5	4	1
3	2	1
1		



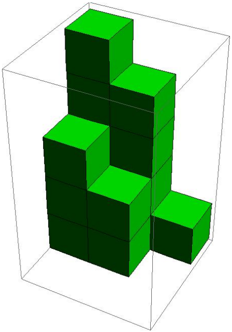
3D Ferrers diagram

5	4	1
3	2	1
1		



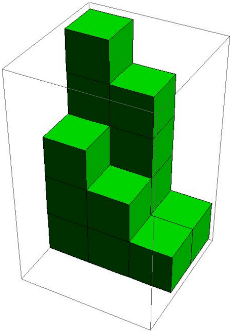
3D Ferrers diagram

5	4	1
3	2	1
1		



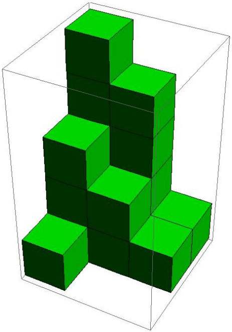
3D Ferrers diagram

5	4	1
3	2	1
1		



3D Ferrers diagram

5	4	1
3	2	1
1		

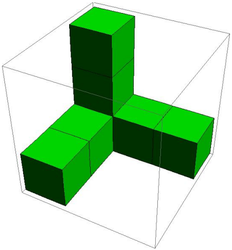


Totally Symmetric Plane Partitions (1)

Definition: A plane partition is *totally symmetric* iff whenever (i, j, k) is occupied (i.e. $\pi_{ij} \geq k$), it follows that all its permutations: $\{(i,k,j),(j,i,k),(j,k,i),(k,i,j),(k,j,i)\}$ are also occupied.

Example:

3	1	1
1		
1		



The TSPP Problem

Theorem: (John Stembridge, 1995)
The number of totally symmetric plane partitions (TSPPs) whose 3D Ferrers diagram is bounded inside the cube $[0, n]^3$ is given by the nice product-formula

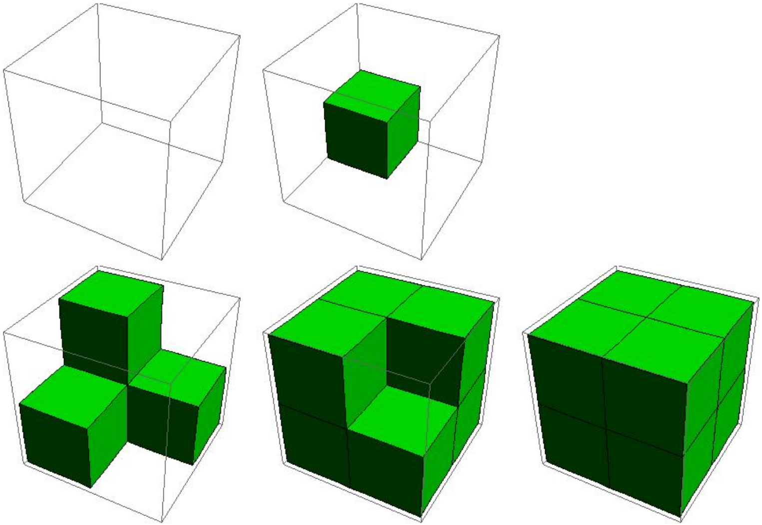
$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Example: For $n = 2$ we obtain

$$\prod_{1 \leq i \leq j \leq k \leq 2} \frac{i+j+k-1}{i+j+k-2} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = 5$$

Totally Symmetric Plane Partitions (2)

Example: All TSPPs for $n = 2$:



q-TSPP

There is a q-analogue:

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

- open problem since 1983
- our approach (to be presented) can be applied without further thinking
- BUT: computational challenge!

Okada's Determinant

Soichi Okada reduced the problem to a determinant evaluation. He proved that the TSPP conjecture is true if

$$\det\left(a(i,j)\right)_{1\le i,j\le n} = \prod_{1\le i\le j\le k\le n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2.$$

where

$$a(i,j) = \binom{i+j-2}{i-1} + \binom{i+j-1}{i} + 2\delta(i,j) - \delta(i,j+1)$$

Zeilberger's Ansatz

In his article

"The HOLONOMIC ANSATZ II. Automatic DISCOVERY(!) and PROOF(!!) of Holonomic Determinant Evaluations",

Doron Zeilberger proposed a method for proving certain determinant evaluations: "Pull out of the hat" a new sequence $B(n,j)$!

Zeilberger's Ansatz

For $i, n \in \mathbb{N}$, check the identities:

$$\begin{aligned} \sum_{j=1}^{n} B(n,j)a(i,j) &= 0, && i < n, \quad \text{(Soichi)} \\ B(n,n) &= 1, && n \geq 1, \\ \sum_{j=1}^{n} B(n,j)a(n,j) &= \frac{\text{Nice}(n)}{\text{Nice}(n-1)}, && n \geq 1. \quad \text{(Okada)} \end{aligned}$$

(the sequence $B(n,j)$ is supposed to be the (n,j)-cofactor divided by the $(n-1)$-determinant)

Zeilberger's prizes

In an e-mail, Doron Zeilberger wrote:

> "...here is a challenge to Christoph. If you can prove (Soichi), I'll give you a prize of $200 (in cash, out of my own pocket). If you can also prove (Okada), then I will give you an additional $100 (in cash, out of my own pocket).
> If you can also do the q-case, then I will give you $1000, during your next trip to the States, where I can pay you as a collaborator (out of my grant)."

How to get the $B(n, j)$

Result of guessing: 65 recurrences for $B(n, j)$
(their total size being about 5MB)

1. Unique, "short" description (leave away redundant recurrences)?
2. Are they consistent?
3. Is the description complete?
4. Do they really describe the desired object $B(n, j)$?

∂-finite description: Gröbner basis of the left ideal generated by them:

- 5 operators (their total size being about 1.6MB)
- leading monomials are $S_j^4, S_j^3 S_n, S_j^2 S_n^2, S_j S_n^3, S_n^4$
- staircase of regular shape
- vector space dimension is 10

$\Longrightarrow$ Answer to questions 1 and 2 is Yes!

Is the description complete?

Recall (univariate case): A P-finite recurrence

$$p_d(n)f(n+d) + \cdots + p_1(n)f(n+1) + p_0 f(n) = 0, \quad p_i \in \mathbb{K}[n]$$

of order d together with d initial values

$$f(0), f(1), \ldots, f(d-1)$$

uniquely defines the sequence $f(n)$ for all $n \in \mathbb{N}$ if the leading coefficient p_d does not have zeros in $\mathbb{N}$!

What about multivariate recurrences?

Investigate leading coefficients (1)

Are there points in $\mathbb{N}^2$ where none of the recurrences can be applied (because the leading term vanishes)?

- in the area $(4,4)+\mathbb{N}^2$, any of the recurrences may be applied
- look for common nonnegative integer solutions of all leading coefficients.
- a Gröbner basis computation reveals that everything goes well:

  $$\{(n-3)^2(n-2)(n-1)^2(2n-3)^2(2n-1)(j+n-1)(j+n),\dots\}$$

- solutions $(0,0)$, $(1,0)$, and $(0,1)$ (are also zeros for the other elements of this Gröbner basis)
- address the cases $n=1,2,3$: plugging these into the remaining polynomials we obtain further common solutions:

  $$\{(1,1),(2,1),(2,2),(3,2),(3,3)\}$$

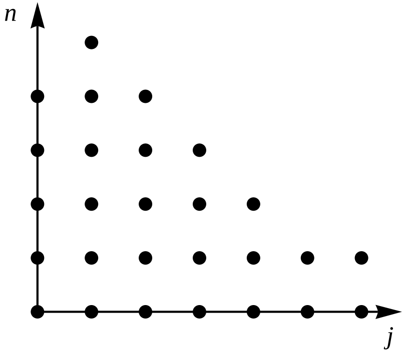
- all are outside of $(4,4)+\mathbb{N}^2$ so we need not to care

Investigate leading coefficients (2)

It remains to look at the lines $j = 0, 1, 2, 3$ and the lines $n = 0, 1, 2, 3\ldots$

Summarizing, the points for which initial values have to be given:

$$\{(j,n) \mid 0 \leq j \leq 6 \wedge 0 \leq n \leq 1\} \cup \{(j,2) \mid 0 \leq j \leq 4\} \cup$$
$$\{(j,3) \mid 0 \leq j \leq 3\} \cup \{(j,4) \mid 0 \leq j \leq 2\} \cup \{(1,5)\}.$$



Proving (Okada)—Getting started (1)

$$\sum_{j=1}^{n} B(n,j)a(n,j) = \frac{\mathrm{Nice}(n)}{\mathrm{Nice}(n-1)}$$

where

$$a(n,j) = \binom{n+j-2}{n-1} + \binom{n+j-1}{n} + 2\delta(n,j) - \delta(n,j+1).$$

Hence we can consider the expression

$$\sum_{j=1}^{n} B(n,j)a'(n,j) + 2B(n,n) - B(n,n-1)$$

with

$$a'(n,j) = \binom{n+j-2}{n-1} + \binom{n+j-1}{n} = \frac{2n+j-1}{n}\binom{n+j-2}{n-1}$$

being hypergeometric.

Proving (Okada)—Getting started (2)

Compute an annihilating ideal for $B(n,j)a'(n,j)$ with CK's package `HolonomicFunctions`:

```
DFiniteTimes[annBnj,
  Annihilator[(2n+j-1)/n*Binomial[n+j-2,n-1],
              {S[j],S[n]}]];
```

(takes just a few seconds).

Unsuccessful tries

Find a recurrence for

$$\sum_{j=1}^{n} B(n,j)a'(n,j).$$

The following algorithms are addressing this problem:

1. Zeilberger's slow algorithm
2. Takayama's algorithm
3. Chyzak's algorithm

But none of them worked in practice (because of computational effort)!

Successful approach (1)

"polynomial" ansatz for a telescoping operator:

$$\underbrace{\sum_{i=0}^{I} c_i(n) S_n^i}_{\text{principal part}} \quad + \quad \underbrace{(S_j - 1) \cdot \sum_{k=0}^{K} \sum_{l=0}^{L} \sum_{m=0}^{M} d_{k,l,m}(n) j^k S_j^l S_n^m}_{\text{delta part}}$$

- polynomial ansatz in j up to some degree
- reduce with the annihilator of the summand
- unknown functions $c_i, d_{k,l,m} \in \mathbb{Q}(n)$ can be computed using pure linear algebra
- no uncoupling needed
- no solving of difference equations needed
- form of the ansatz not as clear as in Chyzak's algorithm

Successful approach (2)

By means of modular computations, we found that an ansatz with

- principal part of order 7
- support of Q being the power products $S_j^l S_n^m$ with $l + m \leq 7$
- j-degree of 5 in the delta part

delivers a solution with nontrivial principal part.

Refined ansatz (after omitting the 0-components): 126 unknowns.

The telescoping relation

- was computed using homomorphic images and rational reconstruction
- quite some effort: rational functions in n with degrees up to 382 in the numerators and denominators
- takes about 5MB of memory (after reducing the delta part to normal form)
- counterchecked it by reducing the whole relation and obtaining 0 as expected

Last steps

∂-finite closure properties deliver a recurrence for

$$\sum_{j=1}^{n} B(n,j)a'(n,j) + 2B(n,n) - B(n,n-1).$$

(no positive integer zeros in the leading coefficients)

The right hand side simplifies to

$$\frac{\mathrm{Nice}(n)}{\mathrm{Nice}(n-1)} = \frac{\prod_{1\le i\le j\le k\le n}\left(\frac{i+j+k-1}{i+j+k-2}\right)^2}{\prod_{1\le i\le j\le k\le n-1}\left(\frac{i+j+k-1}{i+j+k-2}\right)^2} = \frac{4^{1-n}(3n-1)^2(2n)_{n-1}^2}{(3n-2)^2(n/2)_{n-1}^2}.$$

The recurrence for this expression is a right factor of the recurrence for the left hand side; initial values match. q.e.d.

Dear Chrsitoph,

I was about to write you a \$100 check, when I realized that you don't deserve it (yet) The stipulation of the prize was that you FIRST do (Soichi) for \$200, and then if you can also do (Okada) then you get an additional \$100.

(The reason is obvious, since according to Krattenthaler, it is still very possible that (Soichi) is wrong, and in that case, (Okada) is usless).

So, now proving (Soichi) will get you a \$300 prize.

Good luck!

Doron

P.S. This is like Jacob and Laban, before Jacob could marry Rachel, working for seven years, he had to marry Leah, also by working seven years, so the price of Rachel was 14 years of labor, plus an extra, unwanted wife. I am sure that it would take you less than 14 years.

(Soichi)

$$\sum_{j=1}^{n} B(n,j)a(i,j) = 0, \qquad i < n$$

was even harder to get (two surviving variables, need at least two relations, . . .)

Thanks to

- Doron Zeilberger for his advise and a \$300 cheque,
- Manuel Kauers for the guessing and fruitful discussions,
- you for your attention!