Rota-Baxter Type and Differential Type Algebras

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- Let **k** be a unitary commutative ring.
  A *Rota-Baxter* ***k****-algebra* R is a (not necessarily commutative, but associative) **k**-algebra with a **k**-linear operator $P: R \to R$ satisfying the *Rota-Baxter identity* for all $x, y \in R$:

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- Let R be the $\mathbb{R}$-algebra of continuous functions on $\mathbb{R}$ and P the integral operator sending a function $f(x)$ in R to the function

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- Baxter algebras were first introduced by Glenn Baxter in 1960 in fluctuation theory.

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- Examples:
  - RBA of weight λ, for a fixed parameter $\lambda \in \mathbf{k}$:

$$P(x)P(y) = P(P(x)y + xP(y) + \lambda \mathbf{xy}).$$

  - RBTA with *average operator*: $P(x)P(y) = P(xP(y))$.
  - RBTA with the *inverse average operator*: $P(x)P(y) = P(P(x)y)$.
  - RBTA with the *Reynolds operator*:

$$P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y)).$$

Rota's Open Problem

- Rota posed in 1995 the question (paraphrased):

  In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable. A partial (but fairly complete) list of such identities is the following. Besides endomorphisms and derivations, one has averaging operators, Reynolds operators and Baxter operators.

Interest in RBTA

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- Another applications from physics is related to the Yang-Baxter equation and new operators from combinatorial studies such as
  - **Nijenhuis operator**: $P(x)P(y) = P(xP(y) + P(x)y - P(xy))$
  - **Leroux's TD operator** (dendriform trialgebras): $P(x)P(y) = P(xP(y) + P(x)y - xP(1)y)$

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- A *differential* **k**-*algebra* R is (usually commutative and associative) **k**-algebra with a derivation operator $\delta : R \longrightarrow R$, which satisfies the **Leibniz identity** for all $x, y \in R$:

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- As with RBA, the identities may be modified or generalized. For example, we can add a parameter $\lambda \in \mathbf{k}$ and require the identity: $\delta(xy) = \delta(x)y + x\delta(y) + \lambda\delta(x)\delta(y)$.

Identities for Algebras

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- One can generalize this to **rings satisfying polynomial identities with a unary operator** (PIO-ring). For this, we have to construct the **operated polynomial ring**.
- A far more general theory called **variety of algebras** exists. An "**algebra**" is any set with a set of functions (operations), together with some identities perhaps. A **Galois connection** between identities and "variety of algebras" is set up similar to the correspondence between polynomial ideals and algebraic varieties. Thus, differential algebra is one variety of algebra, Rota-Baxter algebra is another, and so on.

What Wikipedia says about Varieties of Algebras

- In mathematics, specifically universal algebra, a *variety of algebras* [or a *finitary algebraic category*] is the class of all algebraic structures of a given **signature** satisfying a given set of **identities**. Equivalently, a variety is a class of algebraic structures of the same signature which satisfies the HSP properties: **closed under the taking of homomorphic images, subalgebras and (direct) products**.

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- Garrett Birkhoff proved equivalent the two definitions of variety, a result of fundamental importance to universal algebra and known as **Birkhoff's theorem** or as the **HSP theorem**. It is simple to see that the class of algebras satisfying some set of equations will be closed under the HSP operations. Proving the converse—classes of algebras closed under the HSP operations must be equational—is much harder.

Understanding Rota's Question

- Rota's question involved a **k**-algebra R with a **k**-linear unary operator P. The operations: addition, multiplication, scalar multiplication, and P, already are required to satisfy certain identities such as the commutative law of addition, the associative laws, the distributive law, and **k**-linearity for P.

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- Rota wants to find "**all possible algebraic identities that can be satisfied by a linear operator on an algebra**" and to "**classify all such identities.**" He also wants "**a complete list of such identities**."
- Clearly, starting with any identity, one can derive, using operations of the algebra, infinitely many **derived** identities. Such identities can be captured with the notion of an operated polynomial ideal.

Hyperidentities: Top Down Approach

- New identities may be obtained using substitutions: these have been called **hyperidentities**. For example, in a commutative algebra, from $xy = yx$, we can have $x^2y = yx^2$ by substituting x^2 for x or $(xz)y = y(xz)$.

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- These have been studied in much more generalities, even for partially defined operations, using **Galois connections and Galois-closed subrelations** (Busaman and Denecke, 2005). The basic structure of the class of algebras is a **lattice of varieties of algebras**, corresponding to a **lattice of identities**. Subclasses can be defined by suitable restrictions on the identities.

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- Interpreted literally, Rota's question may be answered by studying the sublattice of varieties of **k**-algebras with one unary operator and find the minimal varieties in the lattice.

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- The free operated $\mathbf{k}$-algebra $\mathbf{k}\{X\}$ on any set X is constructed using Rota-Baxter words as a basis, and is the **non-commutative operated polynomial ring** over $\mathbf{k}$ with indeterminates X.

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- We restrict the search to $M(U, V) \in \mathbf{k}\{U, V\}$ that will satisfy Eqn. (1) and $N(U, V) \in \mathbf{k}\{U, V\}$ that will satisfy Eqn. (2) through **substitution homomorphisms** (of operated algebras) $\sigma_{x,y} : \mathbf{k}\{U, V\} \to R$, with $(U, V) \mapsto (x, y)$.

Search for $M(U, V)$

- Using properties we developed for Rota-Baxter words, a preliminary result shows that it is nearly sufficient that $M(U, V)$ be associative in the sense that

$$M(M(U, V), W) = M(U, M(V, W)) \qquad \text{mod } \textbf{Eqn. (1)}$$

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- In particular $M(U, V)$ is a linear combination over $\mathbf{k}$ of Rota-Baxter words, in which there is no subexpression of the form $P(x)P(y)$ for any $x, y \in \mathbf{k}\{U, V\}$.

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- We use as an **ansatz** for $M(U, V)$ the linear combination. For ease of reading, we use x, y for U, V and $[x]$ for $P(U)$.

$$\begin{aligned} M(x,y) = {} & a_0 y[x] + a_1 x[y] + b_0[y]x + b_1[x]y + c_0[yx] + c_1[xy] \\ & + d_0 x[1]y + d_1 xy[1] + d_2 y[1]x + d_3 yx[1] + d_4[1]xy + d_5[1]yx \\ & + e_0 yx + e_1 xy \end{aligned}$$

Fourteen Varieties of RBTAs

- We found 14 (including 10 new types in red) distinct (ignoring trivial case when $M(x, y) = 0$) classes of varieties of RBTAs corresponding to the following $M(x, y)$ satisfying associativity.

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- We are in the process of verifying these.
  1. $x[y]$ (RBTA, average operator)
  2. $[x]y$ (RBTA, inverse average operator)
  3. $x[y] + y[x]$
  4. $[x]y + [y]x$
  5. $-[xy] + x[y] + [x]y$ (RBTA, Nijenhuis operator)
  6. $x[y] + [x]y + e_1xy$ (RBA with weight e_1)
  7. $x[y] - x[1]y + e_1xy$
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9. $x[y] + [x]y - x[1]y + e_1 xy$
   (RBTA: generalized Leroux TD operator with weight e_1)
10. $x[y] + [x]y - xy[1] - x[1]y + e_1 xy$
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12. $x[y] + [x]y - x[1]y - [1]xy + e_1 xy$
13. $d_0 x[1]y + e_1 xy$ (RBTA generalizing endomorphisms)
14. $d_2 y[1]x + e_0 yx$

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$$P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y)).$$

- We conjecture these are all the possible ones involving $P(x)P(y)$ on the left hand side of the identity.

(Non-Commutative) Differential Type Algebras

- Similar to RBTAs, DTAs of the form Eqn. (2) need to respect the associativity of multiplication of R. A preliminary result shows this is nearly sufficient:

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- In particular, $N(U, V)$ is a linear combination over $\mathbf{k}$ of Rota-Baxter terms, in which there is no subexpression of the form $P(xy)$ for any $x, y \in \mathbf{k}\{U, V\}$ (note P is δ).

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- We use again (x, y) for (U, V) and $[x]$ for $\delta(x)$ and ansatz for $N(x, y)$ involving the following RB terms:

$$x[y],\ x[[y]],\ [x]y,\ [x]\,[y],\ [x]\,[[y]],\ [[x]]y,\ [[x]]\,[y],\ [[x]]\,[[y]],$$
$$y[x],\ y[[x]],\ [y]x,\ [y]\,[x],\ [y]\,[[x]],\ [[y]]x,\ [[y]]\,[x],\ [[y]]\,[[x]],$$
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- We are in the process of verifying these by hand. Here, $b_1, c_0, c_1, d_0, d_1, d_2, e_1$ are elements of $\mathbf{k}$.
  1. $c_1[x]\,[y] + b_1(x[y] + [x]y) + e_1xy$, where $b_1^2 = b_1 + c_1e_1$ (NDTAs generalizing endomorphisms and derivations)
  2. $c_0[y]\,[x] - c_0e_1(y[x] + [y]x) + e_1xy + c_0e_1^2yx$
  3. $x[y] + [x]y + d_0x[1]y + e_1xy \qquad (d_0 = -1, e_1 = 0)$
  4. $[x]y + d_0(x[1]y - xy[1]) \qquad (d_0 = 0)$
  5. $x[y] + d_0(x[1]y - [1]xy) \qquad (d_0 = 0)$
  6. $d_1xy[1] + d_2[1]xy + e_1xy \qquad (d_1 + d_2 = 1, e_1 = 0)$

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- For NDTA of **Type 2**, if $e_1 = 0$ and $c_0 = 1$, then δ is an anti-morphism. In general, $c_0(\delta - e_1)$ **is an anti-morphism** if and only if $\delta(xy) = N_2(x, y)$.

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- If $[1] = 0$, then Types 3–6 reduce to the three simple cases $N(x, y) = x[y], [x]y$, **or** e_1xy (where $e_1\delta$ is the scalar multiplication by e_1 map).

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- While $N(x, y) = x[y]$ gives a variety of NDTA, the symmetric formula $N(x, y) = y[x]$ does not.

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- A solution of Rota's problem then should identify
  1. **a "complete" list of operator identities** for operated algebras, any other identity would be derived from these.
  2. any subset of identities from this list for which there is **a corresponding variety with free algebras**; and
  3. for each variety in Item 2, particular **interesting algebras in the variety**.

Free Objects and Further Research

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- A variety is **Schreier** if subalgebra of a free algebra in the variety is free. For example, the variety of all groups (resp. abelian groups) is Schreier. A central problem in the theory of varieties is whether it is Schreier. **Which of the varieties of RBTAs or NDTAs is Schreier?**

THANK YOU